HARVARD UNIVERSITY

A GENERAL PROPERTY OF TWO DIMENSIONAL THERMAL STRESS DISTRIBUTION

BY

M. A. BIOT

Reprinted from
Philosophical Magazine
March, 1935

PUBLICATIONS FROM THE HARVARD ENGINEERING SCHOOL

1934–35 No. 146

From the PHILOSOPHICAL MAGAZINE, Ser. 7, vol. xix. p. 540, *March* 1935.

*A General Property of Two-Dimensional Thermal Stress Distribution**. *By* M. A. BIOT, *Harvard University (Cambridge, U.S.A.)* †.

WE consider a two-dimensional temperature distribution in an arbitrary cylindrical body; the temperature is supposed to be the same along any straight line parallel to the generators. We shall also assume that this temperature distribution has reached a state of equilibrium—*i. e.*, the temperature may be different from point to point, but it remains constant at a given point and has any given arbitrary distribution along the boundary of the cross-section (fig. 1).

* See also M. A. Biot, " Propriété générale des tensions thermiques en régime stationnaire dans les corps cylindriques. Application à la mesure photo-elastique de ces tensions," *Ann. de la Soc. Sc. de Bruxelles*, liv. B (janvier 1934).

† Communicated by Prof. R. V. Southwell, M.A., F.R.S.

As we know, such a temperature distribution has the following property: the value of the temperature θ as a function of the coordinates x, y in a cross-sectional plane must satisfy the equation

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0. \qquad (1)$$

We shall show that, owing to this condition, the stresses which may arise from this temperature distribution exhibit remarkable properties stated by the theorem on p. 547. The theorem may also be applied to an elastic

Fig. 1.

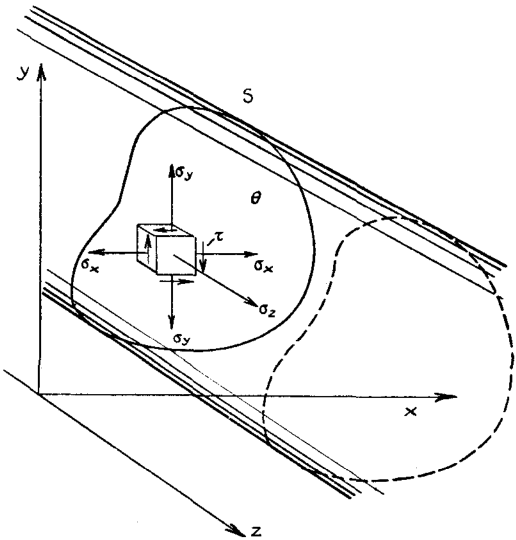


plate for which heat can enter or leave the material only at the boundary. We shall deduce these properties for the case of the cylindrical body: it is easy to see what they become for a plate.

Let us consider a cylindrical elastic body of Young's modulus E and Poisson's ratio ν, the x, y plane being taken along a cross-section (fig. 1). We assume that there is a steady two-dimensional temperature distribution in this cylinder, the same in every cross-section (independent of z), also that the resulting deformation of the cylinder can only be two-dimensional (also independent of z), so that the cross-sections of the cylinder remain parallel and identical with one another. Calling k the coefficient of thermal expansion, let us put $\epsilon = k\theta$. The thermal extension ϵ varies from point to point: it will not give the actual extensions, since such values would generally not be compatible and would give rise to thermal stresses.

The variables entering the problem must satisfy three groups of equations:—

(1) The equilibrium condition for internal stresses $(\sigma_x\, \sigma_y \tau\, \sigma_z)$:

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} &= 0, \\ \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0, \\ \frac{\partial \sigma_z}{\partial z} &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(2) The so-called compatibility equation for the strain components $(\epsilon_x\, \epsilon_y\, \gamma)$:

$$\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2 \gamma}{\partial x\, \partial y}. \quad . \quad . \quad . \quad . \quad (3)$$

(3) The relation between the stress and the strain as a function of the thermal dilatation ϵ:

$$\left.\begin{aligned} &\epsilon_x - \epsilon(1+\nu) = \frac{1}{\mathrm{E}}\left[\sigma_x(1-\nu^2) - \nu(1+\nu)\sigma_y\right], \\ &\epsilon_y - \epsilon(1+\nu) = \frac{1}{\mathrm{E}}\left[\sigma_y(1-\nu^2) - \nu(1+\nu)\sigma_x\right], \\ &\tau = 2\mathrm{G}\gamma, \\ &\sigma_z = -\mathrm{E}\epsilon + \nu(\sigma_x + \sigma_y). \end{aligned}\right\} \quad . \quad (4)$$

From this system we immediately see that the stresses $\sigma_x\, \sigma_y\, \tau$ may disappear under certain conditions. If we put

$$\sigma_x = \sigma_y = \tau = 0,$$

we are left with the conditions

$$\left.\begin{aligned} &\sigma_z = -\mathrm{E}\epsilon, \\ &\epsilon_x - \epsilon(1+\nu) = 0, \\ &\epsilon_y - \epsilon(1+\nu) = 0, \\ &\gamma = 0, \\ &\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2 \gamma}{\partial x\, \partial y}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which, by elimination of ϵ_x ϵ_y, reduce to

$$\sigma_z = -E\epsilon,$$
$$\frac{\partial^2 \epsilon}{\partial x^2} + \frac{\partial^2 \epsilon}{\partial y^2} = 0.$$

Since $\epsilon = k\theta$ is proportional to the temperature θ, we may also write the condition in the form

$$\sigma_z = -E\epsilon,$$
$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0.$$

Comparing with equation (1), we see that this last condition is automatically satisfied in our case because we assumed the temperature distribution to be steady (independent of the time). Hence this steady state of temperature distribution is a necessary condition for the cancellation of the stresses σ_x, σ_y, τ; but, as we shall see, it is a sufficient condition only in the case of a solid cylinder. If the cylinder is hollow more conditions have to be satisfied. In order to establish this conclusion let us investigate the problem further.

Consider the strain components given by equation (5)

$$\left.\begin{aligned} \epsilon_x &= \epsilon(1+\nu) = \epsilon', \\ \epsilon_y &= \epsilon(1+\nu) = \epsilon', \\ \gamma &= 0. \end{aligned}\right\} \quad \dots \dots \quad (6)$$

Due to the deformation a point of coordinates x, y before deformation becomes of coordinates $x+u$, $y+v$ after deformation; moreover, each element undergoes a rotation

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

By introducing the displacements u, v we may write (6) and ω in the form

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \epsilon', \\ \frac{\partial v}{\partial y} &= \epsilon'. \end{aligned}\right\} \quad \dots \dots \dots \quad (7)$$

$$\left.\begin{aligned} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= 0, \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 2\omega. \end{aligned}\right\} \quad \dots \dots \quad (8)$$

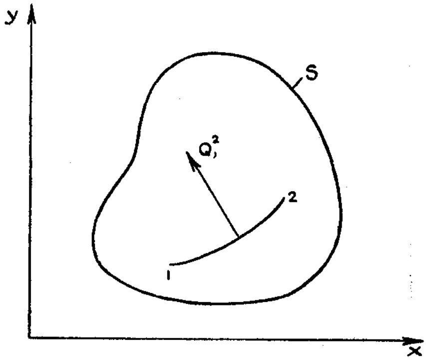
This leads to the relations

$$\left.\begin{aligned}\frac{\partial u}{\partial y}&=-\omega,\\ \frac{\partial v}{\partial x}&=\omega,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\left.\begin{aligned}\frac{\partial \epsilon'}{\partial y}&=-\frac{\partial \omega}{\partial x},\\ \frac{\partial \epsilon'}{\partial x}&=\frac{\partial \omega}{\partial y},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which show that $Z=\epsilon'+i\omega$ is a complex function of $z=x+yi$. From these results we may calculate the value

Fig. 2.



of the rotation ω and the displacements u, v. The difference of the rotation at point 2 and the rotation at point 1 (fig. 2) is given by the integral

$$\omega_2-\omega_1=\int_1^2\left(\frac{\partial\omega}{\partial x}dx+\frac{\partial\omega}{\partial y}dy\right),$$

which, from relation (10), may be written

$$\omega_2-\omega_1=\int_1^2\left(-\frac{\partial\epsilon'}{\partial y}dx+\frac{\partial\epsilon'}{\partial x}dy\right). \quad . \quad . \quad (11)$$

Now since $\left(\frac{\partial\epsilon'}{\partial x}\cdot\frac{\partial\epsilon'}{\partial y}\right)$ is proportional to the temperature gradient, the above difference of rotation $\omega_2-\omega_1$ is proportional to the amount of heat Q_{12} flowing, parallel to the cross-section, between the points 1 and 2. We may write

$$\omega_2-\omega_1=Q_{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

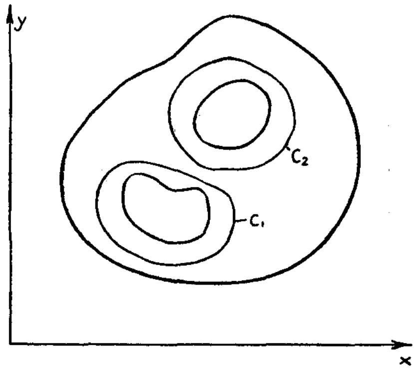
It is easily verified from relations (7) and (9) that the displacement of point 2 relative to point 1 is given by

$$u_2-u_1+i(v_2-v_1)=\int_1^2\left\{\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+i\left(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\right)\right\}$$

$$=\int_1^2(\epsilon'+i\omega)(dx+i\,dy)=\int_1^2 Z(z)dz. \; . \quad (13)$$

Now the condition $\sigma_x=\sigma_y=\tau=0$ for the stresses will be satisfied only if the values of the rotation and the

Fig. 3.



displacements calculated from the above formula and corresponding to such a condition have a physical meaning.

Consider first a solid cylinder, no heat source being present inside; then the functions ϵ' and ω have no singularities inside any closed contour drawn on the area of the cross-section; therefore in this case the rotation and the displacements are single-valued, they have a physical meaning and actually exist. Hence in the case of a solid cylinder under the above condition of heat-flow the stresses σ_x, σ_y, τ actually disappear.

For a hollow cylinder, however (cross-section fig. 3), due to the fact that the holes may contain heat sources or any other kind of singularity of the function Z, we have to complete the previous result. If we take any closed contour C around a hole, the rotation ω given by integral

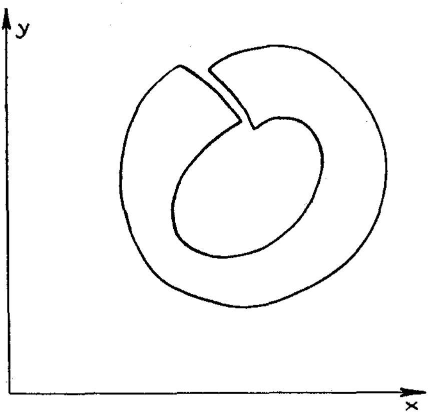
(11) has to be single-valued. From considerations above (12) this means that the total amount of heat flowing out of each hole must be zero. Moreover, the displacements (13) along the same contour must also be single-valued, and this introduces another condition for the following integral taken around each hole:

$$\int_{C_i} (\epsilon' + i\omega) dz = 0,$$

or

$$\left.\begin{aligned} \int_{C_i} (\epsilon' dx - \omega\, dy) &= 0, \\ \int_{C_i} (\omega\, dx + \epsilon'\, dy) &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Fig. 4.



For a hollow cylinder, if the last conditions are not satisfied, the values of the rotation and the displacements may not be single-valued; this means physically that the stresses thus produced are of the "dislocation" type. Consider, for instance, a cylinder with a cross-section having only a single hole. Let us cut this cylinder by a cylindrical slit (cross-section fig. 4), so as to connect the inside with the outside and transform it into a simply connected body. If then the temperature distribution is stationary and two-dimensional, as already mentioned, the two sides of the slit will either stick together without any relative motion or they will separate. This separation will occur whenever the total amount of heat flowing from the inside to the outside is not zero and whenever the conditions (14) are not satisfied. Due to the presence of the slit the cylinder behaves like a solid one without holes, and the stresses σ_x, σ_y, τ are zero. If we bring together the two sides of the slit we produce stresses of a dislocation type.

Hence we may finally state the following theorem :—

If a solid cylinder is heated either uniformly or not, but in such a way that a steady-state temperature distribution exists the same in every cross-section, the only stress produced is a tension (or compression) σ_z, *acting normally to the cross-section and equal to* $\sigma_z = -\mathrm{E}\epsilon = -\mathrm{E}k\theta$, *where* E *is Young's modulus for the cylinder, k its coefficient of thermal expansion, and* θ *the temperature.*

We may say that in such a case *the cylinder expands freely in the plane of its cross-section.*

The same theorem holds for a hollow cylinder provided :

(1) *The total amount of heat flowing in or out each hole is zero.*

(2) *The following contour integrals around each hole are zero :*

$$\int_{\mathrm{C}} (\epsilon' dx - \omega\, dy) = 0,$$

$$\int_{\mathrm{C}} (\omega\, dx + \epsilon' dy) = 0.$$

Applications.

(1) *Circular Tube.*

A circular tube filled with a hot liquid sends out a radial symmetrical flow of heat. The function $\mathrm{Z} = \epsilon + i\omega$ corresponding with this temperature distribution can be written, taking the centre of the tube as origin, as

$$\mathrm{Z} = \mathrm{A} \log z.$$

If we cut a radial slit, the two edges have a relative motion (u, v) given by

$$u + iv = \int_{\mathrm{C}} \mathrm{A} \log z\, dz = [\mathrm{A}z \log z]_{\mathrm{C}} = 2\pi i \mathrm{A}(x + yi),$$

where x, y are the coordinates of a point along the slit. We see that the relative displacement of the two sides of a radial slit is a rotation around the axis of the tube (fig. 5). The stresses in the circular tube are the same as those produced in the slitted tube by bringing the two sides of the slit together again; this coincides with the problem of bending of a curved beam.

Fig. 5.

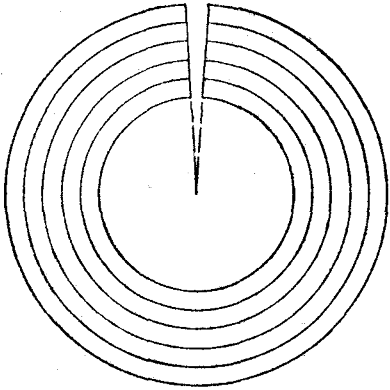
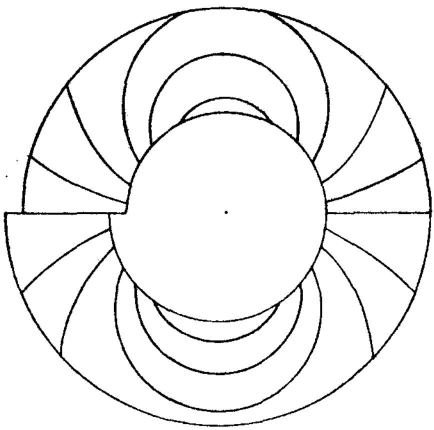


Fig. 6.



If the temperature distribution in the cylinder is such that Z can be represented by

$$Z = \frac{A}{z},$$

it is easily seen that this corresponds with the case of a slit of which the two sides do not separate but simply glide over one another. The isothermal lines are arcs of circles (fig. 6).

Finally, a temperature distribution which can be represented as

$$Z=\frac{A_2}{z^2}+\frac{A_3}{z^3}+\ldots\frac{A_n}{z^n}+\ldots$$

does not produce any dislocation at the slit, hence does not produce any stress σ_x, σ_y, τ.

(2) If a hollow cylinder of arbitrary shape has no heat source inside the holes, and exhibits a two-dimensional steady-state temperature distribution the same in every cross-section, the stresses σ_x, σ_y, τ may be different from zero; but if we fill up the holes with a liquid of same thermal conductivity as that of the material of the cylinder, those stresses disappear, owing to the fact that the function Z cannot have any more singularities inside the holes, and the contour integrals become zero.

(3) *Thermal Stresses measured by Photo-elasticity.*

The identification of the thermal stress problem in a case of steady heat flow with a dislocation problem makes possible the investigation of those thermal stresses by photo-elasticity. All we have to do is to cut a slit in the photo-elastic model and separate the sides of that slit in such a way that the relative displacement of those sides is the same as it would be if due to the given temperature distribution.