THEORY OF ELASTIC SYSTEMS VIBRATING UNDER TRANSIENT IMPULSE WITH AN APPLICATION TO EARTHQUAKE-PROOF BUILDINGS

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THEORY OF ELASTIC SYSTEMS VIBRATING UNDER TRANSIENT IMPULSE WITH AN APPLICATION TO EARTHQUAKE-PROOF BUILDINGS

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Vibrating Systems under Transient Impulse.—The following theory gives a method of evaluating the action of very random impulses on vibrating systems (i.e., effect of static on radio-circuits or earthquakes on buildings). In the following text, we will use the language of mechanics.

Consider a one-dimensional continuous elastic system without damping. The free oscillations are given by the solutions of the homogeneous integral equation[1]

$$y = \omega^2 \int_a^b \rho(\xi)\alpha(x\xi)y(\xi)d\xi.$$

Due to the nature of the kernel there exists an infinite number of characteristic values ω_i of ω and of characteristic functions y_i solutions of this equation. These functions give the shape of the free oscillations of the system. They are orthogonal and have an arbitrary amplitude. This amplitude may be fixed by the condition of normalization,

$$\int_a^b \rho(\xi)y_i^2(\xi)d\xi = 1.$$

We now suppose that certain external forces $f(x)$ are acting on the system, these forces being expressed in such a way that the product of the displacement y by $f(x)$ represents the work done by this force. For example if $f(x)$ is a moment y will be the angle of rotation around the same axis as the moment at that point.

It can be easily proved that the statical deflection of the system is,

$$y = \sum \frac{C_i}{\omega_i^2} y_i$$

where C_i is the Fourier coefficient of the development of $\frac{f(x)}{\rho(x)}$ in a series of the orthogonal functions y_i,

$$\frac{f(x)}{\rho(x)} = \sum C_i y_i$$

hence

$$C_i = \int_a^b f(\xi) y_i(\xi) d\xi.$$

If the applied forces are variable with time and harmonic of the form $f(x)e^{i\omega t}$ the deflection is expressed by the expansion

$$y = \sum \frac{C_i y_i}{\omega_i^2} \frac{1}{1 - \omega^2/\omega_i^2} e^{i\omega t}. \tag{2}$$

The amplitude is composed of each of the terms of the statical deformation (1) *multiplied by a resonance factor* $\frac{1}{1 - \omega^2/\omega_i^2)}$.

The motion due to a sudden application of the forces is of the same type, and can be deduced immediately from the preceding harmonic solution.

By using Heaviside's expansion theorem we get,

$$y_a = \sum \frac{C_i y_i}{\omega_i^2} [1 - \cos \omega_i t]. \tag{3}$$

The amplitude due to a sudden applied force $f(x)$ is composed of a series of oscillations each of which has an amplitude equal to twice the corresponding term of the statical deformation (1).

We will now investigate the action of varying forces of the type $f(x)\psi(t)$; these forces are supposed to start their action at the origin of time and to keep on during a finite time T.

Using the Heaviside method, and the indicial admittance (3), the motion after the impulse has disappeared is given by,

$$y_b = \int_0^T \frac{d}{dt} y_a(t - \tau) . \psi(\tau) d\tau,$$

$$y_b = \sum \frac{C_i y_i}{\omega_i^2} \left\{ \sin \omega_i t [\omega_i \int_0^T \psi(\tau) \cos \omega_i \tau d\tau] - \cos \omega_i t [\omega_i \int \psi(\tau) \sin \omega_i \tau d\tau] \right\}$$

This motion is the superposition of free oscillations. Their respective amplitudes can be physically interpreted as follows:

Put
$$f_1(\nu) = \int_0^T \psi(\tau) \cos 2\pi\nu\tau \, d\tau$$

$$f_2(\nu) = \int_0^T \psi(\tau) \sin 2\pi\nu\tau \, d\tau$$

where ν is the frequency $\nu = \frac{\omega}{2\pi}$. The component free oscillations may then be written,

$$\frac{C_i y_i}{\omega_i^2} \cdot 2\pi\nu_i \sqrt{f_1^2(\nu_i) + f_2^2(\nu_i)}.$$

Now, according to the Fourier integral,

$$\psi(t) = 2 \int_0^\infty f_1(\nu) \cos 2\pi\nu t d\nu + 2 \int^\infty f_2(\nu) \sin 2\pi\nu t d\nu. \qquad (4)$$

This shows that the expression

$$F(\nu) = \sqrt{f_1^2(\nu) + f_2^2(\nu)}$$

may be considered as the "spectral intensity" curve of the impulse.

The amplitude of each free oscillation due to the transient impulse is

$$\frac{C_i y_i}{\omega_i^2} \cdot 2\pi\nu_i F(\nu_i). \qquad (5)$$

The expression $2\pi\nu F(\nu)$ is a dimensionless quantity that we will call "*reduced spectral intensity*." We then have the following theorem: *When a transient impulse acts upon an undamped elastic system, the final motion results from the superposition of free oscillations each of which has an amplitude equal to the corresponding term $C_i y_i/\omega_i^2$ of the statical deformation* (1) *multiplied by the value of the reduced spectral intensity for the corresponding frequency.*

The advantage of this theorem is that for the calculation of the motion it replaces a complicated impulse by a spectral distribution which is always an analytical function of the frequency.

This theorem could also have been established by starting from (2) and using directly the Fourier integral. We will apply this last method in order to generalize the theorem to the case of an elastic system with viscous damping whose motion is defined by the equation,

$$my + a\dot{y} + b^2y = A\psi(t). \tag{6}$$

The impulse is supposed to be given as before by the spectral distribution (4). Introducing a complex spectral distribution,

$$\varphi(\nu) = f_1(\nu) - if_2(\nu)$$

we may write,

$$\psi(t) = \int_{-\infty}^{+\infty} \varphi(\nu)e^{2\pi i\nu t}d\nu, \tag{7}$$

where according to the Fourier integral

$$\varphi(\nu) = \int^{T} \psi(\tau)e^{-2\pi i\nu\tau}d\tau.$$

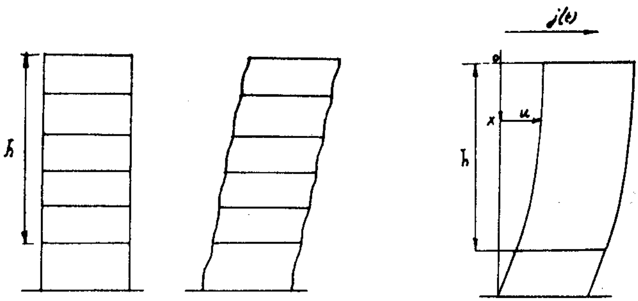

FIGURE 1

The function $\varphi(\nu)$ is holomorphic; its expansion in a power series is,

$$\varphi(\nu) = \sum \frac{A_n\nu^n}{n!}$$

where

$$A_n = (-2\pi i)^n \int_0^T t^n\psi(t)dt.$$

Calling M the largest value of $|\psi(t)|$ and $I(\nu)$ the coefficient of i in the variable ν considered from now on in the complex plane, we have

$$|\varphi(\nu)| < \frac{M}{2\pi T(\nu)}[e^{2\pi T(\nu)T} - 1].$$

This shows that for $t > T$, $|\varphi(\nu)e^{2\pi i\nu t}|$ has an upper limit.

Consider now the elastic system (6) under a harmonic impulse $Ae^{2\pi i\nu t}$, the corresponding motion is

$$x(t) = \frac{Ae^{2\pi i\nu t}}{8\pi^2\beta m}\left[\frac{1}{\nu - \nu_1} - \frac{1}{\nu - \nu_2}\right]. \tag{8}$$

The quantities ν_1 and ν_2 are complex frequencies

$$\nu_1 = \alpha i + \beta, \qquad \nu_2 = \alpha i - \beta.$$

The free oscillation of the system is damped and given by

$$e^{-2\pi\alpha t}\cos 2\pi\beta t.$$

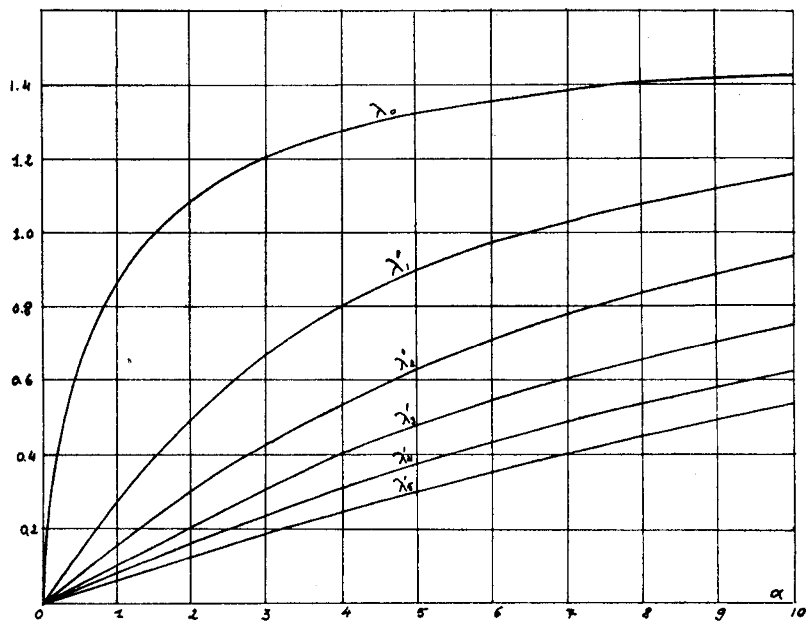

FIGURE 2

According to (7) and (8) the motion due to the impulse $A\psi(t)$ will be,

$$y(t) = \frac{A}{8\pi^2\beta m}\int_{-\infty}^{+\infty}\varphi(\nu)e^{2\pi i\nu t}\left[\frac{1}{\nu - \nu_1} - \frac{1}{\nu - \nu_2}\right]d\nu.$$

We have seen that $\left| F(\omega)e^{i\omega t}\right|$ has an upper limit, and by using then the method of contour integrals and residues, we find,

$$y(t) = \frac{A\pi i}{4\pi^2\beta m}\left[\varphi(\nu_1)e^{2\pi i\nu_1 t} - \varphi(\nu_2)e^{2\pi i\nu_2 t}\right].$$

At the time T when the impulse has ceased, the amplitude is,

$$\left| y(t) \right| = \frac{A}{4\pi^2\beta^2 m} 2\pi\beta \left| \varphi(\alpha i + \beta \right| e^{-2\pi\alpha T}.$$

The quantity $\dfrac{A}{4\pi^2 m(\beta^2 + \alpha^2)}$ is the deflection for the static deformation due to a force A. *This last result generalizes formula* (5) *to the case of damping. We have to consider a complex frequency* $\alpha i + \beta$ *and the analytical prolongation* $\varphi(\alpha i + \beta)$ *of the spectral distribution* $\varphi(\nu)$ *of* $\psi(t)$.

Application to Earthquakes.[2]—The study of seismogram spectral distributions has not yet been made; it is, however, the author's opinion that this study would be of great importance for two reasons:

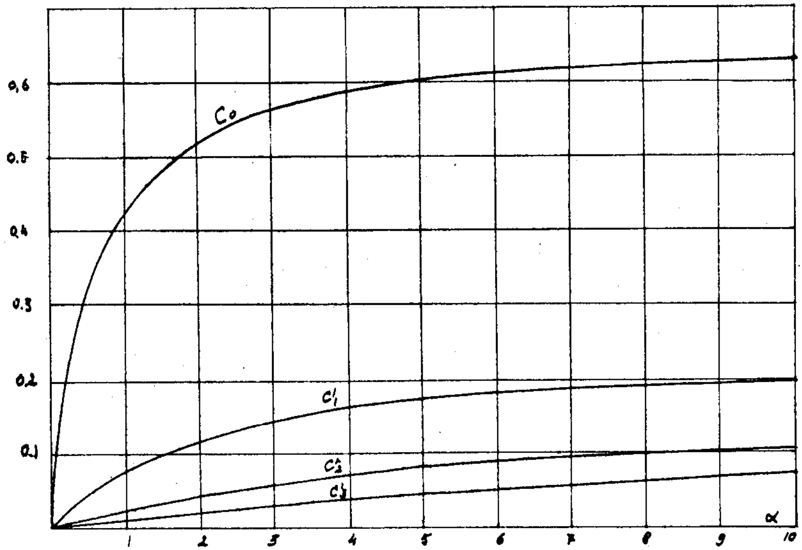

FIGURE 3

(1) The peaks of the spectral curves will reveal the presence of certain characteristic frequencies of the soil at given locations.

(2) By applying the preceding theorems the maximum effect of earthquakes on buildings will be easily evaluated by considering a spectral distribution having larger values than those deduced from known seismograms.

The theory has been applied to evaluate the effectiveness of the so-called "elastic first floor" in earthquake-proof buildings.

The building is supposed to have a simple frame, whose only possible deformation is shear (Fig. 1). All the stories except the first have a uniform mass and shear-rigidity. Call R the ratio of rigidity of the first story to the rigidity of the second, $n + 1$ the number of stories and M the total mass of the n stories above the first one. Put $\alpha = Rn$.

The building is replaced by a continuous beam of same average properties.

The frequencies of the free oscillations are given by

$$\nu_K = \frac{\lambda_K}{2\pi t_0}$$

where t_0 is the time necessary for a shear wave to go from the second floor to the top, and λ_K are functions only of α given by $\lambda \nu g \lambda = \alpha$. The values of λ'_K are plotted in figure 2 with $\lambda_K = k\pi + \lambda'_K$.

Suppose that $j_0\psi(t)$ is the horizontal acceleration seismogram and that we have calculated the spectral intensity $F(\nu) = \sqrt{f_1^2(\nu) + f_2^2(\nu)}$ of $\psi(t)$. The earthquake produces a series of co-existing free oscillations in the building. A total horizontal shear is produced by each of these free oscillations, and the maximum amplitude of each of these shears is,

$$S = j_0 M C_K(\alpha) \frac{F(\nu_K)}{t_0}.$$

The values of $C_K(\alpha)$ are plotted in figure 3.

We see that the fundamental oscillation is by far the most dangerous, and that the influence of the elastic first floor is only important for values of α smaller than 3.

[1] See for instance, F. H. van den Dungen, *Cours de Technique des Vibrations*, Brussels, 1926, and "Les Problèmes Généraux de la Technique des Vibrations," *Mém. Sc. Physiques*, t. 4 (Paris, 1928); K. Hohenemser, *Die Methoden zur angenäherten Lösung von Eigenwertproblemen in der Elastokinetic* (Berlin, Springer, 1932). The function $\alpha(x, \xi)$ is called the influence function.

[2] This study was undertaken at the suggestion of Professor Th. von Kármán. The author wishes to express his appreciation of the continual interest Professor von Kármán and Professor R. R. Martel have taken in its progress.