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Effect of Certain Discontinuities on the Pressure Distribution in a Loaded Soil

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The pressure distribution due to a concentrated load on a semi-infinite elastic body is given by the well-known Boussinesq solution for either the two-dimensional or three-dimensional case. We here investigate the effect on such a pressure distribution, taken at the depth h, of the presence at that depth of a slippery rigid bed (case (b)); of a perfectly rough rigid bed (case (c)); and of a perfectly flexible but inextensible thin layer embedded in the material (case (d)). Both pressure distributions in the two and three-dimensional problems are calculated for each case, Fig. 5 and Fig. 6. Several authors have already investigated case (b) and case (c) in two dimensions (2), (3). Their results are in perfect accordance with ours. The author is indebted to Dr. A. Casagrande for suggesting this investigation as a contribution to the field of soil mechanics.

General Solution of the Equations of Elasticity

In the following paper, solutions of the equations of elasticity are needed which give both the stresses and the displacements. Therefore it will be easier to start from the equations relating the displacements u, v, w.

$$\nabla^2 u+\frac{1}{1-2\nu}\frac{\partial\Theta}{\partial x}=0,\quad \nabla^2 v+\frac{1}{1-2\nu}\frac{\partial\Theta}{\partial y}=0,\quad \nabla^2 w+\frac{1}{1-2\nu}\frac{\partial\Theta}{\partial z}=0 \tag{1}$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, $\Theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$,

ν is the Poisson ratio of the material.

It was established by Neuber[1] that a general solution of these equations may be found as follows.

We call Φ_0 a scalar function of x, y, z satisfying the Laplace equation $\nabla^2\phi_0=0$.

We call $\mathbf{\Phi}$ a vector, such that each of its Cartesian components ϕ_x ϕ_y ϕ_z satisfies the Laplace equation

$$\nabla^2\phi_x=0,\quad \nabla^2\phi_y=0,\quad \nabla^2\phi_z=0.$$

A very general solution of the equations of elasticity (1) can then be expressed as

$$\begin{aligned}u&=-(\partial/\partial x)(\phi_0+x\phi_x+y\phi_y+z\phi_z)+4(1-\nu)\phi_x,\\ v&=-(\partial/\partial y)(\phi_0+x\phi_x+y\phi_y+z\phi_z)+4(1-\nu)\phi_y,\\ w&=-(\partial/\partial z)(\phi_0+x\phi_x+y\phi_y+z\phi_z)+4(1-\nu)\phi_z;\end{aligned} \tag{2}$$

or in vectorial notation

$$\mathbf{u}=-\text{grad }(\phi_0+\mathbf{r}\cdot\mathbf{\Phi})+4(1-\nu)\mathbf{\Phi}.$$

The expression for Θ becomes in terms of the arbitrary functions above

$$\Theta=\text{div }\mathbf{u}=-\text{div grad }\phi_0-\text{div grad }\mathbf{r}\cdot\mathbf{\Phi}+4(1-\nu)\text{ div }\mathbf{\Phi}.$$

Since $\quad \text{div grad }\phi_0=\nabla^2\phi_0=0$

and $\text{div grad }\mathbf{r}\cdot\mathbf{\Phi}=\nabla^2(x\phi_x+y\phi_y+z\phi_z)=2\text{ div }\mathbf{\Phi}$,

we find $\quad \Theta=2(1-2\nu)\text{ div }\mathbf{\Phi}. \quad (3)$

The vertical stress component σ_z and the horizontal shear $\tau_{xz}\tau_{yz}$ are the only stress com-

[1] Neuber, "Ein neuer Ansatz zür Lösung räumlicher Probleme der Elastizitätstheorie," Zeits. f. angew. Math. und Mechanik **14**, 4 (1934).

ponents we shall need in the following theory. They are given by

$$\frac{\sigma_z}{2G}=\frac{\partial w}{\partial z}+\frac{\nu\Theta}{1-2\nu},\quad \frac{\tau_{xz}}{G}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z},$$
$$\frac{\tau_{yz}}{G}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}, \tag{4}$$

where G is the shear modulus of the material and ν the Poisson ratio.

It is important to note that the factor $(1-2\nu)$ disappears in the value of σ_z when using the value of Θ given by (3),

$$\sigma_z/2G=\partial w/\partial z+2\nu \operatorname{div} \Phi. \tag{4_1}$$

This shows that the value of the term $\nu\Theta/(1-2\nu)$ when $\nu=\frac{1}{2}$, $\Theta=0$, tends toward the limit $2\nu \operatorname{div} \Phi$.

The Two-Dimensional Problem

Let us consider first the two-dimensional problem where the load is concentrated on a line and has the value P per unit length (Fig. 1). We shall assume that the ground is elastic, of elasticity modulus E, and shall restrict ourselves to the case where the Poisson ratio is $\nu=\frac{1}{2}$, which means that the material of the ground is supposed to be incompressible. Four cases will be investigated.

(a) The ground is infinitely extended and deep. The pressure distribution at the depth h is given by the well-known Boussinesq solution.

(b) The ground is infinitely extended but of finite depth h. It is resting on a rock base, perfectly rigid; no friction forces are supposed to act between the rock and the upper ground. They can slip with respect to each other without the slightest resistance (Fig. 2). This case has been calculated by various authors.[2] Their result, derived by a different method, is found in proper accordance with ours.

(c) The same case as in (b) but where perfect adhesion is supposed to exist between the upper ground layer and the rock surface. No slippage whatever is allowed to occur; the two materials are assumed to stick together perfectly. This case was investigated by Marguerre[3] for $\nu=0$. Comparison with his paper shows that there is no practical difference between the cases $\nu=0$ and $\nu=\frac{1}{2}$. The pressure distribution is calculated at the rock surface (depth h).

(d) There is a horizontal infinite inextensible but perfectly flexible layer at the depth h. The pressure distribution on that layer due to P is calculated. This case corresponds to the problem of a clay substratum containing a thin horizontal sand layer.

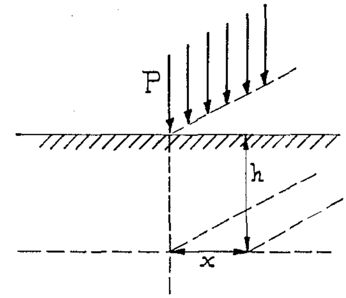

Fig. 1

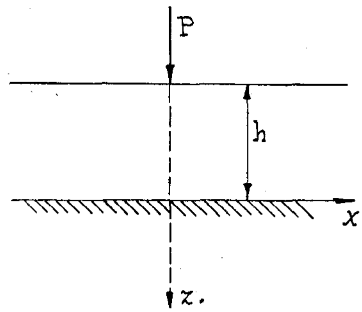

Fig. 2

In cases (a) and (b) the pressure distribution is found to be independent of the elasticity constants of the ground, while it depends on the Poisson ratio ν for cases (d) and (c). Hence the result for cases (a) and (b) is not affected by the restrictive assumption that $\nu=\frac{1}{2}$.

Case (a)

We take the z axis directed positive downward, the x axis being at the surface of the ground, and we assume that all the xz planes are identical. It is possible to find solutions of the elasticity equations for which all the variables are cosine or sine functions of x by the assumption:

$$\phi_0=(Ae^{\lambda z}+Be^{-\lambda z})\cos\lambda x,$$
$$\phi_z=(Ce^{\lambda z}+De^{-\lambda z})\cos\lambda x,\quad \phi_x=\phi_y=0, \tag{5}$$

where A, B, C, D, λ, are arbitrary constants.

By substituting this into Eqs. (2), (4) and (4_1), we obtain;

$$\frac{\sigma_z}{2G}=-\frac{\partial^2\phi_0}{\partial z^2}-z\frac{\partial^2\phi_z}{\partial z^2}+2(1-\nu)\frac{\partial\phi_z}{\partial z},$$
$$\frac{\tau_{xz}}{G}=-\frac{2\partial}{\partial x}\left[\frac{\partial\phi_0}{\partial z}+z\frac{\partial\phi_z}{\partial z}-(1-2\nu)\phi_z\right]. \tag{6}$$

Since we assume incompressibility, $\nu=\frac{1}{2}$,

$$\begin{aligned}\sigma_z/2G&=-\partial^2\phi_0/\partial z^2-z\partial\phi_z/\partial z^2+\partial\phi_z/\partial z,\\ \tau_{xz}/G&=-2(\partial/\partial x)[\partial\phi_0/\partial z+z\partial\phi_z/\partial z].\end{aligned}\tag{7}$$

At infinite depth we must have $\sigma_z=\tau_{xz}=0$ and this is only possible if $A=C=0$. We then have

$$\tau_{xz}/G=2\lambda\sin\lambda x[-B\lambda e^{-\lambda z}-zD\lambda e^{-\lambda z}].$$

By introducing the condition that the surface carries only a normal load and no shear, τ_{xz} must be zero for $z=0$, hence $B=0$. The expression for σ_z then reduces to

$$\sigma_z/2G=-D\lambda e^{-\lambda z}(1+z\lambda)\cos\lambda x.$$

This last relation shows that if the surface $z=0$ is loaded with a normal pressure distribution, $-\sigma_{z0}=q_0\cos\lambda x$, the pressure at the depth $z=h$ is

$$-\sigma_z=q=q_0\,e^{-\lambda h}(1+h\lambda)\cos\lambda x.\tag{8}$$

Now an arbitrary surface loading $p_0(x)$ may be represented by means of the Fourier integral as a superposition of sine loadings of various wave-lengths.

$$p_0(x)=(1/\pi)\int_0^\infty d\lambda\int_{-\infty}^{+\infty}d\xi p_0(\xi)\cos\lambda(x-\xi).$$

The pressure distribution $q(x)$ at the depth h is derived from relation (8) and the above Fourier integral.

$$q(x)=\frac{1}{\pi}\int_0^\infty d\lambda\int_{-\infty}^\infty d\xi p_0(\xi)e^{-\lambda h}(1+h\lambda)\cos\lambda(x-\xi),$$

which can be written

$$q(x)=\frac{1}{\pi}\int_{-\infty}^{+\infty}p_0(\xi)d\xi\int_0^\infty d\lambda e^{-\lambda h}(1+h\lambda)\cos\lambda(x-\xi).$$

If the surface loading $p_0(x)$ is concentrated within a small region $(-\epsilon,+\epsilon)$ such that the total load is

$$P=\int_{-\epsilon}^{\epsilon}p(\xi)d\xi.$$

The pressure distribution at a depth h due to that load is

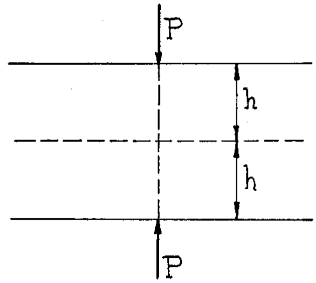

FIG. 3.

$$\begin{aligned}q(x)&=(P/\pi)\int_0^\infty e^{-\lambda h}(1+\lambda h)\cos\lambda x d\lambda,\\ q(x)&=(P/\pi h)\int_0^\infty e^{-\alpha}(1+\alpha)\cos\alpha(x/h)d\alpha.\end{aligned}\tag{9}$$

As will be seen, this yields the well-known Boussinesq solution.

Case (b)

The soft ground is supposed to rest on a rigid rock base, no friction occurring at the surface of contact (Fig. 2). This case is identical with the symmetrical loading illustrated by Fig. 3. In order to proceed as in case (a) and use the Fourier integral, we first have to consider a sinusoidal loading which is symmetrical with respect to the plane at depth h.

We take the x axis at the rock surface, the surface of the ground being at $z=h$ and its symmetrical image at $z=-h$, (Fig. 3). A symmetrical solution of the equations of elasticity is found by putting

$$\phi_0=\cosh\lambda z\cos\lambda x,\quad \phi_z=A\sinh\lambda z\cos\lambda x.$$

The application of formula (7) gives the value of the horizontal shear.

$$\tau_{xz}/G=2\lambda\sin\lambda x[\lambda\sinh\lambda z+Az\lambda\cosh\lambda z].$$

Since no shear is acting at the boundaries $z=\pm h$, we have the condition

$$\sinh\lambda h+hA\cosh\lambda h=0$$

or

$$A=-\sinh\lambda h/h\cosh h\lambda.$$

With this value of A the value of σ_z is given by Eq. (7).

$$\frac{\sigma_z}{2G}=\left[\frac{z}{h}\frac{\sinh\lambda h\sinh\lambda z}{\cosh\lambda h}-\frac{\sinh\lambda h\cosh\lambda z}{\lambda h\cosh\lambda h}-\cosh\lambda z\right]\times\lambda^2\cos\lambda x.$$

This last relation shows that if a pressure $q_0 \cos \lambda x$ is acting at $z=-h$, the pressure q on the plane of symmetry $z=0$, which represents the rock surface, is

$$q = \frac{2(\sinh \lambda h + \lambda h \cosh \lambda h)}{\sinh 2\lambda h + 2\lambda h} q_0 \cos \lambda x.$$

Proceeding as in case (a) by the use of the Fourier integral, we find the pressure distribution at $z=0$ due to a concentrated load P at the surface $z=-h$

$$p = \frac{2P}{\pi h} \int_0^\infty \frac{\alpha \cosh \alpha + \sinh \alpha}{\sinh 2\alpha + 2\alpha} \cos \alpha \frac{x}{h} d\alpha. \quad (10)$$

This checks with the formula found by Melan.[2] It gives the pressure p at the rock surface when no friction occurs as a function of the horizontal distance x from the vertical line of application of the load P.

Case (c)

This case considers the same material, soft ground lying on rigid rock, but here the upper soft material is supposed to stick perfectly to the rock surface.

We take the x axis at the rock surface and the z axis directed positive downward, as in case (b), (Fig. 2), and start from a solution which is sinusoidal along x

$$\phi_0 = (Ae^{\lambda z} + Be^{-\lambda z}) \cos \lambda x,$$
$$\phi_z = (Ce^{\lambda z} + De^{-\lambda z}) \cos \lambda x, \quad (11)$$
$$\phi_x = \phi_y = 0.$$

We have three boundary conditions

$$u=0 \text{ at } z=0, \quad w=0 \text{ at } z=0, \quad \tau_{xz}=0 \text{ at } z=-h. \quad (12)$$

We remember that we assume the soft material to be incompressible ($\nu=\frac{1}{2}$). With this value of the Poisson ratio, we get from formulae (2) and (7)

$$u = \lambda \sin \lambda x[Ae^{\lambda z} + Be^{-\lambda z} + zCe^{\lambda z} + zDe^{-\lambda z}],$$
$$w = -\cos \lambda x[(Ae^{\lambda z} - Be^{-\lambda z})\lambda + Ce^{\lambda z}(\lambda z - 1) - De^{-\lambda z}(\lambda z + 1)],$$
$$\tau_{xz}/G = 2\lambda \sin \lambda x[(Ae^{\lambda z} - Be^{-\lambda z})\lambda + z\lambda(Ce^{\lambda z} - De^{-\lambda z})],$$
$$\sigma_z/2G = \lambda \cos \lambda x[Ce^{\lambda z}(1-\lambda z) - De^{-\lambda z}(1+\lambda z) - \lambda(Ae^{\lambda z} + Be^{-\lambda z}). \quad (13)$$

The three boundary conditions determine three of the constants A B C D in terms of the fourth one. Calculating the coefficients B C D in terms of A, we find that the value of σ_z at the rock surface ($z=0$) is

$$\sigma_z/2G = 2A\lambda^2 \cos \lambda x(\cosh \lambda h + h\lambda \sinh \lambda h)/\lambda h \cosh \lambda h$$

and at the surface of the ground ($z=-h$) it is

$$\sigma_z/2G = 2A\lambda^2 \cos \lambda x(\cosh^2 \lambda h + (h\lambda)^2)/\lambda h \cosh \lambda h.$$

This shows that if a pressure $q_0 \cos \lambda x$ acts at the surface of the ground, a pressure

$$q = (\cosh \lambda h + \lambda h \sinh \lambda)/(\cosh^2 \lambda h + (\lambda h)^2)$$

is acting at the surface of the rock.

Using the Fourier integral as in the previous cases, we find the pressure distribution p at the surface of the rock due to a concentrated load P acting at the surface of the ground.

$$p(x) = \frac{P}{\pi h} \int_0^\infty \frac{\cosh \alpha + \alpha \sinh \alpha}{\cosh^2 \alpha + \alpha^2} \cos \alpha \frac{x}{h} d\alpha.$$

This formula coincides with Marguerre's result for the case $\nu=\frac{1}{2}$.[3]

Case (d)

The soft ground is here supposed to be infinitely deep and to contain at the depth h an inextensible but perfectly flexible thin layer to which the soft ground sticks perfectly so that only vertical motion at that depth is permissible.

We take the x axis at depth h on the inextensible layer and the z axis positive downward, and a sinusoidal solution in x as in case (c). The only difference with the preceding case is the boundary condition (12). The vertical displacement w at $z=0$ is not zero, but is related to the normal stress on that layer. To find this relation,

[2] L. N. G. Filon, Phil. Trans. Roy. Soc. London, 1903; E. Melan, Beton und Eisen 18 (1919).

[3] Marguerre, "Druckverteilung durch eine elastische Schicht auf starrer rauher Unterlage," Ing. Archiv **2** (1931).

let us consider a sinusoidal solution applying to the infinitely deep material located below that layer. This is a case analogous to case (a). Since the material is infinitely deep, we must have $A=C=0$ and the horizontal displacement u deduced from formula (2) with $\nu=\frac{1}{2}$ is found to be

$$u=\lambda \sin \lambda x[Be^{-\lambda z}+zDe^{-\lambda z}].$$

The condition that the horizontal displacement be zero on the inextensible layer ($z=0$) is $B=0$.

The vertical displacement at $z=0$ is

$$w=De^{-\lambda z}(1+\lambda z)\cos \lambda x$$

and the normal stress

$$\sigma_z/2G=-D\lambda e^{-\lambda z}(1+\lambda z)\cos \lambda x.$$

The relation between w and σ_z at the surface of the inextensible layer is

$$\sigma_z/2G=-\lambda w.$$

The boundary conditions for the upper soft ground are then

$$\begin{array}{llll} \text{(a)} & u=0 & \text{at} & z=0, \\ \text{(b)} & \sigma_z/2G=-\lambda w, & & z=0, \\ \text{(c)} & \tau_{xz}=0, & & z=-h. \end{array}$$

As in case (c) we may write the relation (13) and introduce these boundary conditions to find the value of B C D in terms of A.

The value of the normal stress σ_z at the rock surface $z=0$ is found to be

$$\frac{\sigma_z}{2G}=2\lambda^2 A\frac{\cosh \lambda h+\lambda h \sinh \lambda h}{h\lambda e^{h\lambda}}\cdot\cos \lambda x$$

and the value of σ_z at the ground surface is

$$\frac{\sigma_z}{2G}=\frac{2\lambda^2 A}{h\lambda}[h\lambda(h\lambda \cosh h\lambda-h\lambda \sinh h\lambda-\cosh h\lambda)+\cosh h\lambda+h\lambda \sinh h\lambda].$$

From this we conclude that a pressure distribution $q_0 \cos \lambda x$ at the ground surface transmits a pressure on the inextensible layer

$$q=\frac{e^{-h\lambda}}{1+h\lambda[h\lambda/(1+h\lambda \tanh h\lambda)-1]}.$$

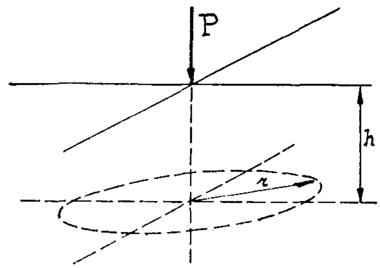

FIG. 4.

By using the Fourier integral as in the previous cases, we find the pressure distribution p transmitted to the inextensible layer when the surface carries a concentrated load P

$$p=\frac{P}{\pi h}\int_0^\infty \frac{e^{-\alpha}}{1-\alpha[1-\alpha/(1+\alpha \tanh \alpha)]}\cos \alpha\frac{x}{h}d\alpha.$$

THE THREE-DIMENSIONAL PROBLEM

The same cases (a) (b) (c) (d) are investigated if the load P is not concentrated on an infinite straight line but concentrated on a point, (Fig. 4). The stress distribution must then be axial-symmetrical around the load. Solutions of the problem may be found by using axial-symmetrical potential functions with cylindrical coordinates r, z.

$$\phi_0=(Ae^{\lambda z}+Be^{-\lambda z})J_0(\lambda r),$$

$$\phi_z=(Ce^{\lambda z}+De^{-\lambda z})J_0(\lambda r),$$

$$\phi_y=0, \quad \phi_x=0.$$

We have a vertical displacement w and a radial displacement u. Eqs. (2) and (3) become

$$u=-(\partial/\partial r)[\phi_0+z\phi_z],$$

$$w=-(\partial/\partial z)[\phi_0+z\phi_z]+4(1-\nu)\phi_z.$$

If we call τ the horizontal shear acting in the radial direction, we have equations entirely similar to Eqs. (4), (4_1).

$$\sigma_z/2G=\partial w/\partial z+\nu\Theta/(1-2\nu),$$

$$\tau/G=\partial w/\partial r+\partial u/\partial z.$$

As before, we shall assume that $\nu=\frac{1}{2}$. These equations show that the problem of determining the constants $A\ B\ C\ D$ is the same as in the two-dimensional case. For instance, if in the two-dimensional case to a load distribution at the surface $q_0 \cos \lambda x$ corresponded a pressure at depth h

$$q = g(\lambda h) q_0 \cos \lambda x$$

we may conclude that in the corresponding three-dimensional case to a load distribution $q_0 J_0(\lambda r)$ at the surface corresponds a pressure at depth h

$$q = g(\lambda h) q_0 J_0(\lambda r)$$

and we do not have to repeat the above calculations to find the function $g(\lambda h)$.

Case (a)

The case of a concentrated load P acting on an infinitely deep ground is found from the distribution $q_0 J_0(\lambda r)$. At a depth h the vertical pressure p transmitted is found by using the solution of the two-dimensional problem $g(h\lambda) = e^{-\lambda h}(1+h\lambda)$

$$p = e^{-h\lambda}(1+h\lambda) q_0 J_0(\lambda r).$$

An arbitrary load $q(r)$ could be represented by using the identity

$$q(r) = \int_0^\infty d\lambda \int_0^\infty J_0(\lambda\rho) J_0(\lambda r) q(\rho) \lambda \rho d\rho.$$

This shows that the corresponding pressure distribution at depth h is

$$p(r) = \int_0^\infty e^{-\lambda h}(1+\lambda h) J_0(\lambda r) \lambda d\lambda \int_0^\infty J_0(\lambda\rho) q(\rho) \rho d\rho.$$

If the load is concentrated at the origin on a circle of radius ϵ

$$P = 2\pi \int_0^\epsilon q(\rho) \rho d\rho.$$

The corresponding pressure distribution at depth h is

$$p(r) = (P/2\pi) \int_0^\infty \lambda e^{-\lambda h}(1+\lambda h) J_0(\lambda r) d\lambda$$

or

$$p(r) = (P/2\pi h^2) \int_0^\infty \alpha e^{-\alpha}(1+\alpha) J_0(\alpha r/h) d\alpha.$$

The same method may be used for all other cases (b), (c), (d). We find:

Case (b)

$$p(r) = \frac{P}{2\pi h^2} \int_0^\infty 2\alpha \frac{\alpha \cosh \alpha + \sinh \alpha}{\sinh 2\alpha + 2\alpha} J_0\left(\alpha \frac{r}{h}\right) d\alpha.$$

Case (c)

$$p(r) = \frac{P}{2\pi h^2} \int_0^\infty \alpha \frac{\cosh \alpha + \alpha \sinh \alpha}{\cosh^2 \alpha + \alpha^2} J_0\left(\alpha \frac{r}{h}\right) d\alpha.$$

Case (d)

$$p(r) = \frac{P}{2\pi h^2} \int_0^\infty \alpha \frac{e^{-\alpha}}{1 - \alpha[1 - \alpha/(1+\alpha \tanh \alpha)]} \times J_0\left(\alpha \frac{r}{h}\right) d\alpha.$$

Numerical Evaluation of the Infinite Integrals

The problem reduces to the evaluation of either

$$(P/\pi h) \int_0^\infty g(\alpha) \cos(\alpha x/h) d\alpha$$

for the two-dimensional problem, or

$$(P/2\pi h^2) \int_0^\infty \alpha g(\alpha) J_0(\alpha r/h) d\alpha$$

in the three-dimensional problem. The function $g(\alpha)$ has the form

$$g(\alpha) = (1+\alpha)e^{-\alpha} \qquad \text{case (a),}$$

$$g(\alpha) = 2\frac{\alpha \cosh \alpha + \sinh \alpha}{\sinh 2\alpha + 2\alpha} \qquad \text{case (b),}$$

$$g(\alpha) = \frac{\cosh \alpha + \alpha \sinh \alpha}{\cosh^2 \alpha + \alpha^2} \qquad \text{case (c),}$$

$$g(\alpha) = \frac{e^{-\alpha}}{1 - \alpha[1 - \alpha/(1+\alpha \tanh \alpha)]} \qquad \text{case (d).}$$

We start from the two identities

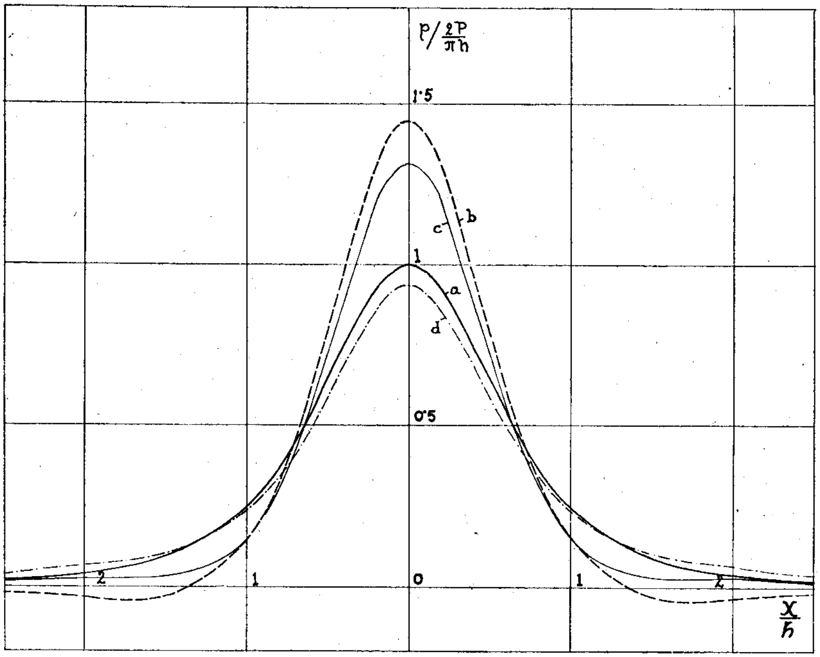

FIG. 5. Pressure distribution $p/(2P/\pi h)$ in the two-dimensional problem.

Case (a) ——— Boussinesq distribution. Maximum value of pressure $p = 2P/\pi h$.
Case (b) — — — Slippery rigid bed. Maximum value of pressure $p = 1.441 \cdot 2P/\pi h$.
Case (c) ——— Rough rigid bed. Maximum value of pressure $p = 1.291 \cdot 2P/\pi h$.
Case (d) —·—·— Inextensible flexible layer. Maximum value of pressure $p = 0.935 \cdot 2P/\pi h$.

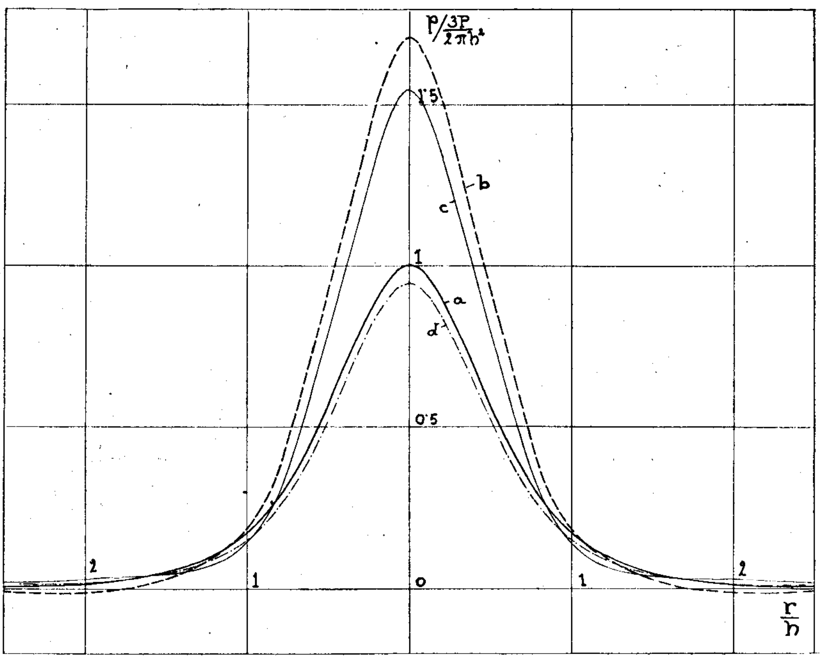

FIG. 6. Pressure distribution $p/(3P/2\pi h^2)$ in the three-dimensional problem.

Case (a) ——— Boussinesq distribution. Maximum value of pressure $p = 3P/2\pi h^2$.
Case (b) — — — Slippery rigid bed. Maximum value of pressure $p = 1.711 \cdot 3P/2\pi h^2$.
Case (c) ——— Rough rigid bed. Maximum value of pressure $p = 1.557 \cdot 3P/2\pi h^2$.
Case (d) —·—·— Inextensible flexible layer. Maximum value of pressure $p = 0.942 \cdot 3P/2\pi h^2$.

$$\int_0^\infty e^{-ax}\cos bx dx = a/(a^2+b^2),$$

$$\int_0^\infty e^{-ax}J_0(bx)dx = (a^2+b^2)^{-\frac{1}{2}}.$$

Taking the derivatives with respect to a, we get

$$\int_0^\infty x^n e^{-ax}\cos bx dx = (-1)^n \frac{d^n}{da^n}\left(\frac{a}{a^2+b^2}\right),$$

$$\int_0^\infty x^n e^{-ax}J_0(bx)dx = (-1)^n \frac{d^n}{da^n}\left(\frac{1}{(a^2+b^2)^{\frac{1}{2}}}\right).$$

This shows that the integrals would be evaluated readily if the functions $g(\alpha)$ where all expressed as a sum of terms of the type $\alpha^n e^{-k\alpha}$, where n is an integer.

Case (a)

In case (a) there is no difficulty in expressing these integrals. For the two-dimensional problem we get

$$p(x) = (P/\pi h)\int_0^\infty (1+\alpha)e^{-\alpha}\cos(\alpha x/h)d\alpha,$$

$$p(x) = (2P/\pi h)(1+x^2/h^2)^{-2}.$$

For the three-dimensional problem

$$p(r) = (P/2\pi h^2)\int_0^\infty \alpha(1+\alpha)e^{-\alpha}J_0(\alpha r/h)d\alpha,$$

$$p(r) = (3P/2\pi h^2)[1+(r/h)^2]^{-5/2}.$$

These are both the well-known Boussinesq solutions.

The two-dimensional distribution

$$p(x)/(2P/\pi h) = (1+x^2/h^2)^{-2}$$

is represented by curve (a) in Fig. 5 and the three-dimensional distribution $p(r)/(3P/2\pi h^2)$ by curve (a) in Fig. 6.

Case (b)

The function $g(\alpha)$ may be represented with an error smaller than 1 percent by

$$(2\alpha\cosh\alpha+\sinh\alpha)/(\sinh 2\alpha+2\alpha) = 2(1+\alpha)e^{-\alpha}-(1+2\alpha)e^{-2\alpha}-5.1\alpha^3e^{-4\alpha}.$$

The pressure distribution for the two-dimensional case is

$$p(x) = \frac{2P}{\pi h}\left[\frac{2}{(1+x^2/h^2)^2}-\frac{0.5}{(1+(x/2h)^2]^2}-0.059\frac{1-6(x/4h)^2+(x/4h)^4}{[1+(x/4h)^2]^4}\right].$$

The dimensionless quantity $p(x)/(2P/\pi h)$ is plotted in Fig. 5 by curve (b).

For the three-dimensional case we have

$$p(r) = \frac{3P}{2\pi h^2}\left[\frac{2}{[1+(r/h)^2]^{5/2}}-\frac{0.25}{[1+(r/2h)^2]^{5/2}}-0.039\frac{1-3(r/4h)^2+\frac{3}{8}(r/4h)^4}{[1+(r/4h)^2]^{9/2}}\right].$$

The ratio $p(r)/(3P/2\pi h^2)$ is represented in Fig. 6 by curve (b).

Case (c)

The function $g(\alpha)$ may be represented with an error smaller than 1 percent by

$$(\cosh\alpha+\alpha\sinh\alpha)/(\cosh^2\alpha+\alpha^2) = (2\alpha\cosh\alpha+\sinh\alpha)/(\sinh 2\alpha+2\alpha)-2.80\alpha^4e^{-3\alpha}-56\alpha^4e^{-8.8\alpha}.$$

The pressure distribution in the two-dimensional case is

$$p(x) = \frac{2P}{\pi h}\left[\frac{2}{[1+x^2/h^2]^2}-\frac{0.5}{[1+(x/2h)^2]^2}-0.059\frac{1-6(x/4h)^2+(x/4h)^4}{[1+(x/4h)^2]^4}-0.138\frac{1-10(x/3h)^2+5(x/3h)^4}{[1+(x/3h)^2]^5}\right.$$
$$\left.-0.012\frac{1-10(x/8.8h)^2+5(x/8.8h)^4}{[1+(x/8.8h)^2]^5}\right].$$

The ratio $p(x)/(2P/\pi h)$ is represented in Fig. 5 by curve (c). The pressure distribution in the three-dimensional case is

$$p(r)=\frac{3P}{2\pi h^2}\left[\frac{2}{[1+(r/h)^2]^{5/2}}-\frac{0.25}{[1+(r/2h)^2]^{5/2}}-0.039\frac{1-3(r/4h)^2+\frac{3}{8}(r/4h)^4}{[1+(r/4h)^2]^{9/2}}\right.$$
$$\left.-0.154\frac{1-5(x/3h)^2+(15/8)(x/3h)^4}{[1+(x/3h)^2]^{11/2}}\right].$$

The ratio $p(r)/(3P/2\pi h^2)$ is represented in Fig. 6, by curve (c).

Case (d)

The function $g(\alpha)$ may be represented with an error less than 1 percent as follows:

$$e^{-\alpha}/\{1-\alpha[1-\alpha/(1+\alpha\tanh\alpha)]\}=(1+\alpha)e^{-\alpha}-1.77\alpha^3e^{-3\alpha}.$$

The pressure distribution in the two-dimensional problem is

$$p(x)=\frac{2P}{\pi h}\left[\frac{1}{(1+x^2/h^2)^2}-0.065\frac{1-6(x/3h)^2+(x/3h)^4}{[1+(x/3h)^3]^4}\right].$$

The ratio $p(x)/(2P/\pi h)$ is represented in Fig. 5 by curve (d).

The pressure distribution in the three-dimensional problem is

$$p(r)=\frac{3P}{2\pi h^2}\left[\frac{1}{(1+r^2/h^2)^{5/2}}-0.058\frac{1-3(r/3h)^2+\frac{3}{8}(r/3h)^4}{[1+(r/3h)^2]^{9/2}}\right].$$

The ratio $p(r)/(3P/2\pi h^2)$ is represented in Fig. 6 by curve (d).