HARVARD UNIVERSITY

UN CAS D'INTÉGRABILITÉ DE L'ÉQUATION NON LINÉAIRE DE LA CHALEUR ET DE LA CONSOLIDATION DES SÉDIMENTS ARGILEUX

LE PROBLÈME DE LA CONSOLIDATION DES MATIÈRES ARGILEUSES SOUS UNE CHARGE

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QUADRATIC WAVE EQUATION—FLOOD WAVES IN A CHANNEL WITH QUADRATIC FRICTION

BY M. A. BIOT

GRADUATE SCHOOL OF ENGINEERING, HARVARD UNIVERSITY

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1. *Equation of Wave Propagation.*—We shall consider a channel of constant cross-section. The depth h of the water is supposed to be the same everywhere.

If there is no friction, the displacement of a cross-sectional area of the water at time t and abscissa x satisfies the well-known wave equation (Fig. 1)

$$\frac{\partial^2 \xi}{\partial t^2} = gh \frac{\partial^2 \xi}{\partial x^2}. \tag{1}$$

The speed of propagation of the waves is $c = \sqrt{gh}$. The increase of altitude η of the water level is

$$\eta = -h \frac{\partial \xi}{\partial x}. \tag{2}$$

It satisfies the same differential equation

$$\frac{\partial^2 \eta}{\partial t^2} = gh \frac{\partial^2 \eta}{\partial x^2}. \tag{3}$$

If we assume that the channel exerts upon the moving water a frictional resistance proportional to the square of the speed $\dfrac{\partial \xi}{\partial t}$ a new term enters the above equation, which then becomes a quadratic hyperbolic equation.

We define a friction coefficient c_f such that the friction force per unit area is $c_f \, \rho \left(\dfrac{\partial \xi}{\partial t}\right)^2$. Then the total friction force on a cross-sectional slab of water of thickness dx and unit width is,

$$-\frac{c_f\rho}{2}\left(\frac{\partial\xi}{\partial t}\right)^2 dx$$

where ρ is the specific mass of the water. The equation of motion is then

$$\frac{\partial^2\xi}{\partial t^2} = -g\frac{\partial\eta}{\partial x} - \frac{c_f}{2h}\left(\frac{\partial\xi}{\partial t}\right)^2.$$

By relation (2) which holds in this case too,

$$\frac{\partial^2\xi}{\partial t^2} = gh\frac{\partial^2\xi}{\partial x^2} - \frac{c_f}{2h}\left(\frac{\partial\xi}{\partial t}\right)^2.$$

Taking as variable $s = ct$ with $c = \sqrt{gh}$, this equation becomes

$$\frac{\partial^2\xi}{\partial s^2} = \frac{\partial^2\xi}{\partial x^2} - \frac{c_f}{2h}\left(\frac{\partial\xi}{\partial s}\right)^2. \tag{4}$$

This is the fundamental equation of the problem. We note that in this case this type of equation is essentially associated with the variable ξ. The altitude increment η does not satisfy a similar equation.

2. *A Solution of the Non-Linear Equation* (4).—In order to solve (4) let us try a solution of the form

$$\xi = f\left(\frac{x}{s}\right).$$

For further simplification, we prefer to put this in the form

$$\xi = \varphi\left(\frac{s+x}{s-x}\right)$$

or $\xi = \varphi(\zeta)$ with $\zeta = \dfrac{s+x}{s-x}$.

We then find

$$\frac{\partial\xi}{\partial s} = \varphi'(\zeta)\frac{1-\zeta}{s-x} \qquad \varphi'(\zeta) = \frac{d}{d\zeta}\varphi(\zeta)$$

$$\frac{\partial^2\xi}{\partial s^2} = \varphi''(\zeta)\left(\frac{1-\zeta}{s-x}\right)^2 - 2\varphi'(\zeta)\frac{1-\zeta}{(s-x)^2}$$

$$\frac{\partial^2\xi}{\partial x^2} = \varphi''(\zeta)\left(\frac{1+\zeta}{s-x}\right)^2 + 2\varphi'(\zeta)\frac{1+\zeta}{(s-x)^2}.$$

Replacing these values in the differential equation (4)

$$\zeta\varphi''(\zeta) + \varphi'(\zeta) = \frac{c_f}{8h}(1-\zeta)^2\varphi'^2(\zeta). \tag{5}$$

This is an ordinary differential equation for $\varphi(\zeta)$. It can be written

$$\frac{\varphi''}{\varphi'^2} + \frac{1}{\zeta}\frac{1}{\varphi'(\zeta)} = \frac{c_f}{8h}\left(\frac{1}{\zeta} - 2 + \zeta\right).$$

Putting

$$y = -\frac{1}{\varphi'(\zeta)}$$

$$y' = \frac{\varphi''(\zeta)}{\varphi'(\zeta)}$$

this equation becomes linear in y,

$$y' - \frac{1}{\zeta}y = \frac{c_f}{8h}\left(\frac{1}{\zeta} - 2 + \zeta\right).$$

Write $y = v\zeta$, with a new unknown v. The equation simplifies into

$$v' = \frac{c_f}{8h}\left(\frac{1}{\zeta^2} - \frac{2}{\zeta} + 1\right).$$

By integration,

$$v = \frac{c_f}{8h}\left(-\frac{1}{\zeta} - 2\log\zeta + \zeta\right) + C$$

where C is a constant of integration.

Going back through the definition of the auxiliary unknowns, we finally get

$$\varphi'(\zeta) = -\frac{1}{y} = -\frac{1}{v\zeta}$$

$$\varphi'(\zeta) = -\frac{1}{C\zeta + \frac{c_f}{8h}(\zeta^2 - 2\zeta\log\zeta - 1)}.$$

Remembering that

$$\zeta = \frac{s+x}{s-x}$$

or $$\zeta = \frac{1+\alpha}{1-\alpha} \quad \text{with} \quad \alpha = \frac{x}{s} = \frac{x}{t\sqrt{gh}}$$

$$\varphi'(\zeta) = -\frac{1}{C\frac{1+\alpha}{1-\alpha} + \frac{c_f}{8h}\left[\left(\frac{1+\alpha}{1-\alpha}\right)^2 - 2\left(\frac{1+\alpha}{1-\alpha}\right)\log\left(\frac{1+\alpha}{1-\alpha}\right) - 1\right]} \tag{6}$$

Putting

$$k = \frac{2}{Ch}$$

$$\varphi'(\zeta) = -\frac{kh}{2\frac{1+\alpha}{1-\alpha} + \frac{c_f k}{8}\left[\left(\frac{1+\alpha}{1-\alpha}\right)^2 - 2\frac{1+\alpha}{1-\alpha}\log\frac{1+\alpha}{1-\alpha} - 1\right]}$$

This value of $\varphi'(\zeta)$ is regular and perfectly defined between the values $\alpha = 0$ and $\alpha = 1$. It is not, however, the only solution of equation (5), which is also obviously satisfied by the singular solution

$$\varphi'(\zeta) = 0.$$

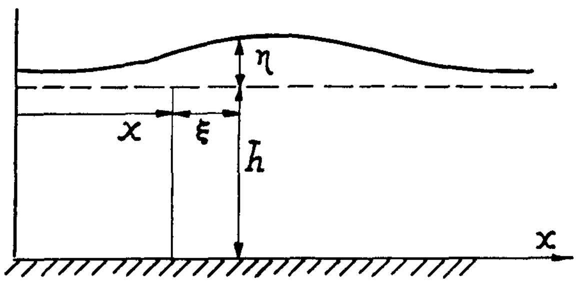

FIGURE 1

We may therefore adopt as a solution of equation (5) a discontinuous function which is

$$\varphi'(\zeta) = -\frac{kh}{2\frac{1+\alpha}{1-\alpha} + \frac{c_f k}{8}\left[\left(\frac{1+\alpha}{1-\alpha}\right)^2 - 2\frac{1+\alpha}{1-\alpha}\log\frac{1+\alpha}{1-\alpha} - 1\right]}$$

$$\text{for } 0 \leq \alpha \leq 1 \tag{7}$$

$$\text{and } \varphi'(\zeta) = 0 \qquad \text{for } \alpha > 1$$

In order to calculate the displacement ξ of the water, we should need, of course, the function $\varphi(\zeta)$ itself. As we shall see, this is not necessary to acquire a physical picture of the solution.

3. *Physical Interpretation of the Solution.*—The elevation η of the water is given by

$$\eta = -h\frac{\partial \xi}{\partial x} = -h\varphi'(\zeta)\frac{d\zeta}{dx}.$$

By using the value (7) of $\varphi'(\zeta)$, noting that

$$\frac{\partial \zeta}{\partial x} = \frac{2}{s(1-\alpha)^2}$$

we get the relative elevation of the water between $\alpha = 0$ and $\alpha = 1$.

$$\frac{\eta}{h} = \frac{h}{s}\,\frac{k}{(1-\alpha^2) + \frac{c_f k}{4}\left[\alpha - {}^1/{}_2(1-\alpha^2)\log\frac{1+\alpha}{1-\alpha}\right]} \tag{8}$$

For $\alpha > 1$ we get, according to (7),

$$\frac{\eta}{h} = 0.$$

This value of η represents the amplitude of a wave as shown in figure 2. For instance, for $k = 1$ and $\frac{c_f k}{4} = 0.5$, the shape of this wave will be the curve $ABCD$. At point B it has a vertical slope. The length BC is the elevation of the wave front, its relative value is given by the general formula (8) for $\alpha = 1$.

$$\frac{\eta}{h} = \frac{4}{c_f}\cdot\frac{h}{s}$$

or, since $s = t\sqrt{gh}$,

$$\frac{\eta}{h} = \frac{4}{c_f}\cdot\frac{h}{t\sqrt{gh}}.$$

It is inversely proportional to time and to the friction coefficient c_f.

The front of the wave corresponds always to $\alpha = 1$ or $\frac{x}{t\sqrt{gh}} = 1$, which means that this wave front moves with the speed

$$c = \sqrt{gh}.$$

At the origin $\alpha = 0$ or $x = 0$, the relative elevation of the wave is

$$\frac{\eta}{h} = k\frac{h}{s} = k\frac{h}{t\sqrt{gh}}. \tag{9}$$

Note too that the area of the wave curve is independent of time. The length varies like $s = t\sqrt{gh}$ and the altitude is proportional to $\frac{1}{s} = \frac{1}{t\sqrt{gh}}$.

For instance, the shape of the wave $ABCD$ (corresponding to $k = 1$ and $\frac{c_f k}{4} = 0.5$) at various instants is illustrated by figure 3.

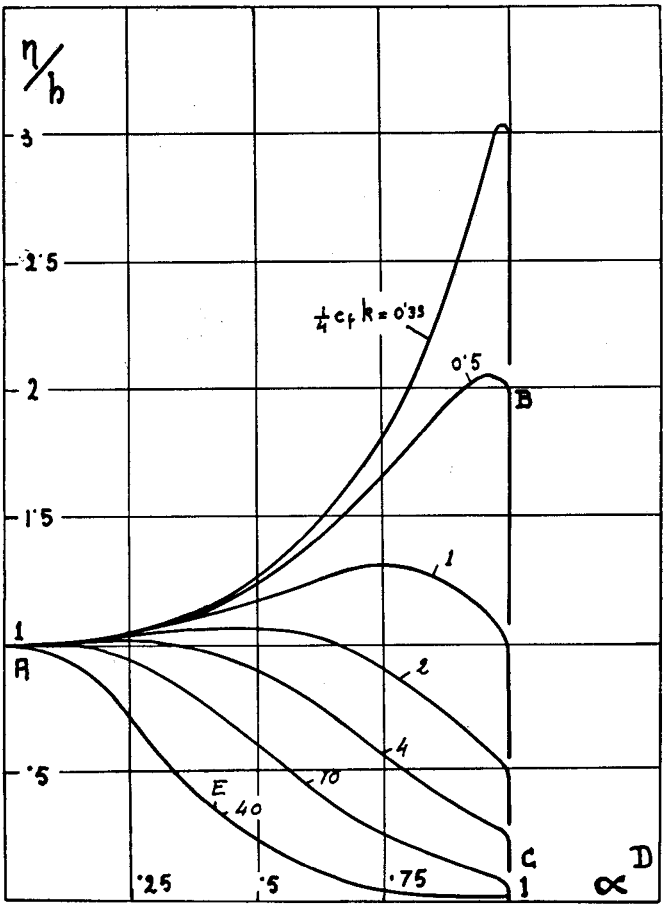

FIGURE 2

Various Shapes of Waves—$\left(\text{Shape Factor } \frac{c_f k}{4}\right)$.—In figure 2 are plotted the curves $\frac{\eta}{h}$ for a value $k = 1$ and different values of $\frac{c_f k}{4}$.

High values of $\frac{c_f k}{4}$ correspond to either high amplitudes or high friction coefficients. The effect is to damp out the wave front. This wave front still propagates with the same speed $c = \sqrt{gh}$ but its amplitude is small compared to that of the average elevation of the wave. The energy of the wave does not propagate as fast as the wave front but only with a fraction of the wave speed. For instance in case $\frac{c_f k}{4} = 40$, point E of altitude $0.38 \frac{h}{t\sqrt{gh}}$ propagates with a speed $c' = 0.4\sqrt{gh}$.

For low values of the friction or of the amplitude, the coefficient $\frac{c_f k}{4}$ is small. In that case, as shown by figure 2, most of the energy is kept in the wave front.

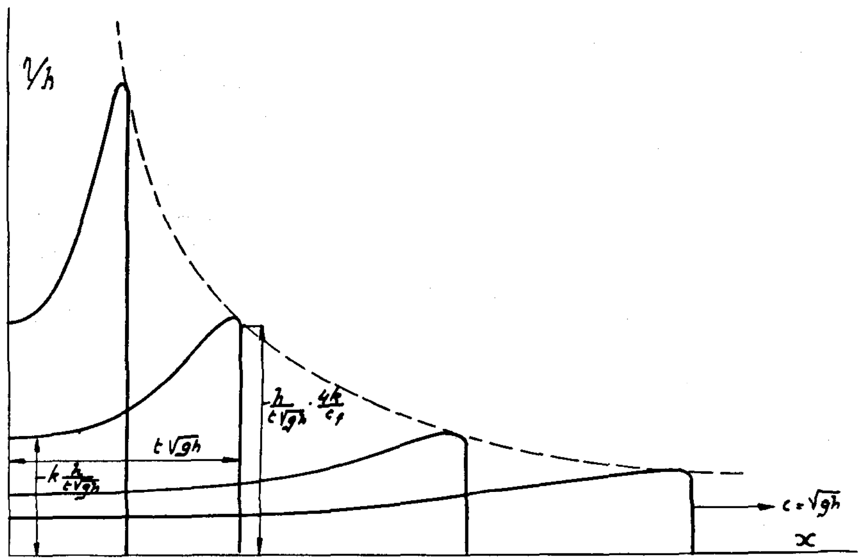

FIGURE 3

We thus get distinctly two types of waves. Waves with a steep front for values of $\frac{c_f k}{4} < 1$ and damped waves with a low front for $\frac{c_f k}{4} > 1$.

This coefficient $\frac{c_f k}{4}$, which may be called the "shape factor" determines the shape of the wave. It is noteworthy to point out the fact that this shape factor is the product of the friction by a factor k on which depends the volume or the average elevation of the wave. The physical meaning of this is clear when we remember that we have here a quadratic friction law: the higher the amplitude of the wave, the higher the speeds and hence the total friction. It is therefore in complete accordance with our natural intuition that in cases of quadratic friction the theory shows an increase of damping effect when the elevation of the wave increases. This is a new feature in the theory of wave propagation; it is introduced by the non-linear character of our wave equarion.

The Determination of k (Volume Factor).—The coefficient c_f is supposed to be given by the physical conditions of the channel. The value of k will then depend entirely on the average elevation of the wave.

An infinite number of waves other than the ones here investigated may exist in a channel, depending on the initial elevations and speeds. It is, however, probable that if we start with a local swelling initially at rest it will propagate and deform as the wave of same average height investigated above. The origin of time to be chosen depends on the height of the wave front as shown by equation (9). The average height or volume determines the coefficient k to be used for the corresponding theoretical curve. The type of initial condition here mentioned may be roughly identified with conditions arising from a sudden input of water at a given point of a river or a channel and propagating as a flood wave.

4. *Conclusion.*—An exact solution of the equation of propagation of waves with quadratic damping has been found. It shows that high amplitude waves are more quickly damped and how this damping effect depends on both the volume of the wave and the friction coefficient of the channel. The solution may be interpreted physically as representing certain types of flood waves.