HARVARD UNIVERSITY

DISTRIBUTED GRAVITY AND TEMPERATURE LOADING IN TWO-DIMENSIONAL ELASTICITY REPLACED BY BOUNDARY PRESSURES AND DISLOCATIONS

BY

M. A. BIOT

ON THE HYDRODYNAMIC ANALOGY OF TORSION

BY

J. P. DEN HARTOG and J. G. McGIVERN

Reprinted from

Journal of Applied Mechanics

June, 1935

PUBLICATIONS FROM THE

HARVARD ENGINEERING SCHOOL

1934–35 No. 150

(*Reprinted from the* JOURNAL OF APPLIED MECHANICS *for June, 1935*)

Distributed Gravity and Temperature Loading in Two-Dimensional Elasticity Replaced by Boundary Pressures and Dislocations

BY M. A. BIOT,[1] CAMBRIDGE, MASS.

This paper deals with two-dimensional stresses due to an important class of body forces. Gravity stresses and temperature stresses are produced by body forces of the type investigated.

In the case of gravity stresses it is proved that these stresses may be calculated by forgetting about the action of gravity on the body and applying instead only external normal loads that are identical with a hydrostatic pressure.

This opens the possibility of applying photoelastic tests to the measurement of gravity stresses; a model loaded by these external pressures would show an isochromatic fringe pattern identical with the one that would appear in a model submitted to an intense gravity field. In Fig. 5 is shown the type of external loading to be applied to find the gravity stresses in a dam.

In the case of two-dimensional temperature stress in cylindrical bodies where the temperature distribution is stationary, it is shown that a distinction is to be made between solid and hollow cylinders. For a solid cylinder the only thermal stress component appearing is an axial stress acting perpendicularly to cross-sectional planes. In other words, the solid cylinder expands freely along the cross-sectional planes. For a hollow cylinder a general stress condition may arise which is identical with the stresses produced by cutting a longitudinal slit of a certain small width in the cylinder wall and sticking together the two sides of the slit (dislocation).

GENERAL THEORY

THE EQUATIONS of equilibrium of a rectangular element in two-dimensional elasticity with body forces are

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y &= 0 \end{aligned}\right\} \qquad [1]$$

If the body forces are derived from a function V such that

$$\left.\begin{aligned} X &= -\frac{\partial V}{\partial x} \\ Y &= -\frac{\partial V}{\partial y} \end{aligned}\right\} \qquad [2]$$

the equations of equilibrium become

$$\left.\begin{aligned} \frac{\partial}{\partial x}(\sigma_x - V) + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial}{\partial y}(\sigma_y - V) + \frac{\partial \tau_{xy}}{\partial x} &= 0 \end{aligned}\right\} \qquad [3]$$

They are satisfied by introducing a "stress function" ϕ

$$\left.\begin{aligned} \sigma_x - V &= \frac{\partial^2 \phi}{\partial y^2} \\ \sigma_y - V &= \frac{\partial^2 \phi}{\partial x^2} \\ \tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} \end{aligned}\right\} \qquad [4]$$

By using Hooke's law for either plane stress or plane strain distribution and the compatibility equation of the strain components, we find an equation of the type[2]

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = C\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \qquad [5]$$

In case the function V satisfies the Laplace equation (potential function)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad [6]$$

Equation [5] reduces to

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \qquad [7]$$

In this case the stress function ϕ satisfies the same equation as if there were no body forces. How will the boundary conditions be expressed?

Let l, m be the cosines of the normal direction to the boundary with the x and y directions and $\overline{X}$, $\overline{Y}$ the components of the force applied per unit length at the boundary. We have

$$\begin{aligned} \overline{X} &= l\sigma_x + m\tau_{xy} \\ \overline{Y} &= m\sigma_y + l\tau_{xy} \end{aligned}$$

[1] Instructor in Applied Mechanics, Graduate School of Engineering, Harvard University. Dr. Biot received at the University of Louvain, Belgium, the degrees of Mining Engineer, 1929, Electrical Engineer, 1930, and Sc.D. in physics and mathematics, 1931. From 1931 to 1933 he was research fellow of the C.R.B. Educational Foundation at the California Institute of Technology under Prof. Th. von Kármán and at the University of Michigan under Prof. S. Timoshenko. In 1932 he received a Ph.D. from the California Institute of Technology. During 1933–1934, as a research associate of the National Research Council of Belgium, he visited the technical schools of Delft and Zürich and the Universities of Göttingen and Cambridge. He has held his present position at Harvard since September, 1934.

Discussion of this paper should be addressed to the Secretary, A.S.M.E., 29 West 39th Street, New York, N. Y., and will be accepted until August 10, 1935, for publication at a later date.

[2] Equation [5] is derived in "Theory of Elasticity," by S. Timoshenko, McGraw-Hill, 1934, pp. 25–26.

or from Equations [4]

$$\left.\begin{aligned} \bar{X} - lV &= l\frac{\partial^2\phi}{\partial y^2} - m\frac{\partial^2\phi}{\partial x\partial y} \\ \bar{Y} - mV &= m\frac{\partial^2\phi}{\partial x^2} - l\frac{\partial^2\phi}{\partial x\partial y} \end{aligned}\right\} \qquad [8]$$

Hence the problem of finding the stresses produced by the body force loading X, Y just considered and the boundary loading

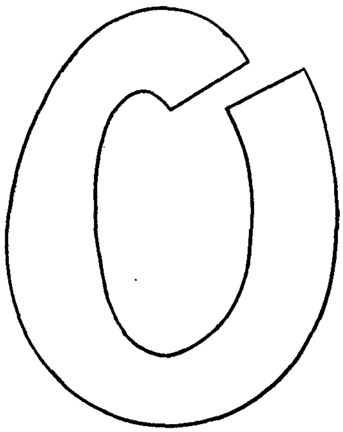

Fig. 1

$\bar{X}$ $\bar{Y}$ reduces to the solution of Equation [7] with the boundary conditions [8]. This is equivalent to finding a stress distribution

$$\left.\begin{aligned} \sigma'_x &= \frac{\partial^2\phi}{\partial y^2} \\ \sigma'_y &= \frac{\partial^2\phi}{\partial x^2} \\ \tau'_{xy} &= -\frac{\partial^2\phi}{\partial x\partial y} \end{aligned}\right\} \qquad [9]$$

produced in the same body free from body forces but submitted to boundary forces $\bar{X} - lV$ and $\bar{Y} - mV$. This boundary load is the actual load $(\bar{X}, \bar{Y})$ plus a normal hydrostatic pressure equal to V.

Apparently, according to Equations [4], the actual stress in the body is the difference of the stresses σ'_x, σ'_y, τ'_{xy}, and the hydrostatic-pressure distribution represented by the potential V.

However, a very important distinction is to be made; *the statement is true only for body forces such that no singularity of the potential V occurs within the body or in any holes if the latter is hollow.*

This distinction arises from the fact that the displacements and rotations of the medium, corresponding to the hydrostatic-stress condition $p = V$ might not be single-valued, i.e., they might not take the same value when we come back to the same point after following a closed path around a singularity of the function V. In order to get single-valued displacements and rotations, we must superpose to the hydrostatic-stress condition a so-called "dislocation"-stress distribution. The meaning of a dislocation is illustrated by the following example. Take a hollow circular cylinder and cut it open by a longitudinal slit of small width (Fig. 8). We shall produce stresses in this cylinder by sticking together the two sides of the slit. This type of deformation is called a dislocation. The stress condition thus generated is not derived from the application of external forces. In our case, for instance, if the body is of hollow shape, let us cut it open so as to connect the inside with the outside. If we now assume that a stress equal to a hydrostatic pressure $p = V$ is distributed in the body, the two edges of the cut may or may not stick together (Fig. 1). If they do separate, we produce dislocation stresses by sticking them together again. As we shall see, the amount of dislocation is determined by two displacements u, v, of one face of the slit with respect to the other and by relative rotation. It can be easily calculated from the value of the potential V.

Let us call ϵ the two-dimensional linear extension of the body produced by the hydrostatic-stress distribution $p = V$. We have

$$\epsilon = -kp = -kV \qquad [10]$$

The corresponding displacements of the material satisfy the equations:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \epsilon \\ \frac{\partial v}{\partial y} &= \epsilon \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= 0 \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 2\omega \end{aligned}\right\} \qquad [11]$$

where ω is the rotation.

We deduce:

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= -\omega \\ \frac{\partial v}{\partial x} &= \omega \\ \frac{\partial \epsilon}{\partial y} &= -\frac{\partial \omega}{\partial x} \\ \frac{\partial \epsilon}{\partial x} &= \frac{\partial \omega}{\partial y} \end{aligned}\right\} \qquad [12]$$

From these equations we see that ϵ and ω are conjugate harmonic functions so that

$$Z(z) = \epsilon + i\omega$$

is an analytic function of $z = x + yi$. The increase of rotation on a closed circuit is

$$\omega_1 - \omega_2 = \oint -\frac{\partial \epsilon}{\partial y}dx + \frac{\partial \epsilon}{\partial x}dy \qquad [13]$$

It is a quantity proportional to the flux of the body force through this closed path.

The displacement is given by

$$u + iv = \int Zdz \qquad [14]$$

so that the increase in displacement on a closed circuit is given by the contour integral

$$\oint Zdz$$

This proves that the two edges of the slit show a relative displacement only when the function Z or, what amounts to the same thing, the potential V has a singularity inside the contour. Yet if the singularity is such that the flux of body force produced by it is zero and if $\oint Zdz = 0$ no dislocation stress has to be introduced.

DISTRIBUTED GRAVITY LOADING

One of the most important cases of potential field body force is the case of the gravity field acting upon a body of uniform density.

Body Resting Under Its Own Weight

Let us consider a two-dimensional elastic body B (Fig. 2) resting under its own weight on a plane surface S. Taking the vertical y axis as positive and calling δ the specific weight of the material, the body force is expressed as $X = 0$, and $Y = -\delta$, and the corresponding potential is $V = \delta y$. This potential presents no singularity whatever, hence no dislocation stress will have to be considered.

The stress condition σ'_x, σ'_y, τ'_{xy} of the above theory is produced by considering a hydrostatic boundary loading due to a pressure

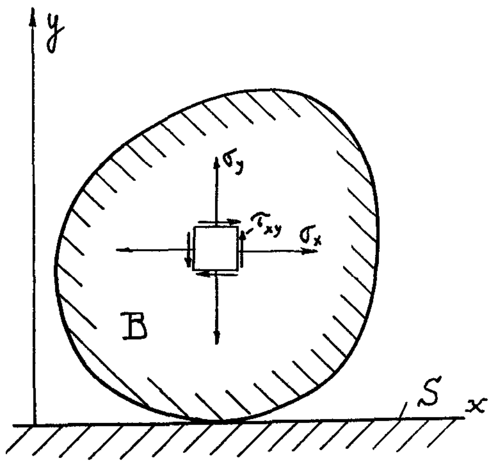

FIG. 2

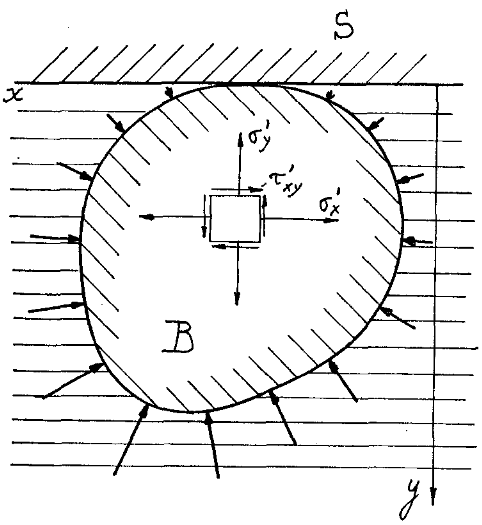

FIG. 3

$p = V = \delta y$. This loading may be applied by turning the body upside down, free from gravity, into a liquid of specific mass δ which is under the action of gravity (Fig. 3). The hydrostatic pressure $p = \delta y$ pushes it upward against the surface S with the same force as would its own weight in Fig. 2 and the contact pressures will be approximately realized. Due to this hydrostatic boundary loading a stress distribution σ'_x, σ'_y, τ'_{xy} is produced in the body B. Since no dislocation stresses have to be introduced, the value of the actual stress σ_x, σ_y, τ_{xy} due to distributed gravity loading in Fig. 2 is the difference of the stress σ'_x, σ'_y, τ'_{xy}, and a pressure $p = y\delta$. Hence

$$\sigma_x = \sigma'_x + y\delta$$
$$\sigma_y = \sigma'_y + y\delta$$
$$\tau_{xy} = \tau'_{xy}$$

Gravity Stress in a Dam

Another example as shown in Fig. 4 is given by the case of a dam which is stressed under its own weight, the specific mass of which is δ.

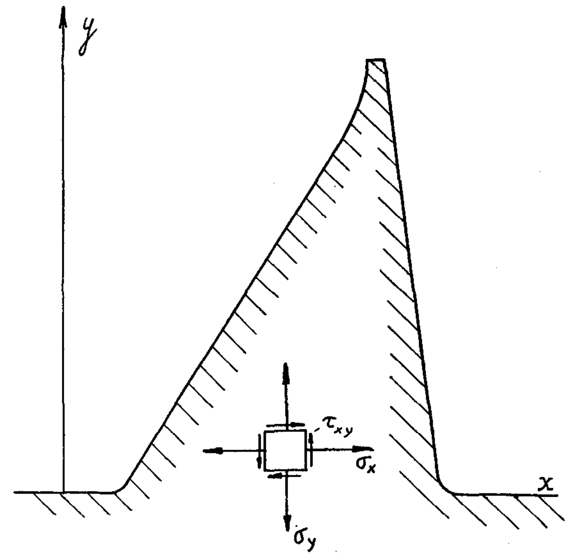

FIG. 4

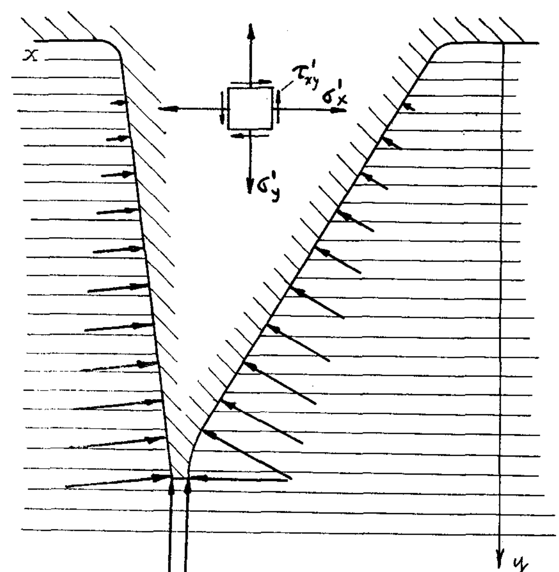

FIG. 5

The stress distribution σ'_x, σ'_y, τ'_{xy} is produced in a dam having no gravity and immersed upside down in a liquid of specific mass δ which is under the action of gravity, Fig. 5. This liquid exerts upon the dam hydrostatic boundary pressure as indicated in this figure. The actual stress σ_x, σ_y, τ_{xy} in the dam resting under its own weight as in Fig. 4, is given by the difference of the stresses σ'_x, σ'_y, τ'_{xy} and a pressure distribution $p = \delta y$ Therefore

$$\sigma_x = \sigma'_x + \delta y$$
$$\sigma_y = \sigma'_y + \delta y$$
$$\tau = \tau'_{xy}$$

Photoelastic Measurement of Gravity Stresses

The present theory shows that gravity stresses may be measured by photoelasticity by applying the proper boundary loading. The fringe pattern and the value of the maximum shear will be the same as if a magnified gravity were acting on the model.

Body With Holes

An interesting property is introduced by considering a body, with a hole, under the action of gravity. This equivalence of hydrostatic boundary loading applies provided we fill the

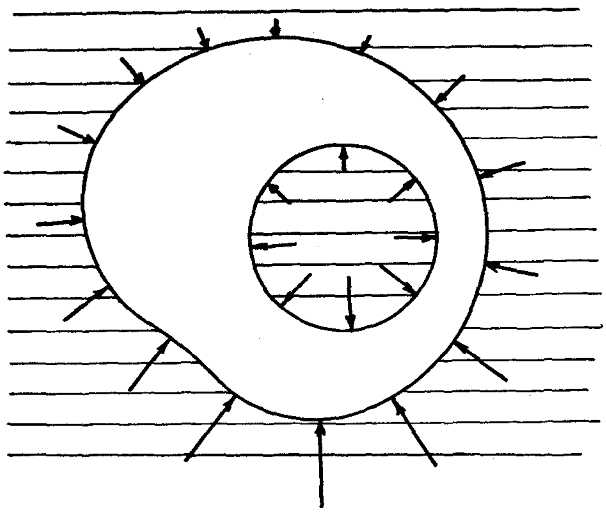

Fig. 6

hole with the weighing liquid and establish a connection between the liquid inside of the hole and the liquid outside such that both are in hydrostatic equilibrium (Fig. 6).

A theorem developed by J. H. Michell[3] shows that for a hollow two-dimensional body the stresses depend on the elasticity constants if the forces applied to the hole have a resultant different from zero. In this case the forces applied to the boundary of the hole have a resultant equal to the weight of the fluid inside. From this we conclude the following theorem:

In a solid two-dimensional homogeneous body the stresses due to gravity do not depend on the elasticity constants of the material; they do, however, in general for a body with holes.

It is important to keep this in mind when investigating gravity stresses by photoelasticity or with gelatine models.

TWO-DIMENSIONAL TEMPERATURE STRESSES RESULTING FROM STEADY HEAT FLOW

We consider a two-dimensional temperature distribution in an arbitrary cylindrical body; the temperature is supposed to be the same along any straight line parallel with the boundary. We also assume that the temperature has reached a state of equilibrium, i.e., the temperature may be different from point to point but remains constant at a given point and has any given arbitrary distribution along the boundary of the cross-section S (Fig. 7). As we know, in such a case the temperature θ in a cross-sectional plane (x, y) must satisfy the potential equation,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \quad\text{[15]}$$

[3] "Theory of Elasticity," by S. Timoshenko, McGraw-Hill, 1934, pp. 113–117.

Let

$$\epsilon = k\theta \quad\text{[16]}$$

where k is the coefficient of linear thermal expansion in three dimensions. It is known that this problem of two-dimensional strain is the same as that of the deformation of the same body under the body forces

$$\left.\begin{aligned} X &= -\frac{2G(1+\nu)}{1-2\nu}\frac{\partial\epsilon}{\partial x} = -\frac{\partial V}{\partial x} \\ Y &= -\frac{2G(1+\nu)}{1-2\nu}\frac{\partial\epsilon}{\partial y} = -\frac{\partial V}{\partial y} \end{aligned}\right\} \quad\text{[17]}$$

and a normal boundary tension

$$\frac{2G(1+\nu)}{1-2\nu}\,\epsilon = V \quad\text{[18]}$$

where G is the modulus of elasticity by shear, and ν is the Poisson ratio.

Since we have a steady-state heat problem, we have, according to Equations [15], [16], and [18]

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

We are thus exactly in the general case considered by the previous theory. The theory is easily applied to this case; we shall state only the conclusions.[4]

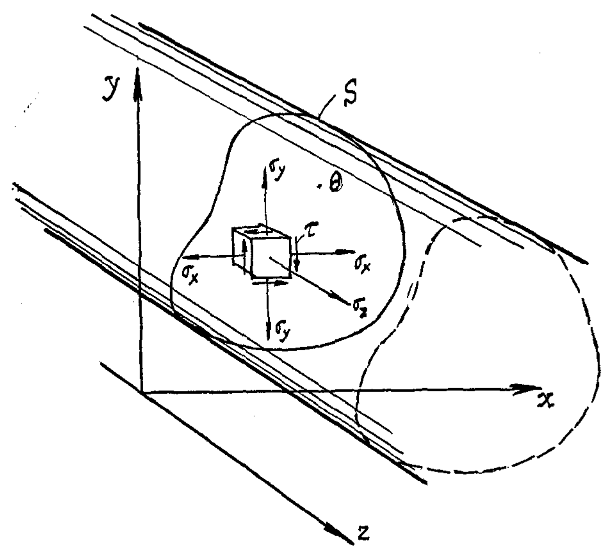

Fig. 7

We have to distinguish between the case where the cylinder is (1) solid or (2) hollow.

(1) If a solid cylinder is heated, either uniformly or not, but in such a way that a steady-state temperature distribution exists the same in every cross-section, the only stress produced is a tension or compression σ_z acting normally to the cross-section and equal to $\sigma_z = -E\epsilon = -Ek\theta$, where E is the Young modulus of the cylinder, k its coefficient of thermal expansion, and θ the

[4] It is expected that the complete analysis of this case by an independent method will be published later in the *Philosophical Magazine:* "A General Property of Two-Dimensional Thermal Stress Distribution," by M. A. Biot. See also, "Propriété generale des tensions thermiques en régime stationnaire dans les corps cylindriques," by M. A. Biot, Annales de la Sociète Scientifique de Bruxelles, vol. 54, series B, 1934.

temperature. We may say that the cylinder expands freely in the plane of its cross-section. This value $\sigma_z = -E\epsilon$ of the longitudinal stress is due to the assumption that longitudinal extension is prevented.

(2) If the cylinder is hollow, we make a longitudinal slit so as to connect the interior with the outside. When heated the two edges will in general separate. The cylinder becomes then a simply connected body without holes and we fall back on the previous case. In this slitted cylinder the temperature distribution will only produce axial stresses σ_z. But if we stick the sides of the slit together again, dislocation stresses σ'_x, σ'_y, τ'_{xy} will appear. Hence in a hollow cylinder stationary temperature

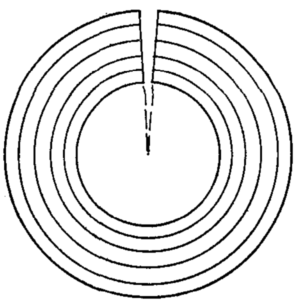

Fig. 8

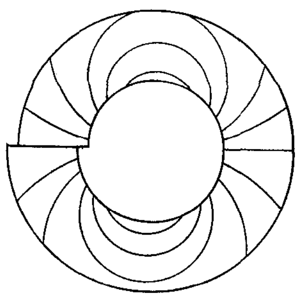

Fig. 9

distribution will generally give rise both to an axial stress σ_z and a dislocation stress system σ'_x, σ'_y, τ'_{xy}. They are the thermal stresses. In order that there should be no relative motion of the edges of the slitted cylinders, hence no stress except σ_z; the temperature distribution must satisfy three conditions, one of them being that the total flow of heat out of the hole is zero. These conditions are automatically satisfied if the temperature distribution has no singularity inside the hole (no source or sink or source-sink doublet).

Photoelastic Thermal-Stress Analysis

Thermal stresses may be measured by photoelasticity. A transient thermal-stress condition is equivalent to a boundary load and a dislocation. As previously shown, a steady-state thermal-stress condition is equivalent to a dislocation only and appears only in a hollow body. We shall have to calculate the amount of dislocation or relative motion of the two edges of a slit connecting the inside with the outside, make a model having that gap, and stick the two edges of this gap together.

Hollow Cylinder With No Heat Source Inside

In steady-state temperature distribution stresses other than σ_z may exist if the cylinder is heated from the outside. Those stresses disappear, however, if we fill the cylinder with a liquid having the same thermal conductivity as the cylinder because then all the singularities of the temperature distribution inside the hole disappear.

Circular Cylinder

For a radial heat flow we get concentric circular isothermal lines. A radial slit opens like that indicated in Fig. 8; no stress appears in the slitted cylinder. The thermal stresses in the non-slitted cylinder are those produced when bringing together the two edges of the slit. This amounts to the bending of a ring. For a doublet heat singularity (source and sink at the center) the isothermal lines are circles (Fig. 9). A radial slit as in the figure does not open up; the edges slide along each other.

Any heat-source singularity of an order higher than a doublet does not produce any heat stress in the cylinder.