BIOT, M.

A FOURIER-INTEGRAL SOLUTION OF THE PROBLEM OF THE BENDING UNDER A CONCENTRATED LOAD OF AN INFINITELY LONG BEAM RESTING ON AN ELASTIC CONTINUUM

1. In the elementary treatment of this problem the value of the maximum bending moment due to a concentrated load P is given by

$$M_{\max} = 0{\cdot}25P\left(\frac{4E_b I}{k}\right)^{\frac{1}{4}} \qquad \ldots\ldots(1)$$

where E_b is the Young modulus of the beam,
I, the inertia moment of its cross-section,
k, the so-called modulus of the foundation.

2. The problem however can be solved exactly for a beam resting on a semi-infinite wall of same width $2b$ as the beam. We start from the fact that a sinusoidal loading $q(x) = q_0 \cos \lambda x$ on the top of this wall gives a deflection

$$w = \frac{q_0}{Eb\lambda} \cos \lambda x$$

where E is the Young modulus of the foundation. From there it can be easily shown that a load per unit length $p(x) = p_0 \cos \lambda x$ on the beam gives a bending moment

$$M(x) = \frac{p_0 \lambda}{\lambda^3 + \dfrac{Eb}{E_b I}} \cos \lambda x.$$

By using the Fourier-Integral we get an expression for the bending moment due to any distribution of load $p(x)$ per unit length,

$$M(x) = \int_{-\infty}^{+\infty} p(\xi)\, d\xi \int_0^{\infty} \frac{d\lambda}{\pi} \frac{\lambda}{\lambda^3 + \dfrac{Eb}{E_b I}} \cos \lambda (x - \xi).$$

In case of a concentrated load $P = \int_{-\infty}^{+\infty} p(\xi)\, d\xi$ at the point $x = 0$ the maximum bending moment (at point $x = 0$) is given by

$$M_{\max} = \frac{P}{\pi} \int_0^{\infty} \frac{\lambda\, d\lambda}{\lambda^3 + \dfrac{Eb}{E_p I}}.$$

Solving this integral

$$M_{\max} = 0{\cdot}385 \left[\frac{E_b}{E} \frac{I}{b^4}\right]^{\frac{1}{3}} Pb.$$

It is noted that this formula differs quite fundamentally from the formula (1) given by the elementary theory.

3. The same problem has been solved for a beam resting on a three-dimensional foundation, starting from the result that a normal loading per unit area on the surface of the foundation $q = q_0 \cos \lambda x \cos ky$ gives a deflection

$$w = \frac{2(1 - \mu^2)}{E\sqrt{\lambda^2 + \kappa^2}} q_0 \cos \lambda x \cos \kappa y.$$

(μ Poisson ratio of the foundation.)

From this can be calculated the mean value of the deflection due to a loading $Q_0 \cos \lambda x$ per unit length and uniformly distributed along a direction parallel to y between the lines $y = \pm b$. We then proceed as in the previous case of a two-dimensional foundation. The final integrations have to be carried out graphically. It is found that the maximum bending moment due to the concentrated load is given with a good approximation by

$$M_{\max} = 0 \cdot 335 \left[(1-\mu^2) \frac{E_b}{E} \frac{I}{b^4} \right]^{0 \cdot 277} Pb.$$

$2b$ is the width of the beam.