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Theory of Vibration of Buildings During Earthquake.

By *M. Biot* in Louvain (Belgien).

Gebäudeschwingungen bei Erdbeben.

Zusammenfassung. Die Bewegungen eines Gebäudes, das von einem Erdbeben getroffen wird, haben den Charakter vorübergehender Schwingungen. Die Untersuchung derartiger Vorgänge nach dem Heaviside-Verfahren ist den Elektro-Ingenieuren geläufig. Für den Fall der Erdbebenschwingungen wird indessen die Anwendung dieser Methode wegen der Unregelmäßigkeit der Erdbewegung zu mühsam. Außerdem braucht man nicht die Bewegung eines Gebäudes genau zu kennen, sondern nur die Scheitelwerte der Beanspruchung.

Die folgende Arbeit enthält eine Theorie der Schwingungen eines Systems unter einem vorübergehenden Anstoß. Der erste Paragraph enthält die theoretischen Grundlagen. Der Hauptgedanke ist die spektrale Zerleguug des Seismogramms, d. h. es wird die Frequenzverteilung des Erdstoßes untersucht. Dann ist es ziemlich einfach, die Wirkung eines solchen Stoßes auf ein Gebäude zu ermitteln.

Im zweiten Paragraphen wird diese Methode auf die Untersuchung der Erdbebensicherheit eines Gebäudes angewendet, das einen elastischen ersten Stock hat. Eine Tabelle ermöglicht die Berechnung der Schwingungsperioden des Gebäudes. Als Endergebnis erhält man die durch die Elastizität des ersten Stockwerks hervorgerufene Verringerung der Beanspruchungen.

1. General Outlines of the Theory[1]. We shall confine ourselves to general outlines of the mathematical theory and we will neglect the damping. A more detailed account of the theory is given in the appendix.

The analysis starts from the fact that any vibration of an elastic undamped system may always be considered as a superposition of harmonics. Although the method is valid for any type of motion, shear, bending, or torsion, let us only consider for the present the horizontal vibration of a building. This building, like any elastic system, has a certain number of so called „normal modes“ of vibration, and to each of them corresponds a certain frequency. When a building vibrates in a normal mode, all the displacements have the same phase, i. e. they all reach the maximum at the same moment. The „shape“ of the oscillation is that of standing waves, and the higher the frequency the higher the number of these waves. We will show that any motion can be calculated when we know these modes of vibration.

Consider the i^{th} floor of mass m_i. The amplitude of oscillation of that floor in the mode of order k is called $A_k y_{ik}$. These quantities contain an arbitrary constant A_k, corresponding to the fact that a free oscillation has an arbitrary amplitude.

In view of the application of the theory we have to give to these amplitudes certain values and determine these constants so as to satisfy the condition

$$\sum^{i} m_i A_k^2 y^2{}_{ik} = 1\,.$$

The amplitudes become then

$$u_{ik} = A_k y_{ik} = \frac{y_{ik}}{\sqrt{\Sigma\, m_i y^2{}_{ik}}}\,.$$

The amplitudes u_{ik} are then said to be „normalized“. We suppose that we also know the frequencies ν_k corresponding to each free oscillation.

The next step is to calculate the statical deflection of the building under a constant horizontal acceleration j_0 that has been applied very slowly. In other words, we have to calculate the deformation that the building would take if the gravity had an horizontal component j_0. In general, this deformation will be very complicated and will be a combined shear, torsion, and bending. The method here developed is valid for any general case, but we will restrict our considerations to the study of horizontal deflections.

By applying the so called property of orthogonality of the deflections u_{ik}, namely,

$$\sum^{i} m_i u_{ih} u_{ik} = 0 \qquad h \neq k$$

[1] See also: M. Biot, Theory of elastic systems vibrating under transient impulse with an application to earthquake-proof buildings, Proc. Nat. Acad. Sc. Vol. 19, No. 2, pp. 262–268, Feb. 1933. — This work was done during the author's stay as Research Fellow at the California Institute of Technology.

it can be proved that the statical deflection $\bar{u}_i$ produced by the acceleration j_0 can be expressed as a sum of amplitudes u_{ik} of the normalized free oscillations by the series

$$\bar{u}_i = \sum^{k} \frac{C_k}{\omega_k^2} u_{ik},$$

where

$$\omega_k = 2\pi\nu_k$$

$$C_k = j_0 \sum^{i} m_i u_{ik}.$$

If, instead of a gradual application of the acceleration j_0, it is suddenly applied at time 0, the building is going to vibrate around a new position of equilibrium which is given by the previously considered statical deflection. Hence, the motion of each floor is given by

$$u_i(t) = \sum^{k} \frac{C_k}{\omega_k^2} u_{ik} [1 - \cos \omega_k t].$$

This shows that when an horizontal acceleration is suddenly applied to a building the maximum deflection is twice the statical one.

From this last formula, we may deduce the motion due to an arbitrary acceleration (see Appendix I). The results are the following:

Let an horizontal acceleration $j_0 \psi(t)$ act upon the building between the instants 0 and T. Consider then the functions of the frequency ν,

$$f_1(\nu) = \int_0^T \psi(\tau) \cos 2\pi\nu\tau \, d\tau$$

$$f_2(\nu) = \int_0^T \psi(\tau) \sin 2\pi\nu\tau \, d\tau$$

and

$$F(\nu) = \sqrt{f_1^2(\nu) + f_2^2(\nu)}.$$

After the instant T the motion of the building is composed of a series of free oscillations each of which gives to the different floors an amplitude[2])

$$\frac{C_k u_k}{\omega_k^2} 2\pi\nu_k F(\nu_k).$$

The function $F(\nu)$ is nothing but the frequency distribution of the impulse, or its spectral distribution. The function $2\pi\nu F(\nu)$ is dimensionless and may be called the reduced spectral distribution of the impulse.

Hence we have the following theorem: When an arbitrary horizontal acceleration $j_0\psi(t)$ of finite duration acts upon a building, the resulting vibration at the end of the impulse is composed of a series of free oscillations each of which has an amplitude equal to the corresponding term $\frac{C_k u_k}{\omega_k^2}$ of the statical deformation due to j_0 and multiplied by the value $2\pi\nu_k F(\nu_k)$ of the reduced spectral distribution of (ψt) for the corresponding frequency.

If we want to know the amplitudes during the earthquake, we must naturally use a function $F(\nu)$ corresponding to an impulse ending at the moment we consider.

It will be noticed that we consider only the amplitudes of the free oscillations and not their phases. This is justified by the fact that we are not interested in the motion itself of the building, but merely in its maximum amplitude. This maximum is the sum of the amplitudes of each separate free oscillation. It will not always be reached because it supposes that an instant exists for which all the free oscillations have their maximum deflection simultaneously. However, this maximum will many times be nearly reached in a short time, and in any case it is the highest possible value.

Remarks. I. The method may be used for any type of deformation. For instance in the case of torsional deformations. Call M_i the torque produced around the axis, at the i^{th} floor, by a unit horizontal acceleration; then $j_0 M_i$ will be the torque due to an acceleration j_0. The normalized angles of rotation of each floor in the torsional normal

[2]) See Appendix I.

modes being Θ_{ik} and I_i being the moments of inertia of the floors, the rotation due to the statical deformation is

$$\overline{\Theta}_i = \sum^{k} \frac{C_k}{\omega_k^2} \Theta_{ik}$$

with

$$C_k = j_0 \sum^{i} M_i \Theta_{ik}.$$

The condition of normalization of the Θ_{ik} becomes here,

$$\sum^{i} I_i \Theta^2_{ik} = 1.$$

In order to get the amplitudes of rotation due to an acceleration we must calculate as before the expressions

$$\frac{C_k}{\omega_k^2} \Theta_{ik} 2\pi \nu_k F(\nu_k).$$

II. The steady state oscillation due to an harmonic horizontal acceleration $j_0 e^{2\pi\nu i t}$ is given for the deflection deformation by

$$y_i = \sum^{k} \frac{C_k u_{ik}}{\omega_k^2} \frac{1}{1-\left(\frac{\omega}{\omega_k}\right)^2} e^{2\pi\nu i t},$$

where $\omega = 2\pi\nu$. This gives an experimental method of measuring the u_{ik} by producing resonance in a model.

III. In order to show the main properties of a spectral curve $F(\nu)$ we calculate its value for a very simple case; we will surpose that $\psi(t)$ is a sine curve $\sin 2\pi N t$ of total length T. The corresponding function $F(\nu)$ has a sharp peak in the vicinity of $\nu = N$ and its approximate value is

$$\frac{1}{2}\left|\frac{\sin\pi(\nu-N)T}{\pi(\nu-N)}\right| \text{ (fig. 1).}$$

This shows that in general a spectral distribution curve will show a certain number of peaks if there exists certain periods in the seismogram (fig. 2).

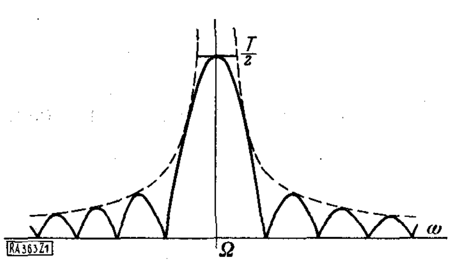

Fig. 1.

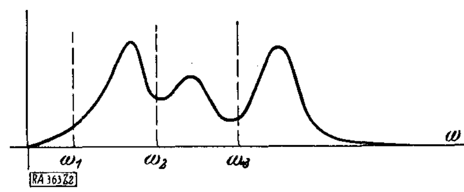

Fig. 2.

According to recent observations[3]) there seems to exist characteristic frequencies of the ground at given locations. These frequencies would be given by the peaks in the spectral curve.

If we possessed a great number of seismogram spectra we could use their envelope as a standard spectral curve for the evaluation of the probable maximum effect on buildings. The spectral curves would be of interest as much to seismologists as to civil engineers.

The direct computation of such spectra might be tedious but automatic electrical methods can be easily imagined such as a photographic record passing in front of a photo-electric cell acting upon a tuned circuit.

2. The Effectiveness of an Elastic First Floor. As an application of the method we will calculate the reduction of stress due to a so called elastic first floor in an earthquake-proof building.

[3]) Suyehiro, Tokyo.

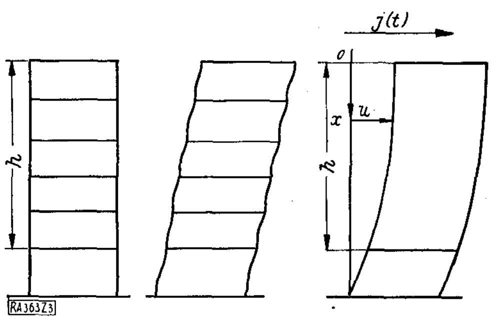

Abb. 3.

Let it be of rectangular shape. We assume that the deformation is an horizontal shear as shown in fig. 3. Furthermore, the shearing rigidity and the mass of each story are supposed to be constant from the second floor to the top. The first floor only will have a different rigidity. In order to simplify the analysis we consider the upper part of the building, i. e. all the stories except the first one as a continuous beam having shearing deformations only.

We use the following notations:

x coordinate measured downwards from the top as origin
u horizontal deflection of the building
h height of the upper part of the building
M mass of the upper part of the building
n number of stories of the upper part
$h_1 = \frac{h}{n}$ height of one story
K force necessary to displace two consecutive floors of a unit length relatively to each other
$m = \frac{M}{n}$ mass density of the equivalent beam
$\mu = K h_1$ rigidity coefficient of the equivalent beam
G force necessary to displace the second floor of a unit length relatively to the ground
$c = \sqrt{\frac{\mu}{m}}$ propagation speed of a shear wave in the beam
$t_0 = \frac{h}{c}$ time necessary for such a wave starting from the bottom of the beam to reach the top
$\frac{t}{t_0} = \tau$
$\frac{x}{h} = \xi$
$\frac{u}{g t_0^2} = y$

According to the general theory we have to consider the normal modes or free oscillations of the equivalent beam, which are given by the equation,

$$\mu \frac{\partial^2 u}{\partial x^2} = m \frac{\partial^2 u}{\partial t^2}.$$

This may be written with dimensionless variables,

$$\frac{\partial^2 y}{\partial \xi^2} = \frac{\partial^2 y}{\partial \tau^2}.$$

By putting $y = z(\xi) e^{i \lambda \tau}$ we get,

$$\frac{d^2 z}{d \xi^2} + \lambda^2 z = 0.$$

The general solution is

$$z = A \cos \lambda \xi + B \sin \lambda \xi.$$

Consider the boundary conditions,

$$\frac{\partial u}{\partial x} = 0 \qquad x = 0$$

$$\mu \frac{\partial u}{\partial x} = - G u \qquad x = h.$$

Put $R = \dfrac{G h_1}{\mu}$ ratio of the rigidity of the first story to that of the others, and $\alpha = R n$, n being the number of these other stories; the boundary conditions become

$$\frac{dz}{d\xi} = 0 \qquad \xi = 0 \quad \ldots \ldots \ldots \ldots \ldots \quad (1),$$

$$\frac{dz}{d\xi} + \alpha z = 0 \qquad \xi = 1 \quad \ldots \ldots \ldots \ldots \ldots \quad (2).$$

From equation (1) $\qquad B = 0 \qquad z = A \cos \lambda \xi$.

From condition (2) $\qquad \lambda \operatorname{tg} \lambda = \alpha$.

The roots λ_k of this equation correspond to the free oscillation frequencies of the building. We choose certain values of α corresponding to certain simple values of R and n as follows:

	R	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$	1
n					
15		1,66	2,50	5	
10		1,11	1,66	3,33	10
5		0,556	0,834	1,66	

The values of λ_k as a function of α are given in the following table:

α	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
0	0	π	2π	3π	4π	5π
0,556	0,68	3,31	6,31	9,48	12,60	15,73
0,834	0,80	3,38	6,41	9,51	12,62	15,75
1,11	0,89	3,45	6,45	9,54	12,65	15,77
1,66	1,03	3,58	6,53	9,59	12,69	15,80
2,50	1,15	3,73	6,65	9,67	12,76	15,85
3,33	1,23	3,86	6,74	9,75	12,82	15,91
5,0	1,32	4,04	6,91	9,90	12,93	16,0
10,0	1,44	4,30	7,22	10,18	13,20	16,24
∞	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	$\frac{5\pi}{2}$	$\frac{7\pi}{2}$	$\frac{9\pi}{2}$	$\frac{11\pi}{2}$

In fig. 4 are plotted the values of λ_k' where $\lambda_k = k\pi + \lambda_k'$. The period T_k corresponding to λ_k is,

$$T_k = \frac{2\pi t_0}{\lambda_k}.$$

It is interesting to compare the fundamental period T_0 to that T_0' which would occur if the building would be perfectly rigid from the second floor to the top, the only elasticity being due to the first floor. We get,

$$M \frac{d^2 u}{d t^2} + G u = 0 \qquad T' = 2\pi \sqrt{\frac{M}{G}} = \frac{2\pi t_0}{\sqrt{\alpha}}.$$

The ratio of frequencies $\dfrac{f}{f_0'} = \dfrac{T_0'}{T} = \dfrac{\lambda_0}{\sqrt{\alpha}}$ is a function of α.

α	$\frac{f_0}{f_0'}$	α	$\frac{f_0}{f_0'}$
0	1	2,50	0,725
0,556	0,910	3,33	0,674
0,834	0,875	5,0	0,590
1,11	0,840	10,	0,455
1,66	0,800	∞	0

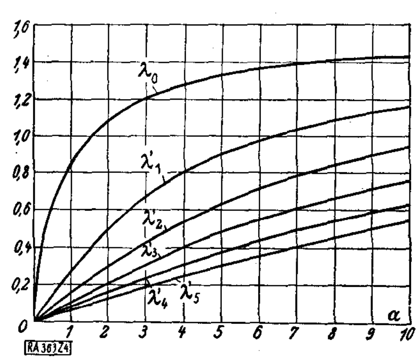

Fig. 4.

The shape of the normal modes is given by

$$u_k(x) = A_k \cos \lambda_k \xi .$$

We have to determine the constants A_k by the condition of normalization which reduces here to the integral

$$m \int_0^h A_k^2 \cos^2 \lambda_k \xi \, d x = 1 \qquad \text{or} \qquad M A_k^2 \int_0^1 \cos^2 \lambda_k \xi \, d \xi = 1 .$$

By integrating and using the equation $\lambda_k \operatorname{tg} \lambda_k = a$

$$A_k = \frac{1}{\sqrt{\frac{M}{2}\left[1 + \frac{a \cos^2 \lambda_k}{\lambda_k^2}\right]}} .$$

The statical deformation of the beam due to an horizontal acceleration g is then

$$\bar{u}(x) = \sum^k \frac{C_k}{\omega_k^2} u_k(x) ,$$

where

$$C_k = g\, m \int_0^h u_k(x)\, d x = \frac{g\, m\, h}{\sqrt{\frac{M}{2}\left[1 + \frac{a \cos^2 \lambda_k}{\lambda_k^2}\right]}} \int_0^1 \cos \lambda_k \xi \, d \xi$$

$$\int_0^1 \cos \lambda_k \xi \, d \xi = \frac{\sin \lambda_k}{\lambda_k} = \frac{a \cos \lambda_k}{\lambda_k^2} .$$

Since $\omega_k = \frac{\lambda_k}{t_0}$, the statical deformation takes the form

$$\bar{u}(x) = 2\, g\, t_0^2\, a \sum^k \frac{\cos \lambda_k}{\lambda_k^4 \left[1 + \frac{a \cos^2 \lambda_k}{\lambda_k^2}\right]} \cos \lambda_k \xi .$$

Put

$$\frac{2\, a \cos \lambda_k}{\lambda_k^4 \left[1 + \frac{a \cos^2 \lambda_k}{\lambda_k^2}\right]} = B_k .$$

These coefficients are dimensionless and functions of a only. Their values are given in the following table:

a	B_0	B_1	B_2	B_3
0	$\infty \frac{1}{a}$	0	0	0
0,556	2,336	— 0,00890	0,000692	— 0,000138
0,834	1,734	— 0,0116	0,000970	— 0,000208
1,11	1,425	— 0,0137	0,00122	— 0,000272
1,66	1,091	— 0,0166	0,00177	— 0,000386
2,50	0,902	— 0,0191	0,00234	— 0,000544
3,33	0,802	— 0,0203	0,00274	— 0,000682
5,0	0,710	— 0,0210	0,00334	— 0,000912
10,0	0,612	— 0,0212	0,00404	— 0,00125
∞	0,520	— 0,0192	0,00417	— 0,00151

The statical deformation is then given by:

$$\bar{u}(x) = g\, t_0^2 \sum^k B_k \cos \lambda_k \xi .$$

By using this result we are able to find the amplitudes of oscillation produced by an acceleration $g\, \psi(t)$.

We build the reduced spectral intensity $2 \pi \nu F(\nu)$ of the function of time $\psi(t)$. The amplitude of each free oscillation of the beam is then

$$V_k(x) = g\, t_0^2\, B_k \cos \lambda_k \xi \cdot 2 \pi \nu_k F(\nu_k) .$$

We are also interested in the total shear produced in this beam. The amplitude of shear produced by each mode is

$$S_k = \mu \frac{\partial U_k}{\partial x} = \frac{\mu}{h} \frac{\partial U_k}{\partial \xi}, \quad S_k = g M \cdot B_k \lambda_k \sin \lambda_k \xi \cdot 2 \pi \nu_k F(\nu_k).$$

Replacing $2\pi\nu_k$ by $\frac{\lambda_k}{t_0}$ we get

$$S_k = -g M B_k \lambda_k^2 \sin \lambda_k \xi \frac{F(\nu_k)}{t_0}.$$

For the fundamental oscillation the shear is maximum at the bottom ($\xi = 1$)

$$S = g M B_0 \lambda_0^2 \sin \lambda_0 \frac{F(\nu_0)}{t_0}.$$

For the oscillations of higher order, the maximum value is

$$S_k = g M B_k \lambda_k^2 \frac{F(\nu_k)}{t_0}.$$

Put

$$B_0 \lambda_0^2 \sin \lambda_0 = C_0(\alpha), \qquad B_k \lambda_k^2 = C_k(\alpha).$$

The maximum shear corresponding to each free oscillation is then given by

$$S_k = g M C_k(\alpha) \frac{F(\nu_k)}{t_0}.$$

The coefficients $C_k(\alpha)$ are dimensionless and functions of α only. Their values are plotted in fig. 5 and given in the following table:

α	C_0	C_1	C_2	C_3
0	0	0	0	0
0,556	— 0,340	0,0486	— 0,0139	0,00622
0,834	— 0,397	0,0666	— 0,0199	0,00942
1,11	— 0,438	0,0818	— 0,0254	0,0123
1,66	— 0,496	0,107	— 0,0377	0,0177
2,50	— 0,544	0,132	— 0,0517	0,0254
3,33	— 0,572	0,150	— 0,0623	0,0324
5,0	— 0,600	0,172	— 0,0812	0,0451
10,0	— 0,629	0,196	— 0,105	0,071
∞	— 0,642	0,212	— 0,129	0,091

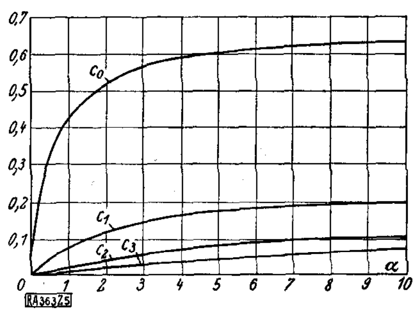

Fig. 5.

We see that the fundamental oscillation is by far the most dangerous, and that, for given values of the $\nu'_\varkappa s$, the influence of an elastic first floor is important only for values of α smaller than 3. For example, it would not be in general of great advantage for a seven story building to build an elastic first floor having only 50% of the rigidity of the other floors.

Appendix I.

Vibrating systems under transient impulse. The following theory gives a method of evaluating the action of very random impulses on vibrating systems (i. e. effect of statics on radio-circuits or earthquakes on buildings). In the following text, we will use the language of mechanics.

Consider a one-dimensional continuous elastic system without damping. The free oscillations are given by the solutions of the homogeneous integral equation,

$$y = \omega^2 \int_a^b \varrho(\xi)\, a(x\,\xi)\, y(\xi)\, d\xi.$$

Due to the nature of the kern there exists an infinite number of characteristic values ω_i of ω and of characteristic functions y_i solutions of this equation. These functions give the shape of the free oscillations of the system. They are orthogonal and have an arbitrary amplitude. This amplitude may be fixed by the condition of normalization,

$$\int_a^b \varrho(\xi)\, y_i^2(\xi)\, d\xi = 1.$$

We now suppose that certain external forces $f(x)$ are acting on the system, these forces being expressed in such a way that the product of the displacement y by $f(x)$ represents the work done by this force. For example, if $f(x)$ is a moment y will be the angle of rotation around the same axis as the moment at that point.

It can be easily proved that the statical deflection of the system is,

$$y = \Sigma \frac{C_i}{\omega_i^2} y_i \qquad (1),$$

where C_i is the Fourier coefficient of the development of $\frac{f(x)}{\varrho(x)}$ in a series of the orthogonal functions y_i,

$$\frac{f(x)}{\varrho(x)} = \Sigma\, C_i\, y_i ,$$

hence

$$C_i = \int_0^b f(\xi)\, y_i(\xi)\, d\xi .$$

If the applied forces are variable with time and harmonic of the form $f(x)\, e^{i\omega t}$ the deflection is expressed by the expansion,

$$y = \Sigma \frac{C_i y_i}{\omega_i^2} \frac{1}{1-\left(\frac{\omega}{\omega_i}\right)^2} e^{i\omega t} \qquad (2).$$

The amplitude is composed of each of the terms of the statical deformation (1) multiplied by a resonance factor.

The motion due to a sudden application of the forces is of the same type, and can be deduced immediately from the preceding harmonic solution.

By using Heaviside's expansion theorem we get,

$$y_a = \Sigma \frac{C_i y_i}{\omega_i^2} [1 - \cos\, \omega_i t] \qquad (3).$$

The amplitude due to a sudden applied force $f(x)$ is composed of a series of oscillations each of which has an amplitude equal to twice the corresponding term of the statical deformation (1).

We will now investigate the action of varying forces of the type $f(x)\, \psi(t)$; these forces are supposed to start their action at the origin of time and to keep on during a finite time T.

Using the Heaviside method, and the indicial admittance (3), the motion after the impulse has disappeared is given by,

$$y_b = \int_0^T \frac{d}{dt} y_a(t-\tau)\, \psi(\tau)\, d\tau ,$$

$$y_b = \Sigma \frac{C_i y_i}{\omega_i^2} \{\sin \omega_i t\, [\omega_i \int_0^T \psi(\tau) \cos \omega_i \tau\, d\tau] - \cos \omega_i t\, [\omega_i \int_0^T \psi(\tau) \sin \omega_i \tau\, d\tau]\} .$$

This motion is the superposition of free oscillations. Their respective amplitudes can be physically interpreted as follows:

Put
$$f_1(\nu) = \int_0^T \psi(\tau) \cos 2\pi\nu\tau\, d\tau ,$$

$$f_2(\nu) = \int_0^T \psi(\tau) \sin 2\pi\nu\tau\, d\tau ,$$

where ν is the frequency $\qquad \nu = \frac{\omega}{2\pi}$.

The amplitudes of composing free oscillations may then be written,

$$\frac{C_i y_i}{\omega_i^2} \cdot 2\pi\nu_i \sqrt{f_1^2(\nu_i) + f_2^2(\nu_i)} .$$

Now, according to the Fourier Integral,

$$\psi(t) = 2\int_0^\infty f_1(\nu) \cos 2\pi\nu t\, d\nu + 2\int_0^\infty f_2(\nu) \sin 2\pi\nu t\, d\nu \qquad (4).$$

This shows that the expression

$$F(\nu) = \sqrt{f_1^2(\nu) + f_2^2(\nu)}$$

may be considered as the „spectral intensity" curve of the impulse.

The amplitude of each free oscillation due to the transient impulse is,

$$\frac{C_i y_i}{\omega_i^2} \cdot 2\pi \nu_i F(\nu_i) \quad \ldots \ldots \quad (5).$$

The expression $2\pi\nu F(\nu)$ is a dimensionless quantity that we will call „reduced spectral intensity". We then have the following theorem: When a transient impulse acts upon an undamped elastic system, the final motion results from the superposition of free oscillations each of which has an amplitude equal to the corresponding term $\frac{C_i y_i}{\omega_i^2}$ of the statical deformation (1) multiplied by the value of the reduced spectral intensity for the corresponding frequency.

The advantage of this theorem is that for the calculation of the motion it replaces a complicated impulse by a spectral distribution which is always an analytical function of the frequency.

This theorem could also have been established by starting from (2) and using directly the Fourier integral. We will apply this last method in order to generalize the theorem to the case of an elastic system with viscous damping whose motion is defined by the equation

$$m\ddot{y} + a\dot{y} + b^2 y = A\psi(t) \quad \ldots \ldots \quad (6).$$

The impulse is supposed to be given as before by the spectral distribution (4). Introducing a complex spectral distribution,

$$\varphi(\nu) = f_1(\nu) - i f_2(\nu)$$

we may write,

$$\psi(t) = \int_{-\infty}^{+\infty} \varphi(\nu) e^{2\pi i \nu t} d\nu \quad \ldots \ldots \quad (7),$$

where according to the Fourier integral

$$\varphi(\nu) = \int_0^T \psi(\tau) e^{-2\pi i \nu \tau} d\tau .$$

The function $\varphi(\nu)$ is holomorphic; its expansion in a power series is

$$\varphi(\nu) = \Sigma \frac{A_n \nu^n}{n!},$$

where

$$A_n = (-2\pi i)^n \int_0^T t^n \psi(t)\, dt .$$

Calling M the largest value of $|\psi(t)|$ and $I(\nu)$ the coefficient of i in the variable ν considered from now on in the complex plane, we have

$$|\varphi(\nu)| < \frac{M}{2\pi I(\nu)} [e^{2\pi I(\nu) T} - 1] .$$

This shows that for $t > T$, $|\varphi(\nu) e^{2\pi i \nu t}|$ has an upper limit.

Consider now the elastic system (6) under an harmonic impulse $A e^{2\pi i \nu t}$, the corresponding motion is

$$x(t) = \frac{A e^{2\pi i \nu t}}{8\pi^2 \beta m} \left[\frac{1}{\nu - \nu_1} - \frac{1}{\nu - \nu_2}\right] \quad \ldots \ldots \quad (8).$$

The quantities ν_1, and ν_2 are complex frequencies

$$\nu_1 = \alpha i + \beta, \qquad \nu_2 = \alpha i - \beta .$$

The free oscillation of the system is damped and given by

$$e^{-2\pi \alpha t} \cos 2\pi\beta t .$$

According to (7) and (8) the motion due to the impulse $A\,\psi(t)$ will be,

$$y(t) = \frac{A}{8\pi^2 \beta m} \int_{-\infty}^{+\infty} \varphi(\nu)\, e^{2\pi i \nu t} \left[\frac{1}{\nu - \nu_1} - \frac{1}{\nu - \nu_2}\right] d\nu .$$

We have seen that $|\varphi(\nu)\, e^{2\pi i \nu t}|$ has an upper limit, and by using then the method of contour integrals and residues, we find

$$y(t) = \frac{A \pi i}{4\pi^2 \beta m} [\varphi(\nu_1)\, {}^{2\pi i \nu_1 t} - \varphi(\nu_2)\, e^{2\pi i \nu_2 t}] .$$

At the time T when the impulse has ceased, the amplitude is

$$|y(t)| = \frac{A}{4\pi^2 \beta^2 m} 2\pi\beta\, |\varphi(\alpha i + \beta)|\, e^{-2\pi\alpha T} .$$

The quantity $\frac{A}{4\pi^2 m(\beta^2+\alpha^2)}$ is the deflection for the static deformation due to a force A. This last result generalizes formula (5) to the case of damping. We have to consider a complex frequency $\alpha i + \beta$ and the analytical prolongation $\varphi(\alpha i + \beta)$ of the spectral distribution $\varphi(\nu)$ of $\psi(t)$.

Appendix II.

A method for computing free oscillations in case of non uniform mass and rigidity. Whenever we know the shape y_k of the free oscillations of the building, we can calculate the corresponding frequencies or ω_k by the following method:

Consider the building as a continuous shear-beam and oscillating freely with the frequency of order k. When the amplitude is maximum the kinetic energy is equal to zero and the potential energy is

$$\frac{1}{2}\int_0^h \mu(x) \left(\frac{d y_k}{d x}\right)^2 d x .$$

On the other hand, the potential energy passes through zero when the strain disappears; at that moment the kinetic energy is maximum and has the value

$$\frac{1}{2}\int_0^h m(x)\, \omega_k^2\, y_k^2\, d x .$$

Equating those two expressions we get the value

$$\omega_k^2 = \frac{\int_0^h \mu(x) \left(\frac{d y_k}{d x}\right)^2 d x}{\int_0^h m(x)\, y_k^2\, d x}$$

which is independent of any arbitrary constant multiplying y_k.

The orthogonal functions y_k representing the shape of the free oscillations may be found by the following semi-empirical method. We note that those functions are defined by equation.

$$\frac{d}{d x}\left[\mu(x) \frac{d y_k}{d x}\right] + m(x)\, \omega^2\, y_k = 0 ,$$

which by a change of the independent variable

$$z = \int \frac{d x}{\mu(x)}$$

becomes

$$\frac{1}{\mu[x(z)]} \frac{d^2}{d z^2} [y_k(z)] + m[x(z)]\, \omega^2\, y_k = 0 .$$

This is the equation of buckling under a load P of a beam of moment of inertia

$$I(z) = \frac{P}{E m \mu \omega^2} .$$

If we consider an elastic strip of uniform thickness h, its moment of inertia will have the value I under the condition that the variable width h satisfies the equation

$$\frac{b\,h^3}{12} = I = \frac{P}{E\,m\,\mu\,\omega^2} = \frac{A}{\mu\,m},$$

where A is an arbitrary constant. We may choose for z any arbitrary scale convenient for we are interested only in the shape of the functions.

In order to rêalize the given boundary conditions, the strip will be repeated symmetrically around a point representing the top of the building (fig. 6). The deformation of the half

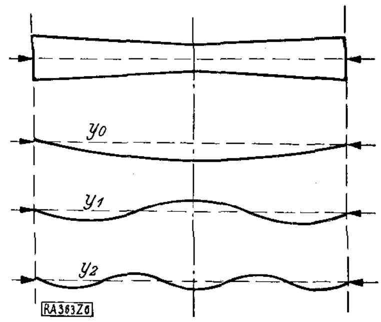

Fig. 6.

strip under various loads will give the corresponding functions y_k. The first deformation only is stable, so that the others will have to be stabilized by a special but very simple device. Knowing the shape of the y_k we then compute the attached frequencies by the method above.

Appendix III.

Method for calculating the acceleration spectrum from a displacement spectrum. Suppose that we know the displacement diagram of the ground, in the form $v\,\varphi(t)$, v being a certain arbitrary length., and T an arbitrary length of time. The acceleration of the ground is

$$j(t) = v\,\varphi''(t) = \frac{v}{T^2}\,\varphi''(t)\,T^2 = j_0\,\psi(t)$$

from there,

$$\varphi''(t)\,T^2 = \psi(t), \qquad j_0 = \frac{v}{T^2}.$$

We can easily deduce the spectral distribution of the unknown acceleration diagram $\psi(t)$ from the spectral distribution of the displacement diagram by the following considerations:

Consider the spectral function $G(\nu)$ of $\varphi(t)$

$$G(\nu) = \sqrt{g_1^2(\nu) + g_2^2(\nu)},$$

$$g_1(\nu) = \int_0^T \varphi(\tau)\cos 2\pi\nu\tau\,d\tau,$$

$$g_2(\nu) = \int_0^T \varphi(\tau)\sin 2\pi\nu\tau\,d\tau.$$

From the Fourier integral we get,

$$\varphi(t) = 2\int_0^\infty g_1(\nu)\cos 2\pi\nu t\,d\nu + 2\int_0^\infty g_2(\nu)\sin 2\pi\nu t\,d\nu,$$

$$\psi(t) = \varphi''(t)\,T^2 = 2\int_0^\infty (2\pi\nu T)^2\,g_1(\nu)\cos 2\pi\nu t\,d\nu + 2\int_0^\infty (2\pi\nu T)^2\,g_2(\nu)\sin 2\pi\nu t\,d\nu.$$

This shows that the spectral distribution $F(\nu)$ of the acceleration diagram $\psi(t)$ is

$$F(\nu) = (2\pi\nu T)^2\,G(\nu).$$

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