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NON-LINEAR THEORY OF ELASTICITY AND THE LINEARIZED CASE FOR A BODY UNDER INITIAL STRESS

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Non-linear Theory of Elasticity and the Linearized Case for a Body under Initial Stress.

By Prof. M. A. BIOT.

Introduction.

IT is well known that the classical Theory of Elasticity is restricted to small deformations and rotations and that this is the underlying reason for its linear character. Attempts have been made (1) to establish a theory for finite deformations starting from the general mathematical viewpoint of tensor theory.

It was thought that a non-linear theory including terms of the first and second order only would yield the essential features due to large deformations which are not explained by a linear theory. Such features are exhibited, for instance, in the flexure of a thin shell.

Our method does not require an explicit formulation of the stress-strain relation, which is a physical problem. The essential idea which led us to our equations was to refer the stress condition to a local system of axis rotating with the material at that point, and to investigate equilibrium conditions for these stress components. These equations will contain explicitly only those second-order terms which are of kinematic origin. This development is made in section 2, while section 1 deals with a more accurate definition of stress and strain. This first section introduces a new definition of the strain components, which are here linearly related to the actual change of distance between two neighbouring points in the material.

These strain components are very important in establishing a correct expression for the potential energy and deriving the equilibrium equations by the variational method. This derivation is made in section 3, and the same equations are found as in the previous method. However, consideration of the strain energy leads to a new interesting viewpoint, as it introduces naturally two forms of representation of the stress—one in which the stresses are referred to the actual areas after deformation, and these are the stresses adopted in the previous section 2, the other in which the stresses are referred to the areas before deformation. Relations are found between these two types of stress components, and equations of equilibrium derived for both.

In section 4 we carry our attention to a linear theory of elasticity for bodies under high initial stress. This covers a wide range of problems, from elastic stability and buckling to the propagation of elastic waves inside the earth and the meaning of the elastic coefficients for a body under high initial stress. The special problem of elastic stability has been the object of many previous researches. R. V. Southwell (2) considers a uniform initial stress, and chooses axes along the principal directions; E. Trefftz (3) uses the variational method, but he chooses an expression for the potential energy which is correct only if the rotation is large with respect to the strain; C. B. Biezeno and H. Hencky (4) have developed a general theory.

The reader will be aware that methods similar to those in the previous sections may be used to establish the linear theory of elasticity of a body under high initial stress. In fact the equations can be derived from the previous theory by linearizing with respect to small increments of stress and the small components of strain and rotation. Those of our equations using the stress components referred to the initial areas are found to coincide with those derived by C. B. Biezeno and H. Hencky (4) by an entirely different method.

The last section (5) deals with the special case when we wish to introduce from the start the assumption that the rotation is large with respect to the strain. Special precautions have to be taken in introducing this assumption, because the strain and the rotation are not independent and satisfy the identities (2.6) Equations are derived which are a generalization of the result obtained by the author (5) in a previous paper.

1. *Strain and Stress.*

The original coordinates x, y, z of a point attached to the material become

$$\xi=x+u,$$
$$\eta=y+v,$$
$$\zeta=z+w$$

after the deformation. The infinitesimal region surrounding this point undergoes a homogeneous deformation defined by the linear transformation of $dx\ dy$, dz into $d\xi$, $d\eta$, $d\zeta$,

$$\left.\begin{aligned} d\xi&=\left(1+\frac{\partial u}{\partial x}\right)dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz,\\ d\eta&=\frac{\partial v}{\partial x}dx+\left(1+\frac{\partial v}{\partial y}\right)dy+\frac{\partial v}{\partial z}dz,\\ d\zeta&=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\left(1+\frac{\partial w}{\partial z}\right)dz,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1.1)$$

or
$$\left.\begin{aligned} d\xi&=(1+e_{xx})\,dx+(e_{xy}-\omega_z)\,dy+(e_{zx}+\omega_y)\,dz,\\ d\eta&=(e_{xy}+\omega_z)\,dx+(1+e_{yy})\,dy+(e_{yz}-\omega_x)\,dz,\\ d\zeta&=(e_{zx}-\omega_y)\,dx+(e_{yz}+\omega_x)\,dy+(1+e_{zz})\,dz,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1.2)$$

with
$$\left.\begin{aligned} e_{xx}&=\frac{\partial u}{\partial x}, & e_{yz}&=\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right), & \omega_x&=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\\ e_{yy}&=\frac{\partial v}{\partial y}, & e_{zx}&=\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right), & \omega_y&=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right),\\ e_{zz}&=\frac{\partial w}{\partial z}, & e_{xy}&=\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right), & \omega_z&=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right).\end{aligned}\right\} . \quad . \quad (1.3)$$

The length element after the deformation is

$$\begin{aligned} ds^2=(1+2g_{xx})\,dx^2&+(1+2g_{yy})\,dy^2+(1+2g_{zz})\,dz^2\\ &+4g_{yz}\,dy\,dz+4g_{zx}\,dz\,dx+4g_{xy}\,dx\,dy, \end{aligned} \quad . \quad . \quad . \quad . \quad (1.4)$$

with
$$\left.\begin{aligned} g_{xx}&=e_{xx}+\tfrac{1}{2}e_{xx}^2+\tfrac{1}{2}(e_{xy}+\omega_z)^2+\tfrac{1}{2}(e_{zx}-\omega_y)^2,\\ g_{yy}&=e_{yy}+\tfrac{1}{2}e_{yy}^2+\tfrac{1}{2}(e_{yz}+\omega_x)^2+\tfrac{1}{2}(e_{xy}-\omega_z)^2,\\ g_{zz}&=e_{zz}+\tfrac{1}{2}e_{zz}^2+\tfrac{1}{2}(e_{zx}+\omega_y)^2+\tfrac{1}{2}(e_{yz}-\omega_x)^2,\\ g_{yz}&=e_{yz}+\tfrac{1}{2}(e_{xy}-\omega_z)(e_{zx}+\omega_y)+\tfrac{1}{2}e_{yy}(e_{yz}-\omega_x)+\tfrac{1}{2}e_{zz}(e_{yz}+\omega_x),\\ g_{zx}&=e_{zx}+\tfrac{1}{2}e_{xx}(e_{zx}+\omega_y)+\tfrac{1}{2}(e_{yz}-\omega_x)(e_{xy}+\omega_z)+\tfrac{1}{2}e_{zz}(e_{zx}-\omega_y),\\ g_{xy}&=e_{xy}+\tfrac{1}{2}e_{xx}(e_{xy}-\omega_z)+\tfrac{1}{2}e_{yy}(e_{xy}+\omega_z)+\tfrac{1}{2}(e_{zx}-\omega_y)(e_{yz}+\omega_x).\end{aligned}\right\} \quad (1.5)$$

The transformation (1.2) contains the *nine* independent coefficients (1.3), while the change of length ds^2 depends only on the six coefficients g. There are therefore three degrees of freedom, leaving unchanged the length element ds^2 and corresponding to the rigid body rotation contained in the general linear transformation (1.2).

One of the first questions arising in the theory of elasticity is to distinguish what part in the general transformation (1.2) is to be considered as a pure strain and what part as a pure rotation. It is well known that when the quantities (1.3) are all small of the first order, and when we neglect quantities of higher order the pure deformation is represented by the coefficients e and the rotation by the vector $\omega_x\,\omega_y\,\omega_z$. However, this is only a first approximation. In the following we are concerned with finding the finite pure deformation contained in the transformation (1.2) and developing an expression for the strain components containing both first and second order terms. We consider therefore the following linear transformation :

$$\left.\begin{aligned} d\xi_1&=(1+\epsilon_{11})\,dx+\epsilon_{12}\,dy+\epsilon_{31}\,dz,\\ d\xi_2&=\epsilon_{12}\,dx+(1+\epsilon_{22})\,dy+\epsilon_{23}\,dz,\\ d\xi_3&=\epsilon_{31}\,dx+\epsilon_{23}\,dy+(1+\epsilon_{33})\,dz,\end{aligned}\right\} . \quad . \quad . \quad . \quad . \quad (1.6)$$

with symmetric coefficients. Such a transformation leaves unchanged the orientation of three rectangular directions which are the principal directions of strain [6]; in other words, the transformation (1.6) is equivalent to elongations along three fixed rectangular directions. It is therefore quite natural to call such a transformation a "pure deformation." We may choose as finite strain components of the deformation the symmetric coefficients

$$\left.\begin{matrix} \epsilon_{11}, & \epsilon_{12}, & \epsilon_{31}, \\ \epsilon_{12}, & \epsilon_{22}, & \epsilon_{23}, \\ \epsilon_{31}, & \epsilon_{23}, & \epsilon_{33}. \end{matrix}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.7)$$

The length of an element after the transformation (1.6) is

$$d\sigma^2=(1+2\gamma_{11})\,dx^2+(1+2\gamma_{22})\,dy^2+(1+2\gamma_{33})\,dz^2$$
$$+4\gamma_{23}\,dy\,dz+4\gamma_{31}\,dz\,dx+4\gamma_{12}\,dx\,dy, \quad . \quad . \quad . \quad (1.8)$$

with

$$\left.\begin{aligned} \gamma_{11}&=\epsilon_{11}+\tfrac{1}{2}(\epsilon_{11}^2+\epsilon_{12}^2+\epsilon_{31}^2), \\ \gamma_{22}&=\epsilon_{22}+\tfrac{1}{2}(\epsilon_{22}^2+\epsilon_{12}^2+\epsilon_{23}^2), \\ \gamma_{33}&=\epsilon_{33}+\tfrac{1}{2}(\epsilon_{33}^2+\epsilon_{31}^2+\epsilon_{23}^2), \\ \gamma_{23}&=\epsilon_{23}+\tfrac{1}{2}(\epsilon_{12}\epsilon_{31}+\epsilon_{22}\epsilon_{23}+\epsilon_{33}\epsilon_{23}), \\ \gamma_{31}&=\epsilon_{31}+\tfrac{1}{2}(\epsilon_{11}\epsilon_{31}+\epsilon_{23}\epsilon_{12}+\epsilon_{33}\epsilon_{31}), \\ \gamma_{12}&=\epsilon_{12}+\tfrac{1}{2}(\epsilon_{11}\epsilon_{12}+\epsilon_{22}\epsilon_{12}+\epsilon_{31}\epsilon_{23}). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1.9)$$

Now the pure deformation (1.6) can be made to represent exactly the same state of strain as that produced by the transformation (1.2), provided the length elements ds^2 and $d\sigma^2$ after deformation are identical. This condition is expressed analytically by the six equations

$$\left.\begin{matrix} g_{xx}=\gamma_{11}, & g_{yz}=\gamma_{23}, \\ g_{yy}=\gamma_{22}, & g_{zx}=\gamma_{31}, \\ g_{zz}=\gamma_{33}, & g_{xy}=\gamma_{12}. \end{matrix}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.10)$$

These equations determine the six strain components (1·7) as functions of the nine quantities (1.3).

The transformations (1.2) and (1.6) thus related represent the same state of strain, and can only differ by a rigid body rotation. The rigid body rotation that we must add to the transformation (1.6) in order to obtain the transformation (1.2) will be called the "*local rotation*" of the material.

The strain components (1.7) have the advantage that they are linearly related to the actual changes of length in the material, while the classical components (1.5) are linearly related to the change of the square of the length. On the other hand, they have the disadvantage that they cannot be expressed rationally by means of the nine quantities (1.3). However, this disadvantage vanishes in one important case—that is, when we assume the *nine quantities* (1.3) *to be small of the first order* and when we consider only the first- and second-order terms in the expression of the strain tensor (1.7) as a function of the nine quantities (1.3). In this case the result is obtained immediately as follows.

We notice from equations (1.10) that e_{xx} and e_{11} differ only by a second-order quantity, the same for e_{xy} and e_{12}, etc., so that we may write with an error of the third order:

$$\left.\begin{aligned}
&e_{xx}^2+e_{xy}^2+e_{zx}^2=\epsilon_{11}^2+\epsilon_{12}^2+\epsilon_{13}^2,\\
&e_{yy}^2+e_{xy}^2+e_{yz}^2=\epsilon_{22}^2+\epsilon_{12}^2+\epsilon_{23}^2,\\
&e_{zz}^2+e_{zx}^2+e_{yz}^2=\epsilon_{33}^2+\epsilon_{31}^2+\epsilon_{23}^2,\\
&e_{xy}e_{zx}+e_{yy}e_{yz}+e_{zz}e_{yz}=\epsilon_{12}\epsilon_{31}+\epsilon_{22}\epsilon_{23}+\epsilon_{33}\epsilon_{23},\\
&e_{xx}e_{zx}+e_{yz}e_{xy}+e_{zz}e_{zx}=\epsilon_{11}\epsilon_{31}+\epsilon_{23}\epsilon_{12}+\epsilon_{33}\epsilon_{31},\\
&e_{xx}e_{xy}+e_{yy}e_{xy}+e_{zx}e_{yz}=\epsilon_{11}\epsilon_{12}+\epsilon_{22}\epsilon_{12}+\epsilon_{31}\epsilon_{23}.
\end{aligned}\right\} \quad . \quad . \quad (1.11)$$

Introducing the approximate identities (1.11) into equations (1.10) we find the following expressions for the strain tensor with an error of the third order :

$$\left.\begin{aligned}
&\epsilon_{11}=e_{xx}+e_{xy}\omega_z-e_{zx}\omega_y+\tfrac{1}{2}(\omega_z^2+\omega_y^2),\\
&\epsilon_{22}=e_{yy}+e_{yz}\omega_x-e_{xy}\omega_z+\tfrac{1}{2}(\omega_x^2+\omega_z^2),\\
&\epsilon_{33}=e_{zz}+e_{zx}\omega_y-e_{yz}\omega_x+\tfrac{1}{2}(\omega_y^2+\omega_x^2),\\
&\epsilon_{23}=e_{yz}+\tfrac{1}{2}\omega_x(e_{zz}-e_{yy})+\tfrac{1}{2}\omega_y e_{xy}-\tfrac{1}{2}\omega_z e_{zx}-\tfrac{1}{2}\omega_y\omega_z,\\
&\epsilon_{31}=e_{zx}+\tfrac{1}{2}\omega_y(e_{xx}-e_{zz})+\tfrac{1}{2}\omega_z e_{yz}-\tfrac{1}{2}\omega_x e_{xy}-\tfrac{1}{2}\omega_z\omega_x,\\
&\epsilon_{12}=e_{xy}+\tfrac{1}{2}\omega_z(e_{yy}-e_{xx})+\tfrac{1}{2}\omega_x e_{zx}-\tfrac{1}{2}\omega_y e_{yz}-\tfrac{1}{2}\omega_x\omega_y.
\end{aligned}\right\} \quad . \quad . \quad (1.12)$$

At this point it is important to stress the physical significance of these components of strain ϵ. If we look at the homogeneous transformation (1.2) of a small region in the vicinity of a point P attached to the material we now see that it can be obtained as follows :—

(1) We rotate this region as a rigid body. This rotation is defined in first approximation by the vector $\omega_x\ \omega_y\ \omega_z$.

(2) A system of rectangular coordinates with point P as origin and parallel with the $x\ y\ z$ directions is rigidly rotated by the same amount as the material, and becomes thereby a system which we call (1, 2, 3). With respect to this system 1, 2, 3 we perform the pure deformation (1.6) with the coefficients (1.12).

Therefore, we may also look at the strain components as representing the pure deformation referred to a rectangular frame (1, 2, 3) originally parallel with the x y z directions and undergoing the same rotation as the material. The strain field is thus referred to a field of rectangular axes whose orientation varies from point to point according to the local rotation of the material. It is important to bear this in mind when we are dealing with the stress, since to correlate stress and strain we must refer them to the same set of axes.

The stress components at a point ξ, η, ζ after deformation referred to the x, y, z directions are denoted by

$$\left.\begin{array}{ccc} \sigma_{xx}, & \sigma_{xy}, & \sigma_{zx}, \\ \sigma_{xy}, & \sigma_{yy}, & \sigma_{yz}, \\ \sigma_{zx}, & \sigma_{yz}, & \sigma_{zz}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.13)$$

The stress components relative to the rotated axis (1, 2, 3) are denoted by

$$\left.\begin{array}{ccc} \sigma_{11}, & \sigma_{12}, & \sigma_{31}, \\ \sigma_{12}, & \sigma_{22}, & \sigma_{23}, \\ \sigma_{31}, & \sigma_{23}, & \sigma_{33}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.14)$$

In view of further application it is interesting to know the relation between the components (1.13) and (1.14) of the stress.

We have introduced the assumption that the quantities (1.3) are of the first order ; we now add the assumption that the *stress components* (1.13) *or* (1.14) *are also of the first order.* Since we are interested in a theory of the second order we will drop all terms of higher order than the second in the relation between the stress components (1.13) and (1.14).

In order to do this we may neglect second-order quantities in the expressions for the direction cosines of the axis x, y, z with respect to 1, 2, 3. The transformation formula for the coordinate x, y, z when we rotate the axis into 1, 2, 3 with coordinates x_1, x_2, x_3 are

$$x=x_1 \cos(x_1 1)+x_2 \cos(x_1 2)+x_3 \cos(x_1 3),$$

$$y=x_1 \cos(y_1 1)+x_2 \cos(y_1 2)+x_3 \cos(y_1 3),$$

$$z=x_1 \cos(z_1 1)+x_2 \cos(z_1 2)+x_3 \cos(z_1 3).$$

Now if the rotation of x, y, z into 1, 2, 3 is represented by the vector ω_x ω_y ω_z we have in first approximation

$$x=x_1-\omega_z x_2+\omega_y x_3,$$

$$y=\omega_z x_1+x_2-\omega_x x_3,$$

$$z=-\omega_y x_1+\omega_x x_2+x_3.$$

Therefore the direction cosines are in first approximation

$$\left.\begin{array}{lll}\cos(x_1\,1)=1, & \cos(x_1\,2)=-\omega_z, & \cos(x_1\,3)=\omega_y,\\ \cos(y_1\,1)=\omega_z, & \cos(y_1\,2)=1, & \cos(y_1\,3)=-\omega_x,\\ \cos(z_1\,1)=-\omega_y, & \cos(z_1\,2)=\omega_x, & \cos(z_1\,3)=1.\end{array}\right\} \quad (1.15)$$

We may use these values of the direction cosines in the transformation formulas of the stress components

$$\sigma_x=\sigma_{11}\cos^2(x_1\,1)+\sigma_{22}\cos^2(x_1\,2)+\sigma_{33}\cos^2(x_1\,3)$$
$$+2\sigma_{23}\cos(x_1\,2)\cos(x_1\,3)+2\sigma_{31}\cos(x_1\,3)\cos(x_1\,1)+2\sigma_{12}\cos(x_1 1)\cos(x_{12})$$

. . . . etc. . . .

Neglecting terms of a higher order than the second, we have

$$\left.\begin{array}{l}\sigma_{xx}=\sigma_{11}+2\sigma_{31}\omega_y-2\sigma_{12}\omega_z,\\ \sigma_{yy}=\sigma_{22}+2\sigma_{12}\omega_z-2\sigma_{23}\omega_x,\\ \sigma_{zz}=\sigma_{33}+2\sigma_{23}\omega_x-2\sigma_{31}\omega_y,\\ \sigma_{yz}=\sigma_{23}+(\sigma_{22}-\sigma_{33})\omega_x-\sigma_{12}\omega_y+\sigma_{13}\omega_z,\\ \sigma_{zx}=\sigma_{31}+(\sigma_{33}-\sigma_{11})\omega_y-\sigma_{23}\omega_z+\sigma_{21}\omega_x,\\ \sigma_{xy}=\sigma_{12}+(\sigma_{11}-\sigma_{22})\omega_z-\sigma_{31}\omega_x+\sigma_{32}\omega_y.\end{array}\right\} \quad . \quad . \quad . \quad (1.16)$$

2. *Equilibrium Equations.*

A point P of the material originally of coordinates x, y, z acquires the coordinates ξ, η, ζ after deformation. An original closed volume V bounded by the surface S becomes after deformation a volume V′ bounded by the surface S′. The x component F_x of the total force acting on the boundary S′ after deformation may be expressed by means of a double integral extended to the same material boundary S before deformation. This is done by using the transformation formula of surface integrals. We have

$$F_x=\iint_{S'}\sigma_{xx}\,d\eta\,d\zeta+\sigma_{xy}\,d\zeta\,d\xi+\sigma_{zx}\,d\xi\,d\eta$$
$$=\iint_S\sigma_{xx}\left[\frac{d(\eta,\zeta)}{d(y,z)}dy\,dz+\frac{d(\eta,\zeta)}{d(z,x)}dz\,dx+\frac{d(\eta,\zeta)}{d(x,y)}dx,dy\right]$$
$$+\iint_S\sigma_{xy}\left[\frac{d(\zeta,\xi)}{d(y,z)}dy\,dz+\frac{d(\zeta,\xi)}{d(z,x)}dz\,dx+\frac{d(\zeta,\xi)}{d(x,y)}dx,dy\right]$$
$$+\iint_S\sigma_{zx}\left[\frac{d(\xi,\eta)}{d(y,z)}dy\,dz+\frac{d(\xi,\eta)}{d(z,x)}dz\,dx+\frac{d(\xi,\eta)}{d(x,y)}dx,dy\right]. \quad (2.1)$$

In this expression $\frac{d(\eta, \zeta)}{d(y, z)}$, etc. are the partial Jacobians of the transformation of x, y, z into ξ, η, ζ. We have, for instance,

$$\frac{d(\eta, \zeta)}{d(y, z)} = \begin{vmatrix} \frac{\partial \eta}{\partial y}, & \frac{\partial \eta}{\partial z}, \\ \frac{\partial \zeta}{\partial y}, & \frac{\partial \zeta}{\partial z}. \end{vmatrix}$$

These Jacobians are the cofactors of the determinant of the differential transformation (1.2).

Since we are interested in a second-order theory, and since these Jacobians appcar multiplied by the stress in expression (2.1), we need only keep the linear terms in their values. We find

$$\left.\begin{array}{lll} \frac{d(\eta, \zeta)}{d(y, z)} = 1+e_{yy}+e_{zz} & \frac{d(\eta, \zeta)}{d(z, x)} = -(e_{xy}+\omega_z) & \frac{d(\eta, \zeta)}{d(x, y)} = -(e_{zx}-\omega_y), \\ \frac{d(\zeta, \xi)}{d(y, z)} = -(e_{xy}-\omega_z) & \frac{d(\zeta, \xi)}{d(z, x)} = 1+e_{xx}+e_{zz} & \frac{d(\zeta, \xi)}{d(x, y)} = -(e_{yz}+\omega)_x, \\ \frac{d(\xi, \eta)}{d(y, z)} = -(e_{zx}+\omega_y) & \frac{d(\xi, \eta)}{d(z, x)} = -(e_{yz}-\omega_x) & \frac{d(\xi, \eta)}{d(x, y)} = 1+e_{xx}+e_{yy}. \end{array}\right\} \tag{2.2}$$

Introducing these values in (2.1), it becomes

$$\mathrm{F}_x = \int_{\mathrm{S}} f_x d\mathrm{S},$$

with

$$\begin{aligned} f_x = {} & \sigma_{xx}\alpha+\sigma_{xy}\beta+\sigma_{zx}\gamma \\ & [\sigma_{xx}(e_{yy}+e_{zz})-\sigma_{xy}(e_{xy}-\omega_z)-\sigma_{zx}(e_{zx}+\omega_y)]\alpha \\ & +[-\sigma_{xx}(e_{xy}+\omega_z)+\sigma_{xy}(e_{xx}+e_{zz})-\sigma_{zx}(e_{yz}-\omega_x)]\beta \\ & +[-\sigma_{xx}(e_{zx}-\omega_y)-\sigma_{xy}(e_{yz}+\omega_x)+\sigma_{zx})e_{xx}+e_{yy})]\gamma. \end{aligned}$$

This expression f_x is the x component of the force per *unit original area*, acting at the boundary after deformation, and α, β, γ are the direction cosines of the outside normal to the *original boundary* S *before deformation.*

Introducing the component of stress (1.14) with respect to the locally rotated axes 1, 2, 3 through equations (1.16), and dropping terms of order higher than the second,

$$\begin{aligned} f_x = {} & \sigma_{11}\alpha+\sigma_{12}\beta+\sigma_{31}\gamma \\ & +(-\sigma_{12}\omega_z+\sigma_{31}\omega_y)\alpha \\ & +(-\sigma_{22}\omega_z+\sigma_{32}\omega_y)\beta \\ & +(-\sigma_{23}\omega_z+\sigma_{33}\omega_y)\gamma \end{aligned}$$

$$+[\sigma_{11}(e_{yy}+e_{zz})-\sigma_{12}e_{xy}-\sigma_{31}e_{xz}]\alpha$$
$$+[-\sigma_{11}e_{xy}+\sigma_{12}(e_{xx}+e_{zz})-\sigma_{31}e_{yz}]\beta$$
$$+[-\sigma_{11}e_{zx}-\sigma_{12}e_{yz}+\sigma_{31}(e_{xx}+e_{yy})]\gamma. \quad . \quad . \quad . \quad . \quad (2.3)$$

We have two other similar expressions for f_y and f_z.

Consider now the body force. We assume that the force per unit mass at a point x, y, z is determined by the components $X(x\,y\,z)$ $Y\,(x\,y\,z)$, $Z(x\,y\,z)$. If ρ is the specific mass before deformation an element of mass $\rho dx\,dy\,dz$ keeps the same mass after deformation, but moves to the point ξ, η, ζ. Therefore the x component of the total body force after deformation is

$$\iiint_V X(\xi, \eta, \zeta)\rho\, dx\, dy\, dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.4)$$

The condition of equilibrium of the volume V after deformation in the x direction is

$$\iint_S f_x\, dS + \iiint_V X(\xi, \eta, \zeta)\rho\, dx\, dy\, dr = 0.$$

By Green's theorem we change the surface integral into a volume integral extended to V ; we then have

$$\iiint_V [A_x+\rho X(\xi, \eta, \zeta)]dx\, dy\, dz = 0.$$

This must be true whatever the volume V, and the equilibrium equations are therefore

$$\begin{aligned} A_x+\rho X(\xi, \eta, \zeta) = {} & \frac{\partial\sigma_{11}}{\partial x}+\frac{\partial\sigma_{12}}{\partial y}+\frac{\partial\sigma_{31}}{\partial z}+\rho X(\xi, \eta, \zeta) \\ & +\frac{\partial}{\partial x}(\sigma_{31}\omega_y)-\frac{\partial}{\partial x}(\sigma_{12}\omega_z)+\frac{\partial}{\partial y}(\sigma_{32}\omega_y)-\frac{\partial}{\partial y}(\sigma_{22}\omega_z) \\ & +\frac{\partial}{\partial z}(\sigma_{33}\omega_y)-\frac{\partial}{\partial z}(\sigma_{23}\omega_z)+\frac{\partial}{\partial x}[\sigma_{11}(e_{yy}+e_{zz})] \\ & +\frac{\partial}{\partial y}[\sigma_{12}(e_{zz}+e_{xx})]+\frac{\partial}{\partial z}[\sigma_{31}(e_{xx}+e_{yy})] \\ & -\frac{\partial}{\partial x}[\sigma_{12}e_{xy}+\sigma_{31}e_{zx}]-\frac{\partial}{\partial y}[\sigma_{11}e_{xy}+\sigma_{31}e_{yz}] \\ & -\frac{\partial}{\partial z}[\sigma_{11}e_{zx}+\sigma_{12}e_{yz}]=0. \quad . \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.5) \end{aligned}$$

There are two other similar equations expressing the equilibrium in the y and z directions. The boundary conditions are determined by expressions (2.3) for the forces $f_x\, f_y\, f_z$ acting per unit original area of the boundary.

It is possible to give equations (2.5) another remarkable form by taking into account the fact deducible from the equations (2.5) themselves that

$$\frac{\partial\sigma_{11}}{\partial x}+\frac{\partial\sigma_{12}}{\partial y}+\frac{\partial\sigma_{13}}{\partial z}+\mathrm{X}\rho,$$

$$\frac{\partial\sigma_{12}}{\partial x}+\frac{\partial\sigma_{22}}{\partial y}+\frac{\partial\sigma_{23}}{\partial z}+\mathrm{Y}\rho,$$

$$\frac{\partial\sigma_{31}}{\partial x}+\frac{\partial\sigma_{23}}{\partial y}+\frac{\partial\sigma_{33}}{\partial z}+\mathrm{Z}\rho$$

are quantities of the second order and that we have the identities

$$\left.\begin{aligned}
\frac{\partial e_{xx}}{\partial z}&=\frac{\partial}{\partial x}[e_{zx}+\omega_y],\\
\frac{\partial e_{yy}}{\partial z}&=\frac{\partial}{\partial y}[e_{yz}-\omega_x],\\
\frac{\partial e_{yy}}{\partial x}&=\frac{\partial}{\partial y}[e_{xy}+\omega_z],\\
\frac{\partial e_{zz}}{\partial x}&=\frac{\partial}{\partial z}[e_{zx}-\omega_y],\\
\frac{\partial e_{zz}}{\partial y}&=\frac{\partial}{\partial z}[e_{yz}+\omega_x],\\
\frac{\partial e_{xx}}{\partial y}&=\frac{\partial}{\partial x}[e_{xy}-\omega_z].
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2.6)$$

The equilibrium equations (2.5) then take the form

$$\begin{aligned}
&\frac{\partial\sigma_{11}}{\partial x}+\frac{\partial\sigma_{12}}{\partial y}+\frac{\partial\sigma_{31}}{\partial z}+\rho\mathrm{X}+\rho(\omega_z\mathrm{Y}-\omega_y\mathrm{Z})\\
&\qquad+(\sigma_{11}-\sigma_{22})\frac{\partial\omega_z}{\partial y}-2\sigma_{12}\frac{\partial\omega_z}{\partial x}+\sigma_{12}\frac{\partial\omega_x}{\partial z}\\
&\qquad-(\sigma_{11}-\sigma_{33})\frac{\partial\omega_y}{\partial z}+2\sigma_{31}\frac{\partial\omega_y}{\partial x}-\sigma_{31}\\
&\qquad+\sigma_{32}\cancel{\frac{\partial\omega_x}{\partial y}}\left(\frac{\partial\omega_y}{\partial y}-\frac{\partial\omega_z}{\partial z}\right)\\
&\qquad+\frac{\partial\sigma_{11}}{\partial x}(e_{yy}+e_{zz})+\frac{\partial\sigma_{12}}{\partial y}(e_{xx}+e_{zz})+\frac{\partial\sigma_{13}}{\partial z}(e_{xx}+e_{yy})\\
&\qquad-\left(\frac{\partial\sigma_{12}}{\partial x}+\frac{\partial\sigma_{11}}{\partial y}\right)e_{xy}-\left(\frac{\partial\sigma_{31}}{\partial y}+\frac{\partial\sigma_{12}}{\partial z}\right)e_{yz}-\left(\frac{\partial\sigma_{11}}{\partial z}+\frac{\partial\sigma_{31}}{\partial x}\right)e_{zx}=0,
\end{aligned}$$

$$\ldots\text{etc.} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.7)$$

We may also expand the functions X, Y, Z with respect to u, v, w and write

$$X(x+u, y+v, z+w) = X(x\,y\,z) + \frac{\partial X}{\partial x}u + \frac{\partial X}{\partial y}v + \frac{\partial X}{\partial z}w,$$
$$\ldots \text{etc.} \qquad (2.8)$$

Considering X Y Z and u v w as quantities of the first order, the above expansion includes the first- and second-order terms.

The terms written on the first line of equation (2.7) are the classical terms of the linear theory and additional terms due to the rotation of the material with respect to the body force. The terms on the second and the third line have their origin in relative change of direction and area of opposite faces of an element of material due to its curvature; we call them the *curvature* terms. On the fourth line is what we call the *torsion* term, because it arises from the torsion of an element. On the last two lines are terms depending on the stress gradient and the strain; they will be called the *buoyancy terms*, because they arise from some kind of buoyancy due to the deformation of an element in its own stress field. The curvature and torsion terms vanish in case of a hydrostatic stress condition, while the buoyancy terms vanish when the stress field is homogeneous.

3. *The Strain-energy.*

The concept of strain-energy and the application of the principle of virtual work throws a new light on the present theory. The first step is to establish a correct expression for the strain-energy. We need only consider a state of homogeneous pure strain defined by the transformation

$$\left.\begin{aligned} \xi &= (1+\epsilon_{11})x + \epsilon_{12}y + \epsilon_{31}z, \\ \eta &= \epsilon_{12}x + (1+\epsilon_{22})y + \epsilon_{23}z, \\ \zeta &= \epsilon_{31}x + \epsilon_{23}z + (1+\epsilon_{33})z. \end{aligned}\right\} \qquad (3.1)$$

The homogeneous stress field associated with this deformation is denoted by the constant components

$$\left.\begin{matrix} \sigma_{11}, & \sigma_{12}, & \sigma_{31}, \\ \sigma_{12}, & \sigma_{22}, & \sigma_{23}, \\ \sigma_{31}, & \sigma_{23}, & \sigma_{33}. \end{matrix}\right\} \qquad (3.2)$$

Consider now a cube of unit volume before deformation, its edges being along x, y, z. After deformation it becomes a parallelopiped. On the side originally perpendicular to the z axis acts now a force of component

$$\tau'_{11}, \qquad \tau'_{12}, \qquad \tau_{13},$$

and on the two other sides will act forces of components

$$\tau'_{21}, \qquad \tau'_{22}, \qquad \tau'_{23},$$

$$\tau'_{31}, \qquad \tau'_{32}, \qquad \tau'_{33}.$$

These forces may be found from the value (2.3) of f_x, f_y, f_z derived above and giving the actual forces at the boundary of a volume. We must introduce ϵ_{11} for e_{xx}, ϵ_{12} for e_{xy}, etc., and put $\omega_x=\omega_y=\omega_z=0$. We find

$$\tau'_{11}=\sigma_{11}(1+\epsilon)-\sigma_{11}\epsilon_{11}-\sigma_{12}\epsilon_{12}-\sigma_{31}\epsilon_{31},$$

$$\tau'_{12}=\sigma_{12}(1+\epsilon)-\sigma_{11}\epsilon_{12}-\sigma_{12}\epsilon_{22}-\sigma_{31}\epsilon_{23},$$

etc.,

or
$$\tau'_{\mu\nu}=\sigma_{\mu\nu}(1+\epsilon)-\sum^{\alpha}\sigma_{\mu\alpha}\epsilon_{\nu\alpha}, \qquad (3.3)$$

with
$$\epsilon=\epsilon_{11}+\epsilon_{22}+\epsilon_{33}.$$

This $\tau'_{\mu\nu}$ is a non-symmetric tensor.

When small increments of strain $\delta\epsilon_{\mu\nu}$ are given to the deformed cube, work is performed by the forces acting on the faces of the parallelipiped. This work, expressing the increment of the strain-energy δW of a unit original volume, is equal to

$$\delta W=\sum^{\mu\nu}\tau'_{\mu\nu}\delta\epsilon_{\mu\nu}. \qquad (3.4)$$

A more convenient form of δW may be found by writing

$$\delta W=\tfrac{1}{2}\Big[\sum^{\mu\nu}\tau'_{\mu\nu}\delta\epsilon_{\mu\nu}+\sum^{\mu\nu}\tau'_{\nu\mu}\delta\epsilon_{\nu\mu}\Big].$$

Since $\delta\epsilon_{\mu\nu}=\delta\epsilon_{\nu\mu}$, we have

$$\delta W=\sum^{\mu\nu}\tfrac{1}{2}\,[\tau'_{\mu\nu}+\tau'_{\nu\mu}]\delta\epsilon_{\nu\mu},$$

or
$$\delta W=\sum^{\mu\nu}\tau_{\mu\nu}\delta\epsilon_{\mu\nu}, \qquad (3.5)$$

with
$$\left.\begin{aligned}\tau_{\mu\nu}&=\tfrac{1}{2}[\tau'_{\mu\nu}+\tau'_{\nu\mu}],\\ \tau_{\mu\nu}&=(1+\epsilon)\sigma_{\mu\nu}-\tfrac{1}{2}\sum^{\alpha}(\sigma_{\alpha\mu}\epsilon_{\alpha\nu}+\sigma_{\alpha\nu}\epsilon_{\alpha\mu}).\end{aligned}\right\} \qquad (3.6)$$

The tensor $\tau_{\mu\nu}$ is the symmetric part of $\tau'_{\mu\nu}$. Explicitly, we have

$$\tau_{11}=(1+\epsilon)\sigma_{11}-\sigma_{11}\epsilon_{11}-\sigma_{12}\epsilon_{12}-\sigma_{31}\epsilon_{31},$$

$$\tau_{22}=(1+\epsilon)\sigma_{22}-\sigma_{12}\epsilon_{12}-\sigma_{22}\epsilon_{22}-\sigma_{23}\epsilon_{23},$$

$$\tau_{33}=(1+\epsilon)\sigma_{33}-\sigma_{31}\epsilon_{31}-\sigma_{23}\epsilon_{23}-\sigma_{33}\epsilon_{33},$$

$$\tau_{23}=(1+\epsilon)\sigma_{23}-\tfrac{1}{2}(\epsilon_{22}+\epsilon_{33})\sigma_{23}-\tfrac{1}{2}(\sigma_{22}+\sigma_{33})\epsilon_{23}-\tfrac{1}{2}(\sigma_{12}\epsilon_{31}+\sigma_{31}\epsilon_{12}),$$

$$\tau_{31}=(1+\epsilon)\sigma_{31}-\tfrac{1}{2}(\epsilon_{33}+\epsilon_{11})\sigma_{31}-\tfrac{1}{2}(\sigma_{33}+\sigma_{11})\epsilon_{31}-\tfrac{1}{2}(\sigma_{12}\epsilon_{23}+\sigma_{23}\epsilon_{12}),$$

$$\tau_{12}=(1+\epsilon)\sigma_{12}-\tfrac{1}{2}(\epsilon_{11}+\epsilon_{22})\sigma_{12}-\tfrac{1}{2}(\sigma_{11}+\sigma_{22})\epsilon_{12}-\tfrac{1}{2}(\sigma_{31}\epsilon_{23}+\sigma_{23}\epsilon_{31}). \qquad (3.6)$$

The symmetric tensor $\tau_{\mu\nu}$ is obviously another representation of the stress. It can easily be seen from the expression of δW that $\tau_{\mu\nu}$ is the symmetric part of the tensor whose components are the *actual forces* acting after deformation on the faces of a parallelopiped which originally was a cube of unit dimension parallel with the coordinate axes. For instance, in the particular case when the coordinate axes are principal directions for both the stress and the strain we have

$$\tau_{11}=(1+\epsilon_{22}+\epsilon_{33})\sigma_{11},$$
$$\tau_{22}=(1+\epsilon_{33}+\epsilon_{11})\sigma_{22},$$
$$\tau_{33}=(1+\epsilon_{11}+\epsilon_{22})\sigma_{33}.$$

In these expressions τ_{11}, for instance, is the stress multiplied by the area $(1+\epsilon_{22}+\epsilon_{33})$ after deformation of the corresponding force of an original unit cube. The tensor $\tau_{\mu\nu}$ might be also called the *stress per unit initial area before deformation* of the material, while the tensor $\sigma_{\mu\nu}$ would be the stress *per unit actual area after deformation.*

The existence of a potential energy imposes that δW be an exact differential with respect to the strain component. It therefore imposes also certain additional conditions on the stress-strain relations. By these relations $\sigma_{\mu\nu}$ or $\tau_{\mu\nu}$ are expressed as functions of the strain component $\epsilon_{\mu\nu}$. Then the stress-strain relations must satisfy the *fifteen conditions*,

$$\frac{\partial\tau_{\mu\nu}}{\partial\epsilon_{ik}}=\frac{\partial\tau_{ik}}{\partial\epsilon_{\mu\nu}}, \quad \ldots \ldots \ldots \quad (3.7)$$

or

$$\frac{\partial\sigma_{\mu\nu}}{\partial\epsilon_{ik}}+\frac{\partial(\epsilon\sigma_{\mu\nu})}{\partial\epsilon_{ik}}-\frac{1}{2}\frac{\partial}{\partial\epsilon_{ik}}\sum^{\alpha}(\sigma_{\alpha\mu}\epsilon_{\alpha\nu}+\sigma_{\alpha\nu}\epsilon_{\alpha\mu})$$
$$=\frac{\partial\sigma_{ik}}{\partial\epsilon_{\mu\nu}}+\frac{\partial(\epsilon\sigma_{ik})}{\partial\epsilon_{\mu\nu}}-\frac{1}{2}\frac{\partial}{\partial\epsilon_{\mu\nu}}\sum^{\alpha}(\sigma_{\alpha i}\epsilon_{\alpha k}+\sigma_{\alpha k}\epsilon_{\alpha i}). \quad . \quad . \quad (3.8)$$

It is possible to derive the equations of equilibrium (2.7) found above by expressing that the sum of the virtual external and internal work vanishes for all possible variations δu, δv, δw of the coordinates.

The variation of the total strain energy in an original volume V is

$$\delta W=-\delta W_i=\iiint_v^{\mu\nu}\Sigma\tau_{\mu\nu}\delta\epsilon_{\mu\nu}dx\,dy\,dz, \quad . \quad . \quad . \quad . \quad (3.9)$$

in which we use expressions (1.12) for the strain components $\epsilon_{\mu\nu}$, if $f_x\, f_y\, f_z$ denotes the components of the force per unit original area acting at the boundary. The virtual work of the external forces is

$$\delta W=\iint_S(fx\,\delta u+fy\,\delta v+fz\,\delta w)dS+\iiint_v[X(\xi,\eta,\zeta)\delta u$$
$$+Y(\xi,\eta,\zeta)\delta v+Z(\xi,\eta,\zeta)\delta w]\rho\,dx\,dy\,dz. \quad . \quad (3.10)$$

In the volume integral (3.9) the variations $\delta e_{\mu\nu}$ introduce such quantities as

$$\delta e_{xx}=\delta\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\delta u,$$

$$\delta\omega_z=\frac{1}{2}\delta\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]=\frac{1}{2}\frac{\partial}{\partial x}\delta v-\frac{1}{2}\frac{\partial}{\partial y}\delta u.$$

It is therefore possible to bring out the factors δu, δv, δw by partial integration. We find an expression of the form

$$\delta \mathrm{W}_i=-\iint_{\mathrm{S}}(p_x\,\delta u+p_y\,\delta v+p_z)d\mathrm{S}+\iiint_v(\mathrm{A}_x\,\delta u+\mathrm{A}_y\,\delta v+\mathrm{A}_z\,\delta w)dx\,dy\,dz. \quad . \quad . \quad . \quad (3.11)$$

For equilibrium the total virtual work must vanish, hence

$$\delta\mathrm{W}_i+\delta\mathrm{W}_e=\iiint_v[(\mathrm{A}_x+\mathrm{X}\rho)\delta u+(\mathrm{A}_y+\mathrm{Y}\rho)\delta v+(\mathrm{A}_z+\mathrm{Z}\rho)\delta w]\,dx\,dy\,dz$$
$$+\iint_{\mathrm{S}}[(f_x-p_x)\delta u+(f_y-p_y)\delta v+(f_z-p_z)\delta w]d\mathrm{S}=0. \quad (3.12)$$

This must be identically zero for arbitrary values of δu, δv, δw, therefore we have the conditions

$$\begin{aligned}\mathrm{A}_x+\mathrm{X}\rho=&\frac{\partial\tau_{11}}{\partial x}+\frac{\partial\tau_{12}}{\partial y}+\frac{\partial\tau_{13}}{\partial z}+\rho\mathrm{X}(\xi,\eta,\zeta)+\frac{\partial}{\partial x}(\tau_{31}\,\omega_y)-\frac{\partial}{\partial x}(\tau_{12}\,\omega_z)\\&+\frac{\partial}{\partial y}(\tau_{23}\omega_y)-\frac{\partial}{\partial y}(\tau_{22}\omega_z)+\frac{\partial}{\partial z}(\tau_{33}\omega_y)-\frac{\partial}{\partial z}(\tau_{23}\omega_z)\\&+\frac{1}{2}\frac{\partial}{\partial y}[(\tau_{22}-\tau_{11})e_{xy}]+\frac{1}{2}\frac{\partial}{\partial z}[(\tau_{33}-\tau_{11})e_{zx}]\\&-\frac{1}{2}\frac{\partial}{\partial y}[\tau_{12}(e_{yy}-e_{xx})]-\frac{1}{2}\frac{\partial}{\partial z}[\tau_{31}(e_{zz}-e_{xx})]\\&+\frac{1}{2}\frac{\partial}{\partial y}[\tau_{23}e_{zx}-\tau_{31}e_{yz}]+\frac{1}{2}\frac{\partial}{\partial z}[\tau_{23}e_{xy}-\tau_{12}e_{yz}]=0, \quad . \quad . \quad . \quad (3.13)\end{aligned}$$

and two other equations as above :

$$\mathrm{A}_y+\mathrm{Y}\rho=0,$$
$$\mathrm{A}_z+\mathrm{Z}\rho=0.$$

These are equilibrium equations expressed by means of the stress $\tau_{\mu\nu}$ referred to the original areas before deformation.

We also derive the boundary conditions

$$\begin{aligned}f_x=p_x=&\tau_{11}\alpha+\tau_{12}\beta+\tau_{31}\gamma+(\tau_{31}\omega_y-\tau_{12}\omega_z)\alpha+(\tau_{23}\omega_y-\tau_{22}\omega_z)\beta\\&+(\tau_{33}\omega_y-\tau_{23}\omega_z)\gamma+[\tfrac{1}{2}(\tau_{22}-\tau_{11})e_{xy}-\tfrac{1}{2}\tau_{12}(e_{yy}-e_{xx})\\&+\tfrac{1}{2}(\tau_{23}e_{zx}-\tau_{31}e_{yz})]\beta+[\tfrac{1}{2}(\tau_{33}-\tau_{11})e_{zx}-\tfrac{1}{2}\tau_{31}(e_{zz}-e_{xx})\\&+\tfrac{1}{2}(\tau_{23}e_{xy}-\tau_{12}e_{yz})]\gamma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.14)\end{aligned}$$

and two other similar equations,

$$f_y = p_y,$$
$$f_z = p_z.$$

We may now introduce the stresses $\sigma_{\mu\nu}$ per unit area after deformation by substituting in the above equation the values (3.6) for $\tau_{\mu\nu}$ and dropping all terms of higher order than the second. Equations (3.13) then become

$$\begin{aligned}
&\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} + \frac{\partial \sigma_{31}}{\partial z} + \rho X(\xi, \eta, \zeta) + \frac{\partial}{\partial x}(\sigma_{31}\omega_y) - \frac{\partial}{\partial y}(\sigma_{12}\omega_z) \\
&\qquad + \frac{\partial}{\partial y}(\sigma_{32}\omega_y) - \frac{\partial}{\partial y}(\sigma_{22}\omega_z) + \frac{\partial}{\partial z}(\sigma_{33}\omega_y) - \frac{\partial}{\partial z}(\sigma_{23}\omega_z) \\
&\qquad + \frac{\partial}{\partial x}[\sigma_{11}(e_{yy}+e_{zz})] + \frac{\partial}{\partial y}[\sigma_{12}(e_{zz}+e_{xx})] + \frac{\partial}{\partial z}[(\sigma_{31}(e_{xx}+e_{yy})] \\
&\qquad - \frac{\partial}{\partial x}[\sigma_{12}e_{xy} + \sigma_{31}e_{zx}] - \frac{\partial}{\partial y}[\sigma_{11}e_{xy} + \sigma_{31}e_{yz}] \\
&\qquad - \frac{\partial}{\partial z}[\sigma_{11}e_{zx} + \sigma_{12}e_{yz})] = 0, \\
&\qquad \text{etc.}, \quad \ldots \ldots \ldots \ldots \qquad (3.15)
\end{aligned}$$

and the boundary conditions (3.14) become

$$\begin{aligned}
f_x = {} & \sigma_{11}\alpha + \sigma_{12}\beta + \sigma_{31}\gamma + (\sigma_{31}\omega_y - \sigma_{12}\omega_z)\alpha + (\sigma_{23}\omega_y - \sigma_{22}\omega_z)\beta \\
& + (\sigma_{33}\omega_y - \sigma_{23}\omega_z)\gamma + \sigma_{11}(e_{yy}+e_{zz})\alpha + \sigma_{12}(e_{xx}+e_{zz})\beta + \sigma_{31}(e_{xx}+e_{zz})\gamma \\
& - (\sigma_{12}e_{xy} + \sigma_{31}e_{zx})\alpha - (\sigma_{11}e_{xy} + \sigma_{31}e_{yz})\beta - (\sigma_{11}e_{zx} + \sigma_{12}e_{yz})\gamma, \\
& \text{etc.} \quad \ldots \ldots \ldots \ldots \qquad (3.16)
\end{aligned}$$

These equilibrium equations and boundary conditions are identical with those (2.7) and (2.3) found above by a different method.

4. *Material under Initial Stress.*

The previous methods may be readily applied to establish a linear theory of elasticity for small deformations in a material under initial stress. The initial state with coordinates $x\ y\ z$ is here associated with initial stresses

$$\left.\begin{matrix}
S_{xx}, & S_{xy}, & S_{zx}, \\
S_{xy}, & S_{yy}, & S_{yz}, \\
S_{zx}, & S_{yz}, & S_{zz}.
\end{matrix}\right\} \quad \ldots \ldots \qquad (4.1)$$

These initial stresses, being in equilibrium, satisfy the following equations:

$$\left.\begin{aligned}\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xy}}{\partial y}+\frac{\partial S_{zx}}{\partial z}+\rho X=0,\\ \frac{\partial S_{xy}}{\partial x}+\frac{\partial S_{yy}}{\partial y}+\frac{\partial S_{yz}}{\partial z}+\rho Y=0,\\ \frac{\partial S_{zx}}{\partial x}+\frac{\partial S_{yz}}{\partial y}+\frac{\partial S_{zz}}{\partial z}+\rho Z=0.\end{aligned}\right\} \quad (4.2)$$

If now the material undergoes small deformations, so that the initial coordinates $x\ y\ z$ become $\xi=x+u$, $\eta=y+v$, $\zeta=z+w$, the stresses undergo slight changes. Referring the stress to axes 1, 2, 3 rotating locally with the material through an amount defined by ω_x, ω_y, ω_z the stresses become

$$\sigma_{\mu\nu}=\left\{\begin{matrix} S_{xx}+s_{11}, & S_{xy}+s_{12}, & S_{zx}+s_{31},\\ S_{xy}+s_{12}, & S_{yy}+s_{22}, & S_{yz}+s_{23},\\ S_{zx}+s_{31}, & S_{yz}+s_{23}, & S_{zz}+s_{33}.\end{matrix}\right\} \quad (4.3)$$

Now we are interested here in a linear theory with respect to u, v, w, the strain, the rotation, and the stress increments, which quantities are all assumed to be small of the first order.

We adopt, therefore, for the components of the strain ϵ the first-order approximation

$$\epsilon=\left\{\begin{matrix} e_{xx}, & e_{xy}, & e_{zx},\\ e_{xy}, & e_{yy}, & e_{yz},\\ e_{zx}, & e_{yz}, & e_{zz}.\end{matrix}\right\} \quad (4.4)$$

Because the local reference axes 1, 2, 3 rotate with the material the stress increments

$$s=\left\{\begin{matrix} s_{11}, & s_{12}, & s_{31},\\ s_{12}, & s_{22}, & s_{23},\\ s_{31}, & s_{23}, & s_{33},\end{matrix}\right\} \quad (4.5)$$

depend only on the strain ϵ. These stress-strain relations may be taken linear in first-order approximation. However, some remark will have to be made later regarding the coefficients, as the properties of the latter are not the same as in the case of no initial stress.

The additional stress may be due to a change in boundary forces Δf_x, Δf_y, Δf_z or in the volume forces ΔX, ΔY, ΔZ, or both. These increments are also considered as first-order quantities. We may proceed exactly along the same lines as in the previous non-linear theory, in which the stress components (1·14) will be replaced by expression (4.3), and then drop in the formulas all the terms which are not linear with respect to the first-order quantities. For instance, the stress component referred to initial directions $x\ y\ z$ is given by expressions (1.16), in which we substitute the stress components (4.3) and drop all second-order terms. We find

$$\left.\begin{aligned}
\sigma_{xx}&=S_{xx}+s_{11}+2S_{zx}\omega_y-2S_{xy}\omega_z,\\
\sigma_{yy}&=S_{yy}+s_{22}+2S_{xy}\omega_z-2S_{yz}\omega_x,\\
\sigma_{zz}&=S_{zz}+s_{33}+2S_{yz}\omega_x-2S_{zx}\omega_y,\\
\sigma_{yz}&=S_{yz}+s_{23}+(S_{yy}-S_{zz})\omega_x-S_{xy}\omega_y+S_{xz}\omega_z,\\
\sigma_{zx}&=S_{zx}+s_{31}+(S_{zz}-S_{xx})\omega_y-S_{yz}\omega_z+S_{yx}\omega_x,\\
\sigma_{xy}&=S_{xy}+s_{12}+(S_{xx}-S_{yy})\omega_z-S_{zx}\omega_x+S_{zy}\omega_y.
\end{aligned}\right\} \quad . \quad . \quad . \quad (4.6)$$

Similarly the same substitution of the components (4·3) instead of σ (1·14) in equations (2.7), and taking into account the initial equilibrium conditions (4.2), yields the following equilibrium conditions

$$\begin{aligned}
&\frac{\partial s_{11}}{\partial x}+\frac{\partial s_{12}}{\partial y}+\frac{\partial \sigma_{13}}{\partial z}+\rho\left(\frac{\partial X}{\partial x}u+\frac{\partial X}{\partial y}v+\frac{\partial X}{\partial z}w\right)+\rho(\omega_z Y-\omega_y Z)\\
&\qquad+(S_{zz}-S_{yy})\frac{\partial \omega_z}{\partial y}-2S_{xy}\frac{\partial \omega_z}{\partial x}+S_{xy}\frac{\partial \omega_x}{\partial z}-(S_{xx}-S_{zz})\frac{\partial \omega_y}{\partial z}\\
&\qquad-2S_{zx}\frac{\partial \omega_y}{\partial x}-S_{zx}\frac{\partial \omega_x}{\partial y}+S_{zy}\left(\frac{\partial \omega_y}{\partial y}-\frac{\partial \omega_z}{\partial z}\right)+\frac{\partial S_{xx}}{\partial x}(e_{yy}+e_{zz})\\
&\qquad+\frac{\partial S_{xy}}{\partial y}(e_{xx}+e_{zz})+\frac{\partial S_{xz}}{\partial z}(e_{xx}+e_{yy})-\left(\frac{\partial S_{xy}}{\partial x}+\frac{\partial S_{xx}}{\partial y}\right)e_{xy}\\
&\qquad-\left(\frac{\partial S_{xz}}{\partial y}+\frac{\partial S_{xy}}{\partial z}\right)e_{yz}-\left(\frac{\partial S_{xx}}{\partial z}+\frac{\partial S_{xy}}{\partial x}\right)e_{zx}=0,\\
&\qquad\text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.7)
\end{aligned}$$

The boundary conditions for the increment of boundary force Δf are

$$\begin{aligned}
\Delta f_x&=s_{11}\alpha+s_{12}\beta+s_{31}\gamma+[S_{zx}\omega_y-S_{xy}\omega_z)\alpha+[S_{yz}\omega_y-S_{yy}\omega_z)\beta\\
&\quad+S_{zz}\omega_y-S_{yz}\omega_z)\gamma+S_{xx}[e_{yy}+e_{zz}]\alpha+S_{xy}(e_{xx}+e_{zz})\beta+S_{zx}(e_{xx}+e_{yy})\gamma\\
&\quad-(S_{xy}e_{xy}+S_{zx}e_{zx})\alpha-(S_{xx}e_{xy}+S_{zx}e_{yz})\beta-(S_{xx}e_{zx}+S_{xy}e_{yz})\gamma. \quad . \quad (4.8)
\end{aligned}$$

We recognize here in equation (4.7) terms of the same physical nature as in equation (2.7) of the non-linear theory. The curvature has no effect when the initial stress is hydrostatic. When the initial stress field is homogeneous the buoyancy terms disappear, and we are left with those of the first line and the curvature terms. It can be seen that the curvature terms are those playing the fundamental rôle in buckling phenomena.

We may also refer the stresses to the original areas, *i. e.*, use the stress tensor $\tau_{\mu\nu}$ instead of $\sigma_{\mu\nu}$. The stress components before the deformation are the same as above (4.1). After deformation the stress $\tau_{\mu\nu}$ is

$$\tau_{\mu\nu}=\left\{\begin{array}{lll} S_{xx}+t_{11}, & S_{xy}+t_{12}, & S_{zx}+t_{31}, \\ S_{xy}+t_{12}, & S_{yy}+t_{22}, & S_{yz}+t_{23}, \\ S_{zx}+t_{31}, & S_{yz}+t_{23}, & S_{zz}+t_{33}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (4.9)$$

However, the stress increments $t_{\mu\nu}$ are not the same as the increments $s_{\mu\nu}$ used before. In fact from relation (3.6) above we derive

$$\begin{aligned}\tau_{\mu\nu}&=s_{\mu\nu}+S_{\mu\nu}e-\tfrac{1}{2}\overset{\alpha}{\Sigma}(S_{\alpha\mu}e_{\alpha\nu}+S_{\alpha\nu}e_{\alpha\mu}),\\ e&=e_{xx}+e_{yy}+e_{zz}.\end{aligned}$$

Explicitly

$$\left.\begin{aligned}
t_{11}&=s_{11}+eS_{xx}-S_{xx}e_{xx}-S_{xy}e_{xy}-S_{zx}e_{zx},\\
t_{22}&=s_{22}+eS_{yy}-S_{xy}e_{xy}-S_{yy}e_{yy}-S_{yz}e_{yz},\\
t_{33}&=s_{33}+eS_{zz}-S_{zx}e_{zx}-S_{yz}e_{yz}-S_{zz}e_{zz},\\
t_{23}&=s_{23}+eS_{yz}-\tfrac{1}{2}(e_{yy}+e_{zz})S_{yz}-\tfrac{1}{2}(S_{yy}+S_{zz})e_{yz}-\tfrac{1}{2}(S_{yz}e_{zx}+S_{zx}e_{xy}),\\
t_{31}&=s_{31}+eS_{zx}-\tfrac{1}{2}(e_{zz}+e_{xx})S_{zx}-\tfrac{1}{2}(S_{zz}+S_{xx})e_{zx}-\tfrac{1}{2}(S_{xy}e_{yz}+S_{yz}e_{xy}),\\
{}_{12}&=s_{12}+eS_{xy}-\tfrac{1}{2}(e_{xx}+e_{yy})S_{xy}-\tfrac{1}{2}(S_{xx}+S_{yy})e_{xy}-\tfrac{1}{2}(S_{zx}e_{yz}+S_{yz}e_{zx}).
\end{aligned}\right\} \quad . \quad (4.10)$$

The equilibrium equations with these components are

$$\begin{aligned}
&\frac{\partial t_{11}}{\partial x}+\frac{\partial t_{12}}{\partial y}+\frac{\partial t_{13}}{\partial z}+\rho\Delta x+\rho\left(\frac{\partial X}{\partial x}u+\frac{\partial X}{\partial y}v+\frac{\partial X}{\partial z}w\right)+\frac{\partial}{\partial x}(S_{zx}\omega_y)\\
&\quad-\frac{\partial}{\partial x}(S_{xy}\omega_z)+\frac{\partial}{\partial y}(S_{yz}\omega_y)-\frac{\partial}{\partial y}(S_{yy}\omega_z)+\frac{\partial}{\partial z}(S_{zz}\omega_y)-\frac{\partial}{\partial z}(S_{yz}\omega_z)\\
&\quad+\frac{1}{2}\frac{\partial}{\partial y}[S_{yy}-S_{xx})e_{xy}]+\frac{1}{2}\frac{\partial}{\partial z}[(S_{zz}-S_{xx})e_{zx}]\\
&\quad-\frac{1}{2}\frac{\partial}{\partial y}[S_{xy}(e_{yy}-e_{xx})]-\frac{1}{2}\frac{\partial}{\partial z}[S_{zx}(e_{zz}-e_{xx})]\\
&\quad+\frac{1}{2}\frac{\partial}{\partial y}[S_{yz}e_{zx}-S_{zx}e_{yz}]+\frac{1}{2}\frac{\partial}{\partial z}[S_{yz}e_{xy}-S_{xy}e_{yz}]=0. \quad . \quad . \quad . \quad (4.11)
\end{aligned}$$

We also have the boundary condition for the increment f of the boundary force

$$\begin{aligned}
\Delta f_x&=t_{11}\alpha+t_{12}\beta+t_{31}\gamma+(S_{zx}\omega_y-S_{xy}\omega_z)\alpha+S_{yz}\omega_y-S_{yy}\omega_z)\beta\\
&\quad+(S_{zz}\omega_y-S_{yz}\omega_z)\gamma+[\tfrac{1}{2}(S_{yy}-S_{xx})e_{xy}-\tfrac{1}{2}(e_{yy}-e_{xx})S_{xy}\\
&\quad+\tfrac{1}{2}(S_{yz}e_{zx}-S_{zx}e_{yz})]\beta+[\tfrac{1}{2}(S_{zz}-S_{xx})e_{zx}-\tfrac{1}{2}(e_{zz}-e_{xx})S_{zx}\\
&\quad+\tfrac{1}{2}(S_{yz}e_{xy}-S_{xy}e_{yz})]\gamma,\\
&\quad\text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.12)
\end{aligned}$$

These equations are readily verified to be equivalent to those found by C. B. Biezeno and H. Hencky[(4)], but derived by an entirely different method.

The same result may be derived from the strain-energy viewpoint. The strain-energy corresponding to a linear theory must contain both the linear and the quadratic terms; therefore we must use here the second-order approximation (1.12) for the strain $\epsilon_{\mu\nu}$. We substitute the component (4.9) for $\tau_{\mu\nu}$ in the potential energy variation (3.5). We find

$$\delta \mathrm{W}=\overset{\mu\nu}{\Sigma} t_{\mu\nu}\delta\epsilon_{\mu\nu}+\overset{\mu\nu}{\Sigma}\mathrm{S}_{\mu\nu}\delta\epsilon_{\mu\nu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.13)$$

The notation S_{11}, S_{12} .... etc. is used for S_{xx}, S_{xy} .... etc.

We have assumed that the stress increments are linear functions of the strains

$$t_{\mu\nu}=\overset{kr}{\Sigma}\mathrm{C}^{kr}_{\mu\nu}\epsilon_{rk}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.14)$$

The sum is extended *to all six combinations of k and r.* The assumption that there exists a strain-energy implies that $\delta\mathrm{W}$ is an exact differential in $\delta\epsilon_{\mu\nu}$, hence that

$$\frac{\partial t_{\mu\nu}}{\partial\epsilon_{kr}}=\frac{\partial t_{rk}}{\partial\epsilon_{\mu\nu}} \quad \text{or} \quad \mathrm{C}^{kr}_{\mu\nu}=\mathrm{C}^{\mu\nu}_{kr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.15)$$

This implies fifteen relations between the coefficients of the stress-strain relation expressed by the fact that the matrix of the coefficients of the stress-strain relation is symmetric. However, it is important to notice that this only holds when we use the $t_{\mu\nu}$ stress components, *i. e.*, the stresses per unit original area before deformation.

If we use the actual stresses $s_{\mu\nu}$ referred to the area after deformation we may write the linear stress-strain relations in the form

$$s_{\mu\nu}=\overset{kr}{\Sigma}\mathrm{B}^{kr}_{\mu\nu}\,\epsilon_{kr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.16)$$

The stress s is related to the stress t by relations (4.10), in which we may indifferently write ϵ instead of e. Using these relations the conditions (4.15) above become

$$\mathrm{B}^{kr}_{\mu\nu}+\mathrm{S}_{\mu\nu}\frac{\partial\epsilon}{\partial\epsilon_{\mu\nu}}-\frac{1}{2}\frac{\partial}{\partial\epsilon_{kr}}\overset{\alpha}{\Sigma}(\mathrm{S}_{\alpha\mu}\epsilon_{\alpha\nu}+\mathrm{S}_{\alpha\nu}\epsilon_{\alpha\mu})$$
$$=\mathrm{B}^{\mu\nu}_{kr}+\mathrm{S}_{kr}\frac{\partial\epsilon}{\partial\epsilon_{\mu\nu}}-\frac{1}{2}\frac{\partial}{\partial\epsilon_{\mu\nu}}\overset{\alpha}{\Sigma}(\mathrm{S}_{\alpha k}e_{\alpha r}+\mathrm{S}_{\alpha r}\epsilon_{\alpha k}). \quad . \quad . \quad . \quad (4.17)$$

We can see that in general the coefficients B do not constitute a symmetric matrix. For instance, in two dimensions the stress-strain relations (4.14) for the stress $t_{\mu\nu}$ referred to the original area is

$$t_{11}=\mathrm{C}^{11}_{11}\epsilon_{11}+\mathrm{C}^{22}_{11}\epsilon_{22}+\mathrm{C}^{12}_{11}\epsilon_{12},$$
$$t_{22}=\mathrm{C}_{22}\epsilon_{11}+\mathrm{C}^{22}_{22}\epsilon_{22}+\mathrm{C}^{12}_{22}\epsilon_{12},$$
$$t_{12}=\mathrm{C}^{11}_{12}\epsilon_{11}+\mathrm{C}^{22}_{12}\epsilon_{22}+\mathrm{C}^{12}_{22}\epsilon_{22},$$

with
$$C^{11}_{22}=C^{22}_{11},$$
$$C^{11}_{12}=C^{12}_{11},$$
$$C^{22}_{12}=C^{12}_{22}.$$

The stress $s_{\mu\nu}$ referred to the actual areas after deformation is related to $t_{\mu\nu}$ by relation (4.10). They are in two dimensions

$$t_{11}=s_{11}+S_{11}\epsilon_{22}-S_{12}\epsilon_{12},$$
$$t_{22}=s_{22}+S_{22}\epsilon_{11}-S_{12}\epsilon_{12},$$
$$t_{12}=s_{12}+\tfrac{1}{2}(e_{11}+\epsilon_{22})S_{12}-\tfrac{1}{12}(S_{11}+S_{22})\epsilon_{12}.$$

The stress-strain relations for the stress s are

$$s_{11}=B^{11}_{11}\epsilon_{11}+B^{22}_{11}\epsilon_{22}+B^{12}_{11}\epsilon_{12},$$
$$s_{22}=B^{11}_{22}\epsilon_{11}+B^{22}_{22}\epsilon_{22}+B^{12}_{22}\epsilon_{12},$$
$$s_{12}=B^{11}_{12}\epsilon_{11}+B^{22}_{12}\epsilon_{22}+B^{12}_{12}\epsilon_{12}.$$

The coefficients B must satisfy the relations (4.17), which become in this case

$$B^{12}_{11}+S_{11}=B^{11}_{12}+S_{22},$$
$$B^{12}_{11}-S_{12}=B^{11}_{12}+\tfrac{1}{2}S_{12},$$
$$B^{12}_{22}-S_{12}=B^{22}_{12}+\tfrac{1}{2}S_{12}.$$

The coefficients in this case will in general not be symmetric except when the initial stress is a hydrostatic pressure

$$S_{11}=S_{22}, \qquad S_{12}=0.$$

Let us now go back to the expression (4.13) for the variation of potential energy δW. The quantity $\delta W'=\overset{\mu\nu}{\Sigma}t_{\mu\nu}\delta\epsilon_{\mu\nu}$ is the differential of an homogeneous quadratic form W' in $\epsilon_{\mu\nu}$. We have

$$\frac{\partial W'}{\partial\epsilon_{\mu\nu}}=t_{\mu\nu}.$$

According to Euler's theorem for homogeneous forms

$$W'=\tfrac{1}{2}\Sigma\frac{\partial W'}{\partial\epsilon_{\mu\nu}}\epsilon_{\mu\nu}=\tfrac{1}{2}\overset{\mu\nu}{\Sigma}t_{\mu\nu}\epsilon_{\mu\nu}$$

Hence the expressions for the strain-energy in a body under initial stress

$$W=\tfrac{1}{2}\overset{\mu\nu}{\Sigma}t_{\mu\nu}\epsilon_{\mu\nu}+\overset{\mu\nu}{\Sigma}S_{\mu\nu}\epsilon_{\mu\nu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.18)$$

The value of the strain $\epsilon_{\mu\nu}$ must be that given by formulas (1.12) above. However, in the quadratic expressions $\Sigma t_{\mu\nu}\epsilon_{\mu\nu}$ we may substitute $e_{\mu\nu}$ for $\epsilon_{\mu\nu}$, as this does not affect the second-order terms, and write

$$W=\tfrac{1}{2}\overset{\mu\nu}{\Sigma}t_{\mu\nu}e_{\mu\nu}+\overset{\mu\nu}{\Sigma}S_{\mu\nu}\epsilon_{\mu\nu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.19)$$

By using the principle of virtual work with this value of the potential energy we derive equations (4.11)

We may also introduce the stress components referred to the actual areas. Using relations (4.10)

$$W=\frac{1}{2}\sum^{\mu\nu} e_{\mu\nu}[s_{\mu\nu}+eS_{\mu\nu}-\sum^{\alpha}(S_{\alpha\mu}e_{\alpha\nu}+S_{\alpha\nu}e_{\alpha\mu})]+\sum^{\mu\nu} S_{\mu\nu}\epsilon_{\mu\nu}.$$

Applying the principle of virtual work with this value of the potential energy we derive equation (4.7).

5. *The Special Case of Large Rotations and Small Strains.*

This case is of special interest because small strains justify the application of Hooke's Law as a stress-strain relation. However, from identities (2.6) we know that the strain and the rotation cannot be independent. Therefore the statement that rotations are large with respect to the strains implies the fulfilment of certain extra conditions, as, for instance, that the thickness of the body is small with respect to the other dimensions.

When introducing this assumption it is most convenient to use the variational method, and introduce it in the expression for the variation of the strain-energy

$$\delta W=\sum^{\mu\nu}\tau_{\mu\nu}\delta\epsilon_{\mu\nu}. \qquad (5.1)$$

We calculate $\delta\epsilon_{\mu\nu}$ from expressions (1.12). For instance,

$$\delta\epsilon_{11}=\delta\epsilon_{xx}+(\epsilon_{xy}+\omega_z)\delta\omega_z+(\omega_y-e_{zx})\delta\omega_y+\omega_z\delta\epsilon_{xy}-\omega_y\delta e_{zx}. \qquad (5.2)$$

If we neglect e_{xy} and e_{zx} against ω_z and ω_y respectively, we may write

$$\delta\epsilon_{11}=\delta\epsilon_{xx}+\omega_z(\delta\omega_z+\delta\epsilon_{xy})+\omega_y(\delta\omega_y-\delta\epsilon_{zx}).$$

It is important to notice that this approximation may not be introduced right away in the expression of ϵ_{11}, but only in $\delta\epsilon_{11}$. In other words, *the principal part of the variatino $\delta\epsilon_{\mu\nu}$ is not the variation of the principal part of $\epsilon_{\mu\nu}$.* This remark holds if we start from the expression W of the potential energy; *the principal part of the variation δW is not the variation of the principal part of* W.

Expressing that the variation of the total internal (3.9) and external (3.10) work vanishes, using expressions such as (5.2) for $\delta\epsilon_{\mu\nu}$, we find the equations of equilibrium

$$\frac{\partial\tau_{11}}{\partial x}+\frac{\partial\tau_{12}}{\partial y}+\frac{\partial\tau_{13}}{\partial z}+\rho X(\xi,\eta,\zeta)+\frac{\partial}{\partial x}(\tau_{31}\omega_y)-\frac{\partial}{\partial x}(\tau_{12}\omega_z)$$
$$+\frac{\partial}{\partial y}(\tau_{23}\omega_y)-\frac{\partial}{\partial y}(\tau_{22}\omega_z)+\frac{\partial}{\partial z}(\tau_{33}\omega_y)-\frac{\partial}{\partial z}(\tau_{23}\omega_z)=0,$$
$$\text{etc.,} \qquad (5.4)$$

with the boundary conditions

$$f(x)=\tau_{11}\alpha+\tau_{12}\beta+\tau_{31}\gamma+(\tau_{31}\omega_y-\tau_{12}\omega_z)\alpha$$
$$+(\tau_{23}\omega_y-\tau_{22}\omega_z)+\beta(\tau_{33}\omega_y-\tau_{23}\omega_z)\gamma.$$

Similar equations are found for the linearized theory of a body under high initial stress. Such equations for the two-dimensional case have been derived already in 1934 by the author[5].

We have written the above equations with the stress components $\tau_{\mu\nu}$ roferred to the initial areas. We may substitute in these equations the values (3.6) of $\tau_{\mu\nu}$ in terms of $\sigma_{\mu\nu}$.

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