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THEORY OF ELASTICITY WITH LARGE DISPLACEMENTS AND ROTATIONS

INTRODUCTION. In the case of two-dimensional strain the original coordinates x, y of a point attached to the elastic body become $\xi = x + u$, $\eta = y + v$ after deformation.

In the classical theory of Elasticity the "strain components"

$$
\begin{aligned}
e_{xx} &= \frac{\partial u}{\partial x} \\
e_{yy} &= \frac{\partial v}{\partial y} \\
e_{xy} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)
\end{aligned}
\tag{1}
$$

and the "rotation"

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

are considered to be all small quantities of the first order so that their squares and products are negligible terms of the second order.

The restriction of the classical theory to small strains justifies in most cases the use of a linear stress strain relation known as HOOKE's law and leads therefore to equations of great practical value by their simplicity and generality. However, there is no necessity to assume that also the rotation ω is small since it may become large with respect to the strain while the latter is still a small quantity.

The question thus naturally arises: What terms must we add to the classical equations of Elasticity when the possibility of large rotations and small strains is taken into account? The present paper is an attempt to answer this question.

We shall be led to equations applicable to such phenomena as the bending of a stiff string with fixed ends or the bending of a thin clamped plate for which the non-linear effects are not negligible and the classical theory of Elasticity breaks down.

We note also that the rotation and the strain are not independent. They are related by the identities

$$
\begin{aligned}
\frac{\partial e_{xx}}{\partial y} &= \frac{\partial}{\partial x}(e_{xy} - \omega) \\
\frac{\partial e_{yy}}{\partial x} &= \frac{\partial}{\partial y}(e_{xy} + \omega)
\end{aligned}
\tag{2}
$$

Therefore the assumption that the rotation is large with respect to the strain implies that the dimension of the material is small in a direction perpendicular to the gradient of the rotation.

As a matter of precaution we have established equations in which appear all the second order terms of geometrical origin whether they contain the rotation or not. We are therefore in a position to deduce as a direct consequence the linear equations of Elasticity for a material under initial stress and solve at the same time the problem of Elastic stability which has already been the object of many investigations by R. V. SOUTHWELL[1], C. B. BIEZENO, H. HENCKY[2], TREFFTZ[3] etc. The form of our stability equations are new; they show the separate effects of shear and stress gradient.

In order to simplify the writing the theory shall be developed for the case of two-dimensional strain. The three-dimensional theory does not involve any new methods or ideas and the equations for this case shall be stated with a short comment at the end of this paper.

STRESS AND STRAIN. A first step is to define clearly what is meant by stress, strain and rotation. We consider an homogeneous deformation such that a square S drawn on the material is transformed into a rectangle R, while the sides keep fixed orientations I, II

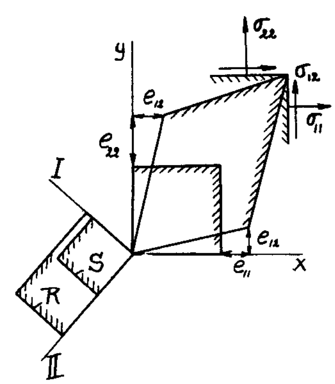

Fig.1a Geometrical interpretation of the coefficients of the homogeneous transformation (3) with symmetrical coefficients. A certain square S is transformed into a rectangle R without rotation of the sides.

(Fig. 1a). Such a deformation is represented by the linear transformation with symmetric coefficients

$$
\begin{aligned}
\xi &= (1 + e_{11})x + e_{12}y \\
\eta &= e_{12}x + (1 + e_{22})y
\end{aligned}
\tag{3}
$$

The coefficients e_{11}, e_{22}, e_{12} define a pure deformation; they are the strain components. The stress components referred to same directions are $\sigma_{11}, \sigma_{22}, \sigma_{12}$, they depend only on the above strain components. For instance if we assume HOOKE's law we have

$$\begin{aligned}\sigma_{11} &= \lambda(e_{11} + e_{22}) + 2Ge_{11}\\ \sigma_{22} &= \lambda(e_{11} + e_{22}) + 2Ge_{22}\\ \sigma_{12} &= 2Ge_{12}\end{aligned} \tag{4}$$

If, after the preceding deformation, we rotate the material through an angle θ (Fig.1b)

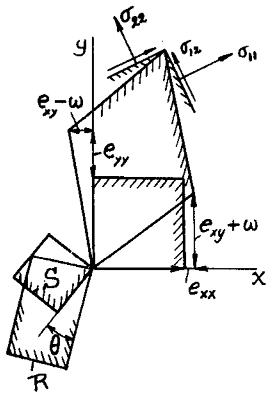

Fig.1b Geometrical interpretation of the coefficients of the homogeneous transformation (5) with non symmetric coefficients. A certain square S is transformed into a rectangle R with a rotation θ of the sides.

the result of these two operations is represented by the relations

$$\begin{aligned}\xi &= (1 + e_{xx})x + (e_{xy} - \omega)y\\ \eta &= (e_{xy} + \omega)x + (1 + e_{yy})y\end{aligned} \tag{5}$$

This is the most general linear transformation. The problem that we have to solve generally in the theory of Elasticity is to find the angle of rotation θ and the coefficients e_{11} e_{22} e_{12} of the pure deformation in terms of the coefficients $e_{xx}, e_{yy}, e_{xy}, \omega$.

We find for the rotation θ,

$$\tan\theta = \frac{\omega}{1 + \frac{1}{2}(e_{xx} + e_{yy})} \tag{6}$$

In first approximation $\theta \cong \omega$

The approximate expressions for the strain components, including the first and second order terms, are

$$\begin{aligned}e_{11} &= e_{xx} + e_{xy}\omega + \tfrac{1}{2}\omega^2\\ e_{22} &= e_{yy} - e_{xy}\omega + \tfrac{1}{2}\omega^2\\ e_{12} &= e_{xy} + \tfrac{1}{2}(e_{yy} - e_{xx})\omega\end{aligned} \tag{7}$$

Identical expressions for the strain will be found if we follow the classical method of considering the length element after deformation

$$ds^2 = g_{11}dx^2 + 2g_{12}dx\,dy + g_{22}dy^2$$

$$\begin{aligned}g_{11} &= 1 + 2\left[e_{xx} + \tfrac{1}{2}e_{xx}^2 + \tfrac{1}{2}(e_{xy} + \omega)^2\right]\\ g_{22} &= 1 + 2\left[e_{yy} + \tfrac{1}{2}e_{yy}^2 + \tfrac{1}{2}(e_{xy} - \omega)^2\right]\\ g_{12} &= e_{xy} + \tfrac{1}{2}e_{xx}(e_{xy} - \omega) + \tfrac{1}{2}e_{yy}(e_{xy} + \omega)\end{aligned} \tag{8}$$

It can be seen that if we neglect the squares and products of e_{xx}, e_{yy} and e_{xy} we find for the strain the same expressions (7).

If we assume the strain to be small but the rotation to be large with respect to the strain, we may neglect the terms $e_{xy}\omega$ and $1/2(e_{yy} - e_{xx})\omega$ with respect to ω^2 and write

$$\begin{aligned}e_{11} &= e_{xx} + \tfrac{1}{2}\omega^2\\ e_{22} &= e_{yy} + \tfrac{1}{2}\omega^2\\ e_{12} &= e_{xy}\end{aligned} \tag{9}$$

The above analysis is valid for a non-homogeneous deformation provided we use the values (1) for the coefficients.

We shall now consider the stress condition. In a non-homogeneous deformation, an infinitesimal region around a point x, y is transported to the point $\xi = x + u$ $\eta = y + v$. In this new position it has undergone a rotation ω and a pure homogeneous deformation. Since the stress condition ignores the rotation it is natural to refer the stress components σ_{11} σ_{22} σ_{12} at the point x + u y + v after deformation to rectangular directions 1, 2, derived from the directions x, y by a rotation ω (fig. 2). The stress components referred to

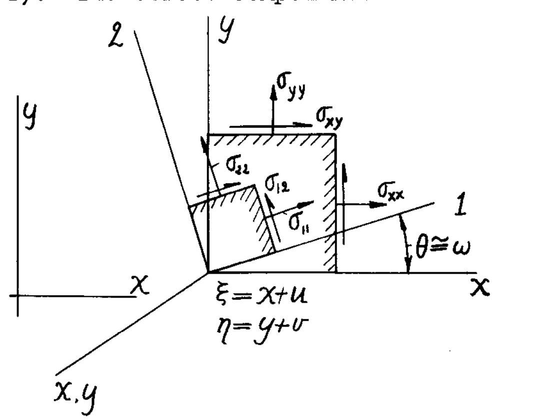

Fig.2 Stress conditions at point ξ, η after deformation with the components σ_{xx}, σ_{yy}, σ_{xy} referred to the original directions x,y and the components σ_{11}, σ_{22}, σ_{12} referred to the rotated directions 1,2.

the directions x, y are given with first and second order terms by

$$\begin{aligned}\sigma_{xx} &= \sigma_{11} - 2\sigma_{12}\omega\\ \sigma_{yy} &= \sigma_{22} + 2\sigma_{12}\omega\\ \sigma_{xy} &= \sigma_{12} + (\sigma_{11} - \sigma_{22})\omega\end{aligned} \tag{10}$$

This is deduced from the well-known transformation relations for the stress tensor when we change the orientation of the axis.

Our procedure amounts to referring the stress-strain condition at a certain point to a set of axes originally parallel to the directions x and y and rotated with the material at that point.

This is associated with the fact that we have performed first the pure deformation (3) and then the rotation. Truly enough we could have performed first the rotation and then the pure deformation and calculated the symmetric coefficients of the latter; they would be different from (7). However they would represent the same pure deformation as (3) but referred this time to the original axes instead of the rotated ones.

We prefer the first method because it leads to the equilibrium equations (19) (instead of 18) in which the second order terms have an intuitive physical meaning.

THE EQUILIBRIUM CONDITIONS. A second step in this analysis will be the development of the equilibrium equations in terms of the stress components σ_{11} σ_{22} σ_{12} and the original coordinates x, y. As a matter of precaution we shall take into account all the terms of the second order whether they contain the rotation or not.

After deformation the material is in equilibrium and therefore the stress components σ_{xx} σ_{yy} σ_{xy} must satisfy the well-known conditions

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial \xi} + \frac{\partial \sigma_{xy}}{\partial \eta} + \mu(\xi,\eta) X(\xi,\eta) &= 0 \\ \frac{\partial \sigma_{xy}}{\partial \xi} + \frac{\partial \sigma_{yy}}{\partial \eta} + \mu(\xi,\eta) Y(\xi,\eta) &= 0 \end{aligned} \tag{11}$$

where $X(\xi,\eta)$ and $Y(\xi,\eta)$ are the components of the body force per unit mass at point ξ, η and $\mu(\xi,\eta)$ the specific mass of the material after deformation.

Now we wish to express these conditions by means of the initial coordinates x, y instead of the coordinates ξ, η after deformation. This is a purely mathematical transformation which is carried through as follows. We have for instance

$$\frac{\partial \sigma_{xx}}{\partial \xi} = \frac{\partial \sigma_{xx}}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \sigma_{xx}}{\partial y}\frac{\partial y}{\partial \xi} \quad \text{etc.} \tag{12}$$

In order to find the partial derivatives $\partial x/\partial \xi$ $\partial y/\partial \xi$ etc. we write

$$\begin{aligned} d\xi &= \left(1+\frac{\partial u}{\partial x}\right)dx + \frac{\partial u}{\partial y}dy \\ d\eta &= \frac{\partial v}{\partial x}dx + \left(1+\frac{\partial v}{\partial y}\right)dy \end{aligned} \tag{13}$$

Solving these equations with respect to dx, dy in terms of $d\xi$, $d\eta$ we find

$$\begin{aligned} dx &= \frac{1}{D}\left(1+\frac{\partial v}{\partial y}\right)d\xi - \frac{1}{D}\frac{\partial u}{\partial y}d\eta \\ dy &= -\frac{1}{D}\frac{\partial v}{\partial x}d\xi + \frac{1}{D}\left(1+\frac{\partial u}{\partial x}\right)d\eta \end{aligned} \tag{14}$$

$$\text{where } D = \begin{vmatrix} 1+\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & 1+\frac{\partial v}{\partial y} \end{vmatrix} \tag{15}$$

is the Jacobian of the transformation of the variables x, y into ξ, η. We deduce

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \frac{1}{D}\left(1+\frac{\partial v}{\partial y}\right) & \frac{\partial x}{\partial \eta} &= -\frac{1}{D}\frac{\partial u}{\partial y} \\ \frac{\partial y}{\partial \xi} &= -\frac{1}{D}\frac{\partial v}{\partial x} & \frac{\partial y}{\partial \eta} &= \frac{1}{D}\left(1+\frac{\partial u}{\partial x}\right) \end{aligned} \tag{16}$$

We also note that the specific mass $\rho(x,y)$ at point x, y before the deformation is

$$\rho(x,y) = D\mu(\xi,\eta) \tag{17}$$

By means of relations (12) (16) and (17) we are now in a position to transform the equilibrium conditions in terms of the original coordinates x, y. We find

$$\begin{aligned} &\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + e_{yy}\frac{\partial \sigma_{xx}}{\partial x} + e_{xx}\frac{\partial \sigma_{xy}}{\partial y} \\ &\quad -(e_{xy}-\omega)\frac{\partial \sigma_{xy}}{\partial x} - (e_{xy}+\omega)\frac{\partial \sigma_{xx}}{\partial y} + \rho X(\xi,\eta) = 0 \\ &\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + e_{yy}\frac{\partial \sigma_{xy}}{\partial x} + e_{xx}\frac{\partial \sigma_{yy}}{\partial y} \\ &\quad -(e_{xy}-\omega)\frac{\partial \sigma_{yy}}{\partial x} - (e_{xy}+\omega)\frac{\partial \sigma_{xy}}{\partial y} + \rho Y(\xi,\eta) = 0 \end{aligned} \tag{18}$$

These equations contain no approximation. Let us now introduce the components of stress σ_{11} σ_{22} σ_{12} with respect to rotated axes. By substituting relations (10) in the above equations and keeping only the terms of the first and second order, we find

$$\begin{aligned} &\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} + \rho X(x,y) \\ &\quad + \rho u\frac{\partial X}{\partial x} + \rho v\frac{\partial X}{\partial y} + \omega\rho Y \\ &\quad - 2\sigma_{12}\frac{\partial \omega}{\partial x} + (\sigma_{11}-\sigma_{22})\frac{\partial \omega}{\partial y} \\ &\quad + e_{yy}\frac{\partial \sigma_{11}}{\partial x} + e_{xx}\frac{\partial \sigma_{12}}{\partial y} - e_{xy}\left(\frac{\partial \sigma_{11}}{\partial y} + \frac{\partial \sigma_{12}}{\partial x}\right) = 0 \\ &\frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} + \rho Y(x,y) \\ &\quad + \rho u\frac{\partial Y}{\partial x} + \rho v\frac{\partial Y}{\partial y} + \omega\rho X \\ &\quad + (\sigma_{11}-\sigma_{22})\frac{\partial \omega}{\partial x} + 2\sigma_{12}\frac{\partial \omega}{\partial y} \end{aligned}$$

$$+ e_{yy}\frac{\partial \sigma_{12}}{\partial x} + e_{xx}\frac{\partial \sigma_{22}}{\partial y} - e_{xy}\left(\frac{\partial \sigma_{12}}{\partial y} + \frac{\partial \sigma_{22}}{\partial x}\right) = 0 \quad (19)$$

These are the equilibrium equations with all the second order terms.

The physical interpretation of these equations is quite obvious. On the first line we have the classical terms. Among the second order terms those written on the third line are quite remarkable as they show that the gradient of the rotation or "curvature" has only an effect through the existence of shear. The effect of the curvature of an element of the material appears for two reasons, first through what we might call the "string effect" because of the analogy with the stress condition in a string under tension, and second through the difference of area of opposite sides of the element (Fig.3.).

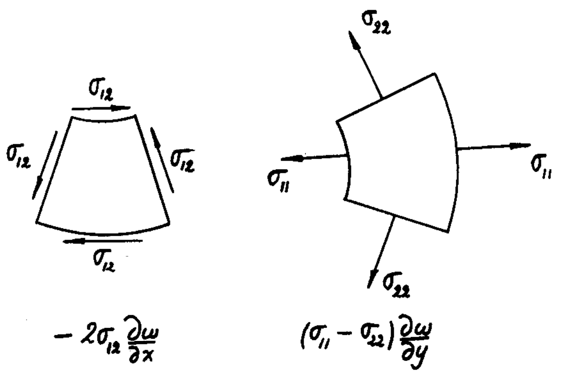

Fig.3 The second order terms due to curvature in the equilibrium condition for the horizontal direction.

On the fourth line are terms depending on the stress gradient and the deformation. They represent some kind of "buoyancy" effect due to the deformation of an element in the stress field. In many applications the buoyancy terms will be negligible.

The boundary conditions are found to be

$$\begin{aligned} &\left[\sigma_{11}(1+e_{yy}) - \sigma_{12}e_{xy} - \sigma_{12}\omega\right]dy \\ &\qquad - \left[\sigma_{12}(1+e_{xx}) - \sigma_{11}e_{xy} - \sigma_{22}\omega\right]dx = dF_x \\ &\left[\sigma_{12}(1+e_{yy}) - \sigma_{22}e_{xy} + \sigma_{11}\omega\right]dy \\ &\qquad - \left[\sigma_{22}(1+e_{xx}) - \sigma_{12}e_{xy} + \sigma_{12}\omega\right]dx = dF_y \end{aligned} \quad (20)$$

where dF_x, dF_y are the components of the force applied to an element of arc dx, dy . If we neglect the strain with respect to the rotation the boundary conditions become

$$\begin{aligned} (\sigma_{11} - \sigma_{12}\omega)\,dy - (\sigma_{12} - \sigma_{22}\omega)\,dx &= dF_x \\ (\sigma_{12} + \sigma_{11}\omega)\,dy - (\sigma_{22} + \sigma_{12}\omega)\,dx &= dF_y \end{aligned} \quad (21)$$

MATERIAL UNDER INITIAL STRESS. Consider a material for which the initial state characterized by the coordinates x and y is already a stressed state. The stress components referred to the directions x and y are denoted by S_{11} S_{22} S_{12}. Since this initial stress system is in equilibrium we have the conditions

$$\begin{aligned} \frac{\partial S_{11}}{\partial x} + \frac{\partial S_{12}}{\partial y} + \rho X &= 0 \\ \frac{\partial S_{12}}{\partial x} + \frac{\partial S_{22}}{\partial y} + \rho Y &= 0 \end{aligned} \quad (22)$$

where X, Y are the components of the body force per unit mass, and ρ is the specific mass in the condition of initial stress.

We now introduce a small deformation of the material so that the coordinates x, y of a point in the material become x + u and y + v, u and v being small increments. The stress components referred to axes originally parallel with the x, y directions and undergoing at every point the same rotation ω as the material, become $S_{11} + \sigma_{11}$, $S_{22} + \sigma_{22}$, $S_{12} + \sigma_{12}$ (fig. 4). The small stress increments σ_{11} σ_{22} σ_{12} are functions only of the strain and this functional relation may generally be taken as linear. Also if the initial shear is not too high the stress-strain relation will be generally isotropic. We also adopt for the strain tensor the linear expressions

$$e_{11} = e_{xx} \qquad e_{22} = e_{yy} \qquad e_{12} = e_{xy}$$

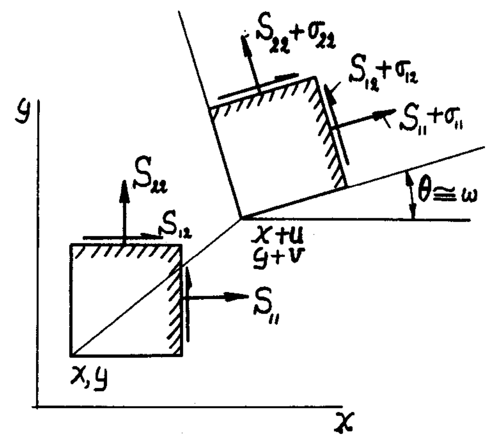

Fig.4 The stress condition after deformation in a material initially under the stress S_{11}, S_{22}, S_{12}.

The total stress must satisfy the equilibrium conditions (19). In these equations we substitute $S_{11} + \sigma_{11}$ for σ_{11} , $S_{22} + \sigma_{22}$ for σ_{22} and $S_{12} + \sigma_{12}$ for σ_{12} . Keeping only those terms which are linear with respect to the increments of stress and coordinates, and taking into account the initial equilibrium conditions (22), we find

$$\begin{aligned} &\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} + \rho u\frac{\partial X}{\partial x} + \rho v\frac{\partial X}{\partial y} + \omega\rho Y \\ &\qquad - 2S_{12}\frac{\partial \omega}{\partial x} + (S_{11}-S_{22})\frac{\partial \omega}{\partial y} \end{aligned}$$

$$+e_{yy}\frac{\partial S_{11}}{\partial x} + e_{xx}\frac{\partial S_{12}}{\partial y} - e_{xy}\left(\frac{\partial S_{11}}{\partial y}+\frac{\partial S_{12}}{\partial x}\right)=0$$

$$\frac{\partial \sigma_{12}}{\partial x}+\frac{\partial\sigma_{22}}{\partial y}+\rho u\frac{\partial Y}{\partial x}+\rho v\frac{\partial Y}{\partial y}+\omega\rho X$$

$$+(S_{11}-S_{22})\frac{\partial\omega}{\partial x}+2S_{12}\frac{\partial\omega}{\partial y}$$

$$+e_{yy}\frac{\partial S_{12}}{\partial x}+e_{xx}\frac{\partial S_{22}}{\partial y}-e_{xy}\left(\frac{\partial S_{12}}{\partial y}+\frac{\partial S_{22}}{\partial x}\right)=0 \qquad (23)$$

The boundary conditions are similarly deduced from (20) and (21).

The terms written on the second line of equations (23) are those arising from the effect of the curvature and the existence of shear in the initial stress. On the third line we have the buoyancy terms due to the deformation of an element in the initial stress field.

We will now consider two types of phenomena for which one of these groups of terms alone has a dominant influence.

Elastic stability. In the case of buckling the buoyancy effect is generally negligible with respect to the effect of shear and curvature. If there is no body force the equilibrium equations are

$$\frac{\partial \sigma_{11}}{\partial x}+\frac{\partial\sigma_{12}}{\partial y}-2S_{12}\frac{\partial\omega}{\partial x}+(S_{11}-S_{22})\frac{\partial\omega}{\partial y}=0$$

$$\frac{\partial \sigma_{12}}{\partial x}+\frac{\partial\sigma_{22}}{\partial y}+(S_{11}-S_{22})\frac{\partial\omega}{\partial x}+2S_{12}\frac{\partial\omega}{\partial y}=0 \qquad (24)$$

We note that the additional terms responsible for elastic instability depend only on the maximum initial shear and disappear if the latter is zero.

The classical terms are of the order of the product $G\epsilon$ of the shear modulus by the strain ϵ while the additional terms are of the order of the product ωT of the rotation by the shear $T = \sqrt{4S_{12}^2 + (S_{11} - S_{22})^2}$. In the case of buckling the additional terms are of the same order as the classical terms and the rotation will be large with respect to the strain in the ratio

$$\frac{\omega}{\epsilon} \cong \frac{G}{T} \qquad (25)$$

We have applied equations (24) to the case of a plate of thickness t under a uniform compression σ buckling in cylindrical waves. The problem is one of two-dimensional strain in the x, y plane of the cross section as illustrated by Fig.5.

Using HOOKE's law (4) the buckling equations are

$$G\nabla^2 u + (G+\lambda)\frac{\partial e}{\partial x} - \sigma\frac{\partial\omega}{\partial y} = 0$$

$$G\nabla^2 v + (G+\lambda)\frac{\partial e}{\partial y} - \sigma\frac{\partial\omega}{\partial x} = 0$$

$$e = e_{xx} + e_{yy}$$

These equations are satisfied by the solution

$$u = -\sin ax\left[A\sinh ay + Ck(1-\beta)\sinh aky\right]$$

$$v = \cos ax\left[A\cosh ay + C(1-k^2\beta)\cosh aky\right]$$

$$\text{with } k^2 = \frac{G-\frac{\sigma}{2}}{G+\frac{\sigma}{2}} \qquad \beta = \frac{G+\frac{\sigma}{2}}{\lambda+2G}$$

This solution represents a sinusoidal deformation of the plate and corresponds to the phenomenon of buckling under the compression σ. The constants A and C must be determined by the

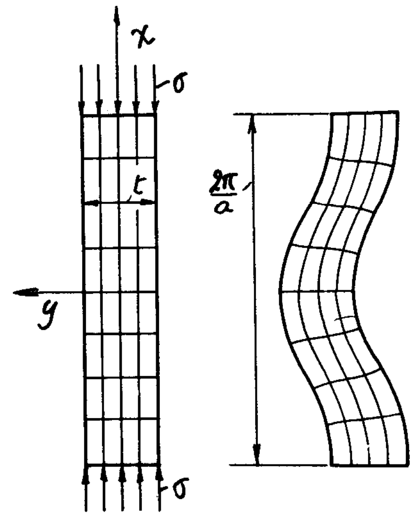

Fig.5 Cross section of a plate of thickness t buckling under a compression σ with a wave-length $\frac{2\pi}{a}$.

boundary condition that the stress is zero at the surface $y = \pm t/2$ of the plate. We can only satisfy this boundary condition if

$$\frac{1+\left(\frac{\sigma}{2G}\right)^2(1-2\nu)}{1-\left(\frac{\sigma}{2G}\right)^2} = \frac{\tanh\frac{akt}{2}}{\tanh\frac{at}{2}}$$

(ν = POISSON ratio)

This equation gives the critical compressive load σ necessary to produce a deformation of wave length $2\pi/a$ in a plate of thickness t.

Expanding both sides of the equation with respect to $\sigma/2G$ and at, and keeping the higher order terms we find

$$\sigma = \frac{a^2t^2}{6}\frac{G}{1-\nu}$$

which is the EULER load.

Elastic Waves in a material under hydrostatic pressure due to gravity. For a state of hydrostatic pressure, $S_{11} = S_{22}, S_{12} = 0$, the curvature terms disappear from the equilibrium equations. The buoyancy terms, however, may play a role if the pressure is not uniform. We consider the case where the pressure is due to gravity.

$$\frac{\partial S_{11}}{\partial y} = \frac{\partial S_{22}}{\partial y} = \rho g = -\rho Y$$

It is easy to write the equilibrium equations in this case and add the terms representing the inertia force. Assuming an isotropic HOOKE's law we find

$$G\nabla^2 u + (G+\lambda)\frac{\partial e}{\partial x} - \rho g \frac{\partial v}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}$$

$$G\nabla^2 v + (G+\lambda)\frac{\partial e}{\partial y} + \rho g \frac{\partial u}{\partial x} = \rho \frac{\partial^2 v}{\partial t^2}$$

Introducing the dilatation e and the rotation ω these equations become

$$(2G+\lambda)\nabla^2 e - 2\rho g \frac{\partial \omega}{\partial x} = \rho \frac{\partial^2 e}{\partial t^2}$$

$$G\nabla^2 \omega + 2\rho g \frac{\partial e}{\partial x} = \rho \frac{\partial^2 \omega}{\partial t^2}$$

They show that the effect of gravity introduces a coupling between the transversal and longitudinal waves. The physical reason for this coupling is illustrated in Fig.6 . The

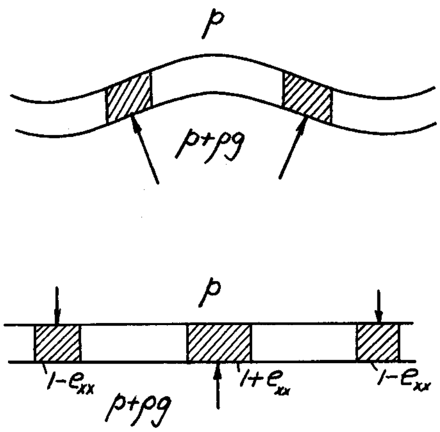

Fig.6. Illustration of the second order effect of buoyancy producing coupling between transversal and longitudinal elastic waves.

coupling terms are generally negligible, except in the case of large wave lengths. The effect will be important for instance in the case of tidal waves of a 1000 Km length in the earth's crust.

EQUATIONS FOR THREE-DIMENSIONS. In three dimensions there are at every point three directions at right angles which undergo a solid rotation. This is what we call the rotation of the material. If the coordinates x y z become x + u, y + v, z + w, after deformation the components of the rotation to a first order approximation may be represented by

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

The equilibrium conditions are found as in the two-dimensional theory. Including all the first and second order terms we find for instance in case of no body force one of the three equations to be

$$\begin{aligned}
&\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} + \frac{\partial \sigma_{13}}{\partial z} \\
&\quad + (\sigma_{11} - \sigma_{22})\frac{\partial \omega_z}{\partial y} - 2\sigma_{12}\frac{\partial \omega_z}{\partial x} + \sigma_{12}\frac{\partial \omega_x}{\partial z} \\
&\quad - (\sigma_{11} - \sigma_{33})\frac{\partial \omega_y}{\partial z} - 2\sigma_{31}\frac{\partial \omega_y}{\partial x} - \sigma_{31}\frac{\partial \omega_x}{\partial y} \\
&\quad + \sigma_{32}\left(\frac{\partial \omega_y}{\partial y} - \frac{\partial \omega_z}{\partial z}\right) \\
&+ \frac{\partial \sigma_{11}}{\partial x}(e_{yy} + e_{zz}) + \frac{\partial \sigma_{12}}{\partial y}(e_{xx} + e_{zz}) + \frac{\partial \sigma_{13}}{\partial z}(e_{xx} + e_{yy}) \\
&- \left(\frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{11}}{\partial y}\right)e_{xy} - \left(\frac{\partial \sigma_{13}}{\partial y} + \frac{\partial \sigma_{12}}{\partial z}\right)e_{yz} - \left(\frac{\partial \sigma_{11}}{\partial z} + \frac{\partial \sigma_{13}}{\partial x}\right)e_{zx} = 0
\end{aligned}$$

If we look for the physical interpretation of the terms in this equation we recognize most of them to have the same meaning as in the two-dimensional case. However the term

$$\sigma_{32}\left(\frac{\partial \omega_y}{\partial y} - \frac{\partial \omega_z}{\partial z}\right)$$

introduces something peculiar to the three-dimensional case; the influence of the twist of an element.

The strain components e_{11} e_{22} ....etc. are found by using the expression ds^2 for the square of the length element after deformation. Writing the classical expression for the finite strain tensor and neglecting as in the two-dimensional case all second order terms except the products and squares of the rotation components, we find

$$e_{11} = e_{xx} + \frac{1}{2}(\omega_z^2 + \omega_y^2) \qquad e_{12} = e_{xy} - \frac{1}{2}\omega_y\omega_x$$

$$e_{22} = e_{yy} + \frac{1}{2}(\omega_x^2 + \omega_z^2) \qquad e_{23} = e_{yz} - \frac{1}{2}\omega_z\omega_y$$

$$e_{33} = e_{zz} + \frac{1}{2}(\omega_y^2 + \omega_x^2) \qquad e_{31} = e_{zx} - \frac{1}{2}\omega_x\omega_z$$

where

$$e_{xx} = \frac{\partial u}{\partial x} \qquad etc.$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad etc.$$

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