HARVARD UNIVERSITY

BENDING OF AN INFINITE BEAM ON AN ELASTIC FOUNDATION

BY

M. A. BIOT

Reprinted from

Journal of Applied Mechanics

March, 1937

PUBLICATIONS FROM THE

GRADUATE SCHOOL OF ENGINEERING

1936–37 No. 203

(*Reprinted from* JOURNAL OF APPLIED MECHANICS *for March, 1937*)

Bending of an Infinite Beam on an Elastic Foundation

BY M. A. BIOT,[1] CAMBRIDGE, MASS.

The elementary theory of the bending of a beam on an elastic foundation is based on the assumption that the beam is resting on a continuously distributed set of springs,[2] the stiffness of which is defined by a "modulus of the foundation" k. Very seldom, however, does it happen that the foundation is actually constituted this way.

Generally, the foundation is an elastic continuum characterized by two elastic constants, a modulus of elasticity E, and a Poisson ratio ν. The problem of the bending of a beam resting on such a foundation has been approached already by various authors.[3]

The author attempts to give in this paper a more exact solution of one aspect of this problem, i.e., the case of an infinite beam under a concentrated load. A notable difference exists between the results obtained from the assumptions of a two-dimensional foundation and of a three-dimensional foundation.

Bending-moment and deflection curves for the two-dimensional case are shown in Figs. 4 and 5. A value of the modulus k is given for both cases by which the elementary theory can be used and leads to results which are fairly acceptable. These values depend on the stiffness of the beam and on the elasticity of the foundation.[4]

APPROXIMATE THEORY

IN THE approximate theory it is assumed that the effect of the foundation is the same as that of a great number of small springs and therefore that the reaction of the foundation is proportional to the local deflection. A deflection w of the beam gives rise to a reaction of the foundation upon the beam of value q per unit length

$$q = kw \qquad [1]$$

The coefficient k which has the dimension of a modulus of elasticity is called the "modulus of the foundation."

[1] Instructor in Applied Mechanics, Graduate School of Engineering, Harvard University.

[2] Die Lehre von der Elastizität und Festigkeit, by E. Winkler, Prag, 1867, p. 182.

"Die Berechnung des Eisenbahn Oberbaues," by H. Zimmermann, Berlin, 1888.

"Strength of Materials," by S. Timoshenko, D. Van Nostrand Company, New York, N. Y., 1934, vol. 2, pp. 401–407.

[3] Über den Balken auf Nachgiebiger Unterlage," by K. Wieghardt, *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 2, no. 3, June, 1922, pp. 165–184.

"Zur Theorie elastische gelagerter Konstruktionen," by W. Prager, *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 7, no. 5, October, 1927, pp. 354–360.

[4] For a short abstract of this paper see "A Fourier-integral solution of the problem of the bending under a concentrated load of an infinitely long beam resting on an elastic continuum," by M. A. Biot, Proceedings Fourth International Congress Applied Mechanics, 1934

Discussion of this paper should be addressed to the Secretary, A.S.M.E., 29 West 39th Street, New York, N. Y., and will be accepted until May 10, 1937, for publication at a later date. Discussion received after this date will be returned.

The differential equation of the deflection of a beam of stiffness E_bI resting on such a foundation and under the action of a distributed load $p(x)$ is

$$E_bI\frac{d^4w}{dx^4} + kw = p(x) \qquad [2]$$

It can be proved that if a concentrated load P acts upon an infinitely long beam resting on such a foundation, the bending moment M produced at a distance x from the load P, when $x \geq 0$, is

$$M = 0.353Ple^{-\frac{x}{l\sqrt{2}}}\left(\cos\frac{x}{l\sqrt{2}} - \sin\frac{x}{l\sqrt{2}}\right) \qquad [3]$$

where $l = \left(\frac{E_bI}{k}\right)^{1/4}$ is a "fundamental length"

E_b = Young's modulus of the beam

I = moment of inertia of the cross section of the beam

k = modulus of the foundation as given by Equation [1]

We may also write

$$M = Pl\varphi\left(\frac{x}{l}\right) \qquad [4]$$

with

$$\varphi(\zeta) = 0.353e^{-\frac{\zeta}{\sqrt{2}}}\left(\cos\frac{\zeta}{\sqrt{2}} - \sin\frac{\zeta}{\sqrt{2}}\right) \qquad [5]$$

The maximum value of the bending moment occurs right under the load, that is, when $x = 0$. Therefore

$$M_{\max} = 0.353Pl \qquad [6]$$

or

$$M_{\max} = 0.353P\left(\frac{E_bI}{k}\right)^{1/4} = 0.353Pb\left(\frac{E_bI}{kb^4}\right)^{1/4} \qquad [7]$$

when $x \geq 0$, the deflection curve is given by

$$w = 0.645\frac{Pl^3}{E_bI}e^{-\frac{x}{l\sqrt{2}}}\left(\cos\frac{x}{l\sqrt{2}} + \sin\frac{x}{l\sqrt{2}}\right) \qquad [8]$$

We may also write

$$w = \frac{Pl^3}{E_bI}\psi\left(\frac{x}{l}\right) \qquad [9]$$

with

$$\psi(\zeta) = 0.645e^{-\frac{\zeta}{\sqrt{2}}}\left(\cos\frac{\zeta}{\sqrt{2}} + \sin\frac{\zeta}{\sqrt{2}}\right) \qquad [10]$$

A serious objection can be made to the simplifying assumptions on which this elementary theory is based, because it is obvious that the reaction q of the foundation on the beam does not depend upon the local deflection w alone but is also a function of all the other deflections of the foundation surface occurring at that moment.

The elementary theory assumes the possibility of negative pressures between the foundation and the beam. These negative pressures are generally small and the same assumption will be made in the following paragraphs. Moreover, it may be added that in practical cases the dead weight of the beam introduces a uniform positive pressure which may be of sufficient magnitude to prevent the actual occurrence of negative pressures on the foundation.

Bending of an Infinite Beam on a Two-Dimensional Foundation

Assume that the beam is resting on top of a wall infinitely high and long. The width of the wall has the value $2b$ and the beam is supposed to be in contact with the wall along the full width as shown in Fig. 1. This wall may be considered as a two-dimensional foundation. A concentrated load P acts on the beam.

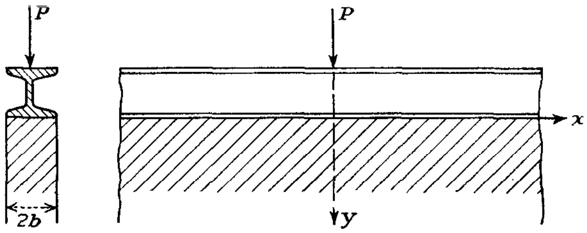

Fig. 1

Sine-Wave Loading. We shall first disregard the presence of the beam and study the effect of a sinusoidal load of value Q per unit length acting directly on top of the wall as shown in Fig. 2c. Under this condition the value of the load is

$$Q = Q_0 \cos \lambda x$$

To find the corresponding deflection w of the top of the wall is a two-dimensional elasticity problem. Taking the Y-axis directed downward and the X-axis along the ridge of the wall, the stress components σ_x, σ_y, and τ in the foundation, due to the load Q, are given by the stress function F satisfying the equation

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0$$

we have

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2 F}{\partial y^2} \\ \sigma_y &= \frac{\partial^2 F}{\partial x^2} \\ \tau &= -\frac{\partial^2 F}{\partial x \partial y} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots [11]$$

The boundary conditions are

$$\sigma_x = \sigma_y = \tau = 0 \quad \text{for } y = \infty$$

$$\sigma_y = -\frac{Q_0}{2b}\cos \lambda x, \quad \tau = 0 \text{ for } y = 0$$

The corresponding stress function is

$$F = \frac{Q_0}{2b\lambda^2}\cos \lambda x e^{-\lambda y}(1 + \lambda y) \ldots\ldots\ldots\ldots [12]$$

The coordinates x, y of a point before deformation, become $x + u$ and $y + v$ after deformation and the functions u, v are related to the stresses by the law of elasticity

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} \\ \frac{\partial v}{\partial y} &= \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots [13]$$

where E is the modulus of elasticity and ν the Poisson ratio of the wall.

By integrating the second Equation [13] and using Equations [11] and [12], the vertical deflection w of the ridge of the wall can be found as follows

$$w = -\int_0^\infty \frac{\partial v}{\partial y}\,dy$$

$$w = -\frac{1}{E}\int_0^\infty \frac{\partial^2 F}{\partial x^2}\,dy + \frac{\nu}{E}\left[\frac{\partial F}{\partial y}\right]_0^\infty$$

$$w = \frac{Q_0}{Eb\lambda}\cos \lambda x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [14]$$

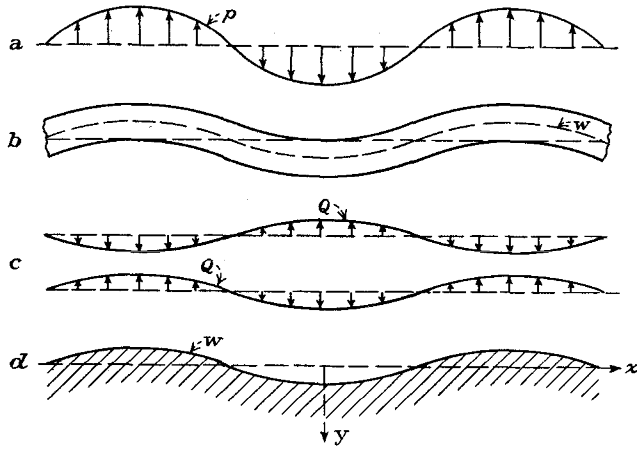

Fig. 2

We conclude that a sine-wave loading

$$Q = Q_0 \cos \lambda x$$

per unit length at the ridge produces a sine-wave deflection given by Equation [14]. For this type of loading, porportionality between load and deflection of the foundation actually holds, and we may write

$$\left.\begin{aligned} Q &= Eb\lambda w \\ Q &= kw \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots [15]$$

where $k = Eb\lambda$. The proportionality factor k may be considered as the modulus of the foundation *for a sine-wave loading.* We see that for a fixed value of the maximum load Q_0 its magnitude is inversely proportional to the wave length of the load. This is obviously in gross contradiction to the hypothesis of the elementary theory which assumes that k is independent of the wave length.

Now consider a beam under the action of two sine-wave loadings as shown in Fig. 2. In Fig. 2a, the load acting on top of the beam is

$$p = p_0 \cos \lambda x$$

and in Fig. 2c the reaction acting upward is

$$Q = Q_0 \cos \lambda x$$

The deflection w shown in Fig. 2b of the beam, according to the beam theory, is given by the equation

$$E_bI\frac{d^4w}{dx^4} = p - Q \qquad [16]$$

where E_b is the modulus of elasticity of the beam and I its moment of inertia.

The reaction Q is now supposed to be due to a sine-wave deflection w of the foundation as shown in Fig. 2d, the same as that of the beam, so that this reaction and deflection are related by Equation [15].

From Equations [15] and [16] after eliminating the reaction Q, we conclude that the load $p = p_0 \cos \lambda x$ on the beam resting on the wall produces a deflection of the beam $w = w_0 \cos \lambda x$ as shown in Fig. 3. This deflection is given by the relation

$$w = \frac{p_0 \cos \lambda x}{E_b\lambda\left[1 + \frac{E_bI}{Eb}\lambda^3\right]} \qquad [17]$$

The bending moment in the beam due to the load $p = p_0 \cos \lambda x$ is

$$M = E_bI\frac{d^2w}{dx^2}$$

$$M = \frac{p_0\lambda}{\lambda^3 + \frac{Eb}{E_bI}}\cos \lambda x \qquad [18]$$

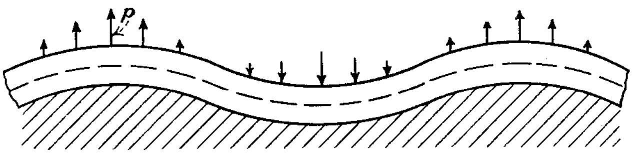

FIG. 3

Concentrated Load P and Bending Moments. By means of Equation [18] we may calculate the bending moment due to any loading using the superposition principle and the Fourier integral.

An arbitrary loading $p(x)$ may be represented as the superposition of an infinite number of sine loads by the equation

$$p(x) = \frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^{+\infty} p(\zeta)\cos\lambda(x - \zeta)d\zeta \qquad [19]$$

Each elementary sine loading of Equation [19]

$$(1/\pi)d\lambda d\zeta p(\zeta)\cos\lambda(x - \zeta)$$

gives a bending moment (see Equation [18])

$$dM(x) = \frac{1}{\pi}d\lambda d\zeta\frac{\lambda\, p(\zeta)}{\lambda^3 + \frac{Eb}{E_bI}}\cos\lambda(x - \zeta)$$

and the total bending moment due to the load $p(x)$ will be the superposition of all these elementary bending moments

$$M(x) = \int_{-\infty}^{+\infty} p(\zeta)d\zeta \int_0^\infty \frac{d\lambda}{\pi}\frac{\lambda}{\lambda^3 + \frac{Eb}{E_bI}}\cos\lambda(x - \zeta)$$

In particular, the bending moment due to a concentrated load

$$P = \int_{-\epsilon}^{+\epsilon} p(\zeta)d\zeta$$

acting at the origin $x = 0$ and localized on a small width 2ϵ is given by

$$M(x) = \frac{P}{\pi}\int_0^\infty \frac{\lambda\cos\lambda x}{\lambda^3 + \frac{Eb}{E_bI}}d\lambda$$

As in the elementary theory, we may define a "fundamental length" as

$$a = \left[\frac{E_bI}{Eb}\right]^{1/3} \qquad [20]$$

The bending moment can be expressed as

$$M(x) = Pa\frac{1}{\pi}\int_0^\infty \frac{\alpha\cos\left(\alpha\frac{x}{a}\right)}{\alpha^3 + 1}d\alpha \qquad [21]$$

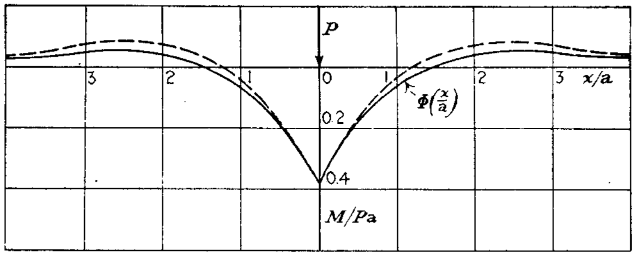

FIG. 4 BENDING-MOMENT CURVES

(The solid curve was drawn by the exact theory for two-dimensional foundations. The dashed curve was drawn by the elementary theory with a value of the modulus k adjusted so that the maximum bending moment has the correct value.)

The integral in Equation [21] has been evaluated partly graphically and partly by the method of residues for various values of (x/a).

The bending-moment curve calculated from Equation [21] can be denoted by

$$M(x) = Pa\Phi(x/a) \qquad [22]$$

The function $\Phi(x/a)$ is represented by the full line in Fig. 4. For very large values of (x/a) the function $\Phi(x/a)$ is asymptotic to $\frac{1}{(x/a)^2}$ and *does not oscillate.*

The maximum bending moment occurs at the point of loading $(x = 0)$. Its value is

$$M_{max} = Pa\frac{1}{\pi}\int_0^\infty \frac{\alpha}{\alpha^3 + 1}d\alpha \qquad [23]$$

This integral can be evaluated exactly and gives a check on the graphical method. We find

$$M_{max} = \frac{2}{3\sqrt{3}}Pa = 0.385\ Pa \qquad [24]$$

and replacing a by its explicit value given in Equation [20]

$$M_{max} = 0.385\ Pb\left[\frac{E_bI}{Eb}\right]^{1/3} \qquad [25]$$

Equations [24] and [25] may be compared with Equations [6] and [7] of the elementary theory.

Equation [25] differs fundamentally from Equation [7] of the elementary theory. The maximum bending moment is found to be actually proportional to the one-third power of the beam stiffness E_bI instead of the one-fourth power.

It is interesting to know what value of the foundation modulus k must be chosen in order to obtain the same maximum bending moment by the approximate theory as by the exact one. If for instance, Equations [6] and [24] are compared, it is seen that in order to obtain the same maximum bending moment by both equations, a value of the fundamental length l of the approximate theory must be chosen such that

$$0.353\ Pl = 0.385\ Pa \qquad [26]$$

or $l = 1.09a$ when a is the fundamental length of the exact theory. This also amounts to choosing a modulus k given by

$$k = 0.710\left[\frac{Eb^4}{E_bI}\right]^{1/3} . . E \qquad [27]$$

or

$$k = 0.710[b/a]E$$

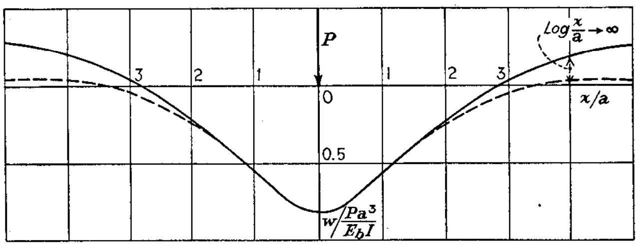

FIG. 5 DEFLECTION CURVES CORRESPONDING TO THE BENDING MOMENT CURVES OF FIG. 4

(In this figure the shapes of the curves are compared and not absolute values since the deflection deduced from the exact theory is everywhere infinite.)

The value of the modulus k turns out to be proportional to a dimensionless ratio

$$\frac{b}{a} = \left[\frac{Eb^4}{E_bI}\right]^{1/3}$$

which is the ratio of the half-width b of the wall to the fundamental length a.

It is quite natural to expect that the elementary theory is approximately verified by choosing l or k such that the maximum bending moment coincides with the correct value given by Equation [24]. This is justified by the fact that, with the proper value for k as given by Equation [15], the elementary theory is correct in case of a sine-wave deflection, and that the elementary theory yields a deflection curve which is roughly of sinusoidal shape. Moreover, it can be verified that if the maximum bending moments are made to coincide by proper choice of k or l, the bending-moment diagrams are practically the same. By using Equation [4] of the approximate theory, with a value $l = 1.09a$ given by Equation [26], we have

$$M = 1.09Pa\varphi(x/1.09a)$$

The curve of $M/Pa = 1.09\varphi(x/1.09a)$ as a function of x/a and compared with the exact one $\Phi(x/a)$ is represented by the dashed line in Fig. 4.

Deflection. The deflection curve is found by double integration of the bending-moment curve. The absolute value of the deflection is infinite everywhere, as can be shown. This is derived from the fact that the bending-moment curve goes to zero as the expression $(1/x^2)$ when x approaches infinity. However, we may find the shape of the deflection in the vicinity of the load by double integration of the relation

$$E_bI\frac{d^2w}{dx^2} = M(x)$$

or

$$w = \frac{Pa^3}{E_bI}\int_0^{\frac{x}{a}}\int_0^{\frac{x}{a}}\Phi(\zeta)d\zeta^2 \qquad [28]$$

Since $\Phi(\zeta)$ is asymptotic to $(1/\zeta^2)$ for large values of ζ, the ordinates of the deflection curve become infinite at infinite distance and are asymptotic to log (x/a). The shape of the deflection curve is represented by the full line in Fig. 5. The ordinates are dimensionless deflections $w\Big/\frac{Pa^3}{E_bI}$ as a function of x/a.

We may calculate here also an approximate deflection curve by using Equations [9] and [10] of the elementary theory but with a value of $l = 1.09a$, Equation [26], such that the value of the maximum bending moment will coincide with the exact one. This means that the approximate deflection curve and the exact one will have the same curvature under the load P. The dimensionless deflection $w\Big/\frac{Pa^3}{E_bI}$ thus given by the approximate theory is represented by the dashed line in Fig. 5.

BENDING OF AN INFINITE BEAM ON A THREE-DIMENSIONAL FOUNDATION

Consider the bending of an infinitely long beam under a concentrated load P. The beam is supposed to rest on a three-

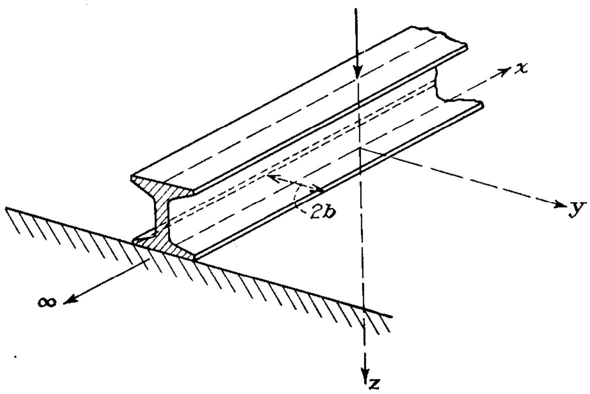

FIG. 6

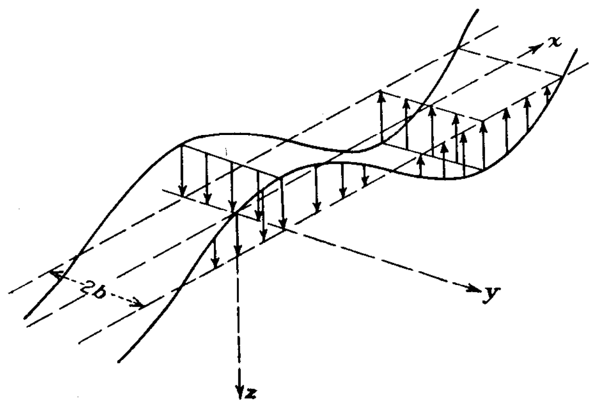

FIG. 7

dimensional semi-infinite elastic continuum. The area of contact between the beam and the surface of the foundation is a strip of width $2b$ as shown in Fig. 6.

The Z-axis is taken downward, as shown in Fig. 7, and is positive, and the X- and Y-planes coincide with the surface of the foundation.

Double Sine-Wave Loading on Foundation. Similarly to the case of a two-dimensional foundation, let us first study the deflection due to a double sine-wave loading, where

$$q = q_0 \cos \lambda x \cos \kappa y$$

The displacement components u, v, and w, of a point inside the foundation satisfy the equations of elasticity

$$\Delta^2 u + \frac{1}{1-2\nu}\frac{\partial e}{\partial x} = 0$$

$$\Delta^2 u + \frac{1}{1-2\nu}\frac{\partial e}{\partial y} = 0$$

$$\Delta^2 w + \frac{1}{1-2\nu}\frac{\partial e}{\partial z} = 0$$

where

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \quad \Delta^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and ν is the Poisson ratio of the foundation. A solution of these equations has to be found which (1) is doubly sinusoidal in x and y; (2) produces no shear at the surface $z = 0$; and (3) goes to zero and produces zero stresses at $z = \infty$.

It may be verified that a solution satisfying these conditions is

$$u = \frac{A}{\kappa}\left[z - \frac{1-2\nu}{\sqrt{(\lambda^2+\kappa^2)}}\right]e^{-z\sqrt{(\lambda^2+\kappa^2)}} \sin \lambda x \cos \kappa y$$

$$v = \frac{A}{\lambda}\left[z - \frac{1-2\nu}{\sqrt{(\lambda^2+\kappa^2)}}\right]e^{-z\sqrt{(\lambda^2+\kappa^2)}} \cos \lambda x \sin \kappa y$$

$$w = \frac{A}{\lambda\kappa}\left[z\sqrt{(\lambda^2+\kappa^2)} + 2(1-\nu)\right]e^{-z\sqrt{(\lambda^2+\kappa^2)}} \cos \lambda x \cos \kappa y$$

where A is an arbitrary constant.

The vertical deflection of the surface of the foundation w_0 (w at $z = 0$) is

$$w_0 = \frac{2A}{\lambda\kappa}(1-\nu)\cos \lambda x \cos \kappa y$$

and the corresponding normal load is

$$q = -\sigma_z = \frac{AE}{1+\nu}\frac{\sqrt{\lambda^2+\kappa^2}}{\lambda\kappa}\cos \lambda x \cos \kappa y$$

This yields, between q and w_0, the relation

$$q = \frac{1}{2}\frac{E}{1-\nu^2} w_0 \sqrt{(\lambda^2+\kappa^2)} \qquad [29]$$

where E is the modulus of elasticity of the foundation.

Simple Sine-Wave Loading. Referring to Fig. 7, let us now find the deflection produced by a loading located in a strip of width $2b$ between the lines $y = \pm b$. This load is supposed to be constant in the Y-direction and have a sine distribution along the X-direction. It can be represented as

$$q_1(xy) = q_0(y) \cos \lambda x \qquad [30]$$

where $q_0(y)$ is a function of y such that it is equal to zero when $y < -b$, $y > b$, and equal to q_0 when $-b < y < b$ as indicated in Fig. 8.

The deflection $W_1(x,y)$ of the foundation surface corresponding to the load $q_1(x, y)$ may be expressed in exactly the same way

$$W_1(x, y) = W_0(y) \cos \lambda x \qquad [31]$$

where $W_0(y)$ represents the deflection of the foundation along a cross section parallel to the Y-axis.

The problem is to derive the value of $W_0(y)$ from the knowledge of $q_0(y)$. This can be done by the use of the Fourier integral and Equation [29], as shown in the Appendix. In fact, what is finally calculated is the ratio of average loads and deflections. The average load is taken as

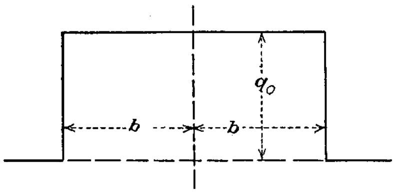

FIG. 8

$$Q_{avg} = \frac{1}{2b}\int_{-b}^{+b} q_1(y)\, dy$$

and the average deflection

$$W_{avg} = \frac{1}{2b}\int_{-b}^{+b} W_0(y)\, dy$$

It is shown that the ratio (Q_{avg}/W_{avg}) varies only about 10 per cent when the distribution of pressure $q_0(y)$ changes from a uniform one to one giving constant W_0 along the width $2b$.

This ratio can be expressed as

$$\frac{Q_{avg}}{W_{avg}} = \frac{E}{C(1-\nu^2)}\beta\Psi(\beta) \qquad [32]$$

where, as shown in the Appendix, Ψ is a numerically calculated function of $\beta = b\lambda$, and C is a coefficient varying from 1 for uniform pressure distribution to 1.13 for uniform deflection W_0.

Sine-Wave Loading on the Beam. The preceding result may be applied to the bending of a beam in a manner similar to that for the two-dimensional foundation.

Assume that a load

$$p = p_0 \cos \lambda x$$

acts on top of a beam and that on the bottom a reaction of average value is acting across the width, or

$$Q = Q_{avg} \cos \lambda x$$

The corresponding sine-wave deflection W of the beam, according to the beam theory, is given by the equation

$$E_b I \frac{d^4 W}{dx^4} = p - Q \qquad [33]$$

where E_b is the modulus of elasticity of the beam and I is the moment of inertia of the beam.

On the other hand, if the force ($Q = Q_{avg} \cos \lambda x$) acting under the beam is due to a sine-wave deflection of the foundation of average value across the width

$$W = W_{avg} \cos \lambda x$$

from Equation [32] we may write

$$Q = W\frac{E}{C(1-\nu^2)}\beta\Psi(\beta) \qquad [34]$$

Assuming that the deflection W in Equation [33] is the same as in Equation [34], we may eliminate Q between the two relations. This leads to the conclusion that a load ($p = p_0 \cos \lambda x$) on the beam resting on the foundation produces a deflection

$$W = \frac{p_0 \cos \lambda x}{E_b I \lambda^4 + \dfrac{E}{C(1-\nu^2)}\beta\Psi(\beta)}$$

The corresponding bending moment is

$$M = E_b I \frac{d^2W}{dx^2}$$

$$M = \frac{\lambda p_0 \cos \lambda x}{\lambda^3 + \frac{1}{C(1-\nu^2)} \frac{E}{E_b} \frac{b}{I} \Psi(\beta)}$$

Concentrated Load P and Bending Moments. By using the last given Equation and the Fourier integral given in Equation [19] we may derive, as in the case of the two-dimensional foundation, the bending-moment curve due to a concentrated load P.

This bending moment is

$$M(x) = \frac{P}{\pi} \int_0^\infty \frac{\lambda \cos \lambda x \, d\lambda}{\lambda^3 + \frac{1}{C(1-\nu^2)} \frac{E}{E_b} \frac{b}{I} \Psi(\beta)}$$

As before, we may introduce here a fundamental length

$$c = \left[C(1-\nu^2) \frac{E_b I}{Eb} \right]^{1/3} \quad \text{[35]}$$

and then

$$M(x) = Pc \frac{1}{\pi} \int_0^\infty \frac{\alpha \cos \left(\alpha \frac{x}{c}\right)}{\alpha^3 + \Psi\left(\frac{b}{c}\alpha\right)} d\alpha$$

The maximum bending moment occurs right under the load ($x = 0$). Its value is

$$M_{max} = Pc \frac{1}{\pi} \int_0^\infty \frac{\alpha \, d\alpha}{\alpha^3 + \Psi\left(\frac{b}{c}\alpha\right)} d\alpha$$

This integral has been evaluated partly graphically and partly analytically for six values of b/c, ranging from 0.01 to 1. If the results are plotted on logarithmic paper, it is found that the integral can be represented with an approximation of the order of 2 to 3 per cent, as

$$\frac{1}{\pi} \int_0^\infty \frac{\alpha \, d\alpha}{\alpha^3 + \psi\left[\frac{b}{c}\alpha\right]} = 0.332 \left(\frac{c}{b}\right)^{0.831}$$

so that finally the maximum bending moment due to a concentrated load P may be expressed by

$$M_{max} = 0.332 \, Pc \left(\frac{c}{b}\right)^{0.831} \quad \text{[36]}$$

or, replacing c by its explicit value as given by Equation [35]

$$M_{max} = 0.332 Pb \left[C(1-\nu^2) \frac{E_b I}{Eb^4} \right]^{0.277} \quad \text{[37]}$$

Comparing Equation [36] with Equation [25] obtained in the case of the two-dimensional foundation, we see that they differ quite fundamentally by the presence of the factor $(c/b)^{0.831}$. A great similarity, however, exists between Equation [37] and Equation [7] of the elementary theory. In Equation [37] the maximum bending moment is found to be proportional to the 0.277 power of the beam stiffness $E_b I$, while in the approximate theory it is proportional to the 0.25 power of the same quantity.

Hence, the elementary theory approximates more closely the exact theory for the three-dimensional foundation than it does for the two-dimensional case. We may carry over from the two-dimensional theory the conclusion that if we were to use a value of k or l giving a correct value for the maximum bending moment, good approximate values are to be found by applying the approximate theory.

By identifying Equations [6] and [36] we find

$$l = 0.94c(c/b)^{0.831} \quad \text{[38]}$$

and from Equations [7] and [37]

$$k = 1.23 \left[\frac{1}{C(1-\nu^2)} \frac{Eb^4}{E_b I} \right]^{0.11} \frac{E}{C(1-\nu^2)} \quad \text{[39]}$$

or

$$k = 1.23 \left[\frac{b}{c} \right]^{0.33} \frac{E}{C(1-\nu^2)}$$

Equations [38] and [39] for the fundamental length l and the modulus k of a three-dimensional foundation show also fundamental differences from Equations [27] and [28] of the two-dimensional theory.

Deflections. In this case the deflection is found to be finite. The exact deflection can be derived from the bending moment and expressed as

$$w(x) = \frac{Pc^3}{E_b I} \frac{1}{\pi} \int_0^\infty \frac{\cos \left(\alpha \frac{x}{c}\right) d\alpha}{\alpha^4 + \alpha^4 \left(\frac{b}{c}\alpha\right)}$$

From Equation [43] of the Appendix, it can be seen that as α approaches 0, the integrand is asymptotic to a logarithm, and

$$\frac{2c}{\pi b} \log \frac{c}{b\alpha}$$

This shows that the integral is finite in spite of the infinite value of the integrand for $\alpha = 0$.

It is natural to assume that the deflection wave computed by Equation [8] of the elementary theory with a value of k or l given by Equation [38] or Equation [39] is a good approximation to the actual value.

Appendix

The function $q_0(y)$ is a discontinuous function which can be represented as a sum of sine functions by means of

$$q_0(y) = \frac{q_0}{\pi} \int_0^\infty \frac{d\kappa}{\kappa} [\sin \kappa(y+b) - \sin \kappa(y-b)]$$

Applying Equation [15] to each of the sine waves under the integral sign, we find

$$W_0(y) = \frac{2q_0}{\pi} \frac{1-\nu^2}{E} \left[\int_0^\infty \frac{d\kappa}{\kappa} \frac{\sin \kappa(y+b)}{\sqrt{(\lambda^2+\kappa^2)}} - \int_0^\infty \frac{d\kappa}{\kappa} \frac{\sin \kappa(y-b)}{\sqrt{(\lambda^2+\kappa^2)}} \right]$$

Putting $2q_0 b = Q_0$, $\lambda b = \beta$, and $\kappa b = \alpha$, the last given equation becomes

$$W_0(y) = \frac{Q_0}{\pi} \frac{1-\nu^2}{E} \left[\int_0^\infty \frac{d\alpha}{\alpha} \frac{\sin\left(\frac{y}{b}+1\right)\alpha}{\sqrt{(\alpha^2+\beta^2)}} - \int_0^\infty \frac{d\alpha}{\alpha} \frac{\sin\left(\frac{y}{b}-1\right)\alpha}{\sqrt{(\alpha^2+\beta^2)}} \right] \quad \text{[40]}$$

We have to evaluate the integral

$$\int_0^\infty \frac{d\alpha}{\alpha} \frac{\sin \gamma\alpha}{\sqrt{(\alpha^2 + \beta^2)}} = I \ldots\ldots\ldots\ldots [41]$$

We note that its derivative with respect to γ is a known function

$$\frac{dI}{d\gamma} = \int_0^\infty \frac{\cos \gamma\alpha}{\sqrt{(\alpha^2 + \beta^2)}} d\alpha = K_0(\gamma\beta)$$

where $K_0(u)$ is the zero-order Bessel function of the third kind (Hankel function)[5,6] sometimes also denoted by $(\pi/2)iH_0^{(1)}(ix)$.

By integration with respect to γ

$$I = \frac{1}{\beta}\int_0^{\gamma\beta} K_0(u)du$$

We put

$$\int_0^\zeta K_0(u)du = \varphi(\zeta) \ldots\ldots\ldots\ldots [42]$$

This integral has been calculated graphically and analytically in the vicinity of point $\zeta = 0$ by using the asymptotic formula

$$K_0(u)_{u\to 0} = 0.1159 - \log u$$

We find the following values

ζ	$\varphi(\zeta)$
0	0
0.1	0.3417
0.2	0.5467
0.5	0.9237
1	1.237
2	1.468
3	1.528
4	1.544
5	1.550
∞	$\pi/2$

For values of ζ smaller than 0.1, the function $\varphi(\zeta)$ is approximately equal to

$$\varphi(\zeta) = \zeta[1.116 - \log \zeta]$$

We have introduced a constant Q_0; it is such that the load Q_1 per unit length along the x direction is

$$Q_1 = Q_0 \cos \lambda x$$

This load is supposed to be uniformly distributed along the Y-direction in the width $2b$.

According to the derivation of Equations [40], [41], and [42] this load produces a deflection $W_1(x,y) = W_0(y) \cos \lambda x$, such that in the Y-direction

$$W_0(y) = \frac{Q_0}{\pi}\frac{1-\nu^2}{E}\frac{1}{\beta}\left\{\varphi\left[\left(\frac{y}{b}+1\right)\beta\right] - \varphi\left[\left(\frac{y}{b}-1\right)\beta\right]\right\} \ldots\ldots [43]$$

We may deduce from this the average deflection W_{avg} along the width $2b$ where

$$W_{avg} = \frac{1}{2b}\int_{-b}^{+b} W_0(y)dy$$

This can be evaluated graphically and it is found that

$$\frac{Q_0}{W_{avg}} = \frac{E}{1-\nu^2}\Psi(\beta)$$

where $\Psi(\beta)$ is the function here tabulated.

β	$\Psi(\beta)$
0.1	4.80
0.5	1.90
1	1.42
3	1.13
8	1.04
∞	1

For $\beta < 0.1$ the function $\Psi(\beta)$ is given by the asymptotic expression

$$\Psi(\beta) = \frac{2}{\pi\beta}\left[\log\frac{1}{\beta} + 0.923\right]^{-1} \ldots\ldots\ldots [44]$$

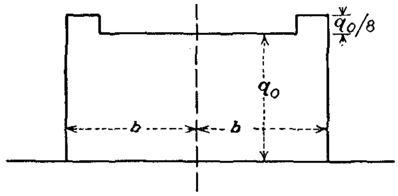

FIG. 9

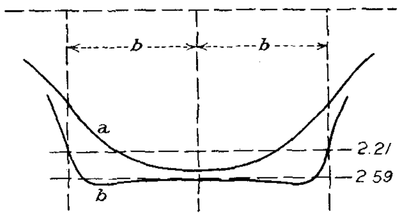

FIG. 10

The effect of changing the distribution of the loading in the Y-direction $q_0(y)$ has also been investigated in case $\beta = 1$. Calling Q_{avg} the average loading in the Y-direction, we have

$$Q_{avg} = \frac{1}{2b}\int_{-b}^{+b} q_0(y)dy$$

If the loading is constant, $q_0 = (Q_{avg}/2b)$ in the Y-direction between $y = -b$ and $y = +b$; as in Fig. 8, the corresponding deflection is given by curve a, Fig. 10.

Let us apply a loading made of the superposition of the previous rectangular loading and two rectangular loadings at the edge of width $(b/4)$ and intensity $(q_0/8)$ as shown in Fig. 9. We get a deflection shown by curve b in Fig. 10. This deflection is found simply by applying Equation [43] to each rectangular loading and superposing the deflections. We have increased the average loading by the relative amount $^1/_{32}$ or 3.1 per cent and the average deflection by 17 per cent. The shape of curve B in Fig. 10 shows nearly constant deflection. We see that between the case where q_0 is a constant and the case where the deflection $W_0(y)$ is a constant, the ratio (Q_{avg}/W_{avg}) can become $(1.17/1.03) = 1.13$ times as great, showing a relative variation of 13 per cent.

This shows that the ratio (Q_{avg}/W_{avg}) can differ from (Q_0/W_{avg}) by as much as 13 per cent when $\beta = 1$. To calculate a better approximation for the average deflection when the load distribution $q(y)$ is not rectangular, is very complicated and beyond practical interest. We shall write

$$\frac{Q_{avg}}{W_{avg}} = \frac{Q_0}{CW_{avg}} = \frac{E}{C(1-\nu^2)}\beta\Psi(\beta) \ldots\ldots\ldots [45]$$

where C is a coefficient having values between 1 and 1.13. Rigorously, C is a function of β. The interval of variation of 13 per cent holds only in case $\beta = 1$. The margin of variation of C is generally much smaller and goes to zero for $\beta = 0$ or $\beta = \infty$.

[5] "Treatise on the Theory Bessel Functions," by G. N. Watson, Cambridge University Press, London, 1924, p. 77.

[6] See Watson, Bessel Functions, p. 77. Also "Functionentafeln mit formeln und kurven," by E. Jahnke and E. Emde, Teubner, Leipzig, 1933, p. 286.