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Coupled Oscillations of Aircraft Engine-Propeller Systems

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ABSTRACT

The coupled oscillations of a flexible propeller and an aircraft engine with arbitrary number of cranks is investigated. In the first part of this paper a practical method is developed to determine the complete dynamical characteristics of a rotating vibrating propeller of arbitrary shape, and the legitimacy of the approximations is checked with a numerical example. In the second part the first nine torsional frequencies of a V-12 engine, with a flexible propeller are calculated and compared to the frequencies of the free-wheeling propeller and those obtained by considering the propeller as rigid. The simultaneous influence of the engine speed on all nine frequencies and the general procedure to compute the complete critical speed spectrum are also indicated.

INTRODUCTION

THE call for light weight and highly efficient metal propellers, together with the high speeds and power of aircraft engines, has considerably increased the danger of propeller fatigue failure. In fact the vibration characteristics of the coupled engine-propeller system has become one of the primary factors which govern the choice of a propeller for a given engine. The calculation of these oscillations is a rather complicated problem and it is made especially difficult in case the propeller is coupled with an in-line engine as will probably occur more frequently in the future.

In a previous paper[1] the torsional oscillations of a crankshaft-propeller system with an arbitrary number of cranks has been calculated with the assumption that the propeller is rigid. In the present paper the flexibility of the propeller is taken into account, the crankshaft and the propeller being considered as two coupled vibrating systems, and a method is developed by which it becomes practically feasible to predict with fair accuracy the vibrational behavior of the system for all speeds of the engine.

It is clear that the propeller is a dynamical system with an infinite number of degrees of freedom, and that the problem of coupled oscillations implies the knowledge of the dynamical characteristics of the propeller. They are completely defined for the present purpose by a function of the frequency referred to hereafter as the "dynamic modulus" of the propeller. This concept is very similar to that of impedance in electrical engineering, and it may also be referred to as the mechanical impedance, as was done in the previous paper.[1] However, in view of its physical significance as a generalization of the concept of spring modulus the designation above is preferred.

The dynamic characteristics of a propeller are rather complicated and the dynamic modulus in this case is tedious to evaluate by either calculation or measurement. To increase these difficulties it also depends on the speed of rotation. Before considering the problem of oscillations of the coupled crankshaft-propeller system a semi-empirical method which leads to a rather simple evaluation of the dynamic modulus of a rotating propeller is developed.

Previous authors [2, 3, 10] have considered coupled oscillations of crankshaft and propellers from the theoretical viewpoint but limit their investigation to a rigid crankshaft and a non-rotating engine. As pointed out in the text, some of the dynamic modulus curves calculated by Meyer are erroneous and lead to paradoxical conclusions. Extensive experimental work on propeller oscillations in actual flight has been done by C. M. Kearns.[4]

THE CONCEPT OF DYNAMIC MODULUS

Consider an elastic shaft AB (Fig. 1a) rigidly clamped at B. A torque T applied at the other end A produces

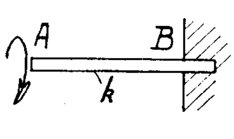

FIG. 1a. Illustrating the spring modulus of a clamped shaft.

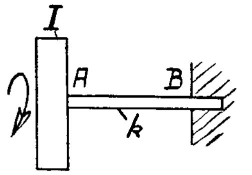

FIG. 1b. Illustrating the dynamic modulus of a mass on a shaft.

at that point a rotation θ (radians) with the proportionality relation

$$T = k\theta \tag{1}$$

The constant k which is called the "spring constant" or "modulus" of the shaft is a measure of its torsional stiffness. If instead of a constant torque, an alternating torque $T = T_0 \sin \omega t$ is applied the rotation at A follows exactly the variation of the torque and may be written $\theta = \theta_0 \sin \omega t$. At every instant relation (1) holds, and the ratio $T/\theta = k$ is a constant independent of the frequency $\omega/2\pi$.

This would of course not be the case if the shaft carried masses. Consider that the shaft carries a disc of moment of inertia I at the driving point A (Fig. 1b).

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In that case a harmonic oscillation $\theta = \theta_0 \sin \omega t$ of the disc will be produced by a harmonic torque of magnitude

$$T = (-I\omega^2 + k)\theta \tag{2}$$

This may be written $T = K\theta$ with

$$K = -I\omega^2 + k \tag{3}$$

For a given frequency the amplitude of oscillation is proportional to the torque and the system is equivalent to a massless shaft of modulus K. This equivalent modulus, however, depends on the frequency, being equal to k for small frequencies, vanishing if $\omega = \sqrt{k/I}$ and becoming negative for $\omega > \sqrt{k/I}$. This leads to the concept of a spring with a negative spring constant. The quantity K may be considered as a generalization of the concept of spring stiffness to the case of harmonic oscillations of a system possessing both elasticity and inertia. This quantity is referred to as a "dynamic modulus." Putting $k = 0$ in Eq. (3), the dynamic modulus of a disc free to rotate about its axis is

$$K = -I\omega^2 \tag{4}$$

The dynamic modulus of a pure mass is always negative.

There are simple rules by which the dynamic modulus of a chain of discs connected by elastic shafts may be rapidly evaluated. Consider a free-wheeling system (Fig. 2) and suppose that for a torque T acting at B the dynamic modulus is K. If an elastic shaft AB of

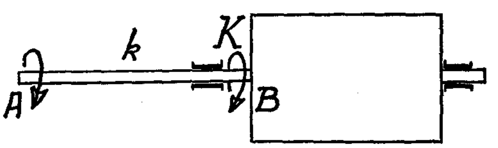

FIG. 2. Illustrating the relation between dynamic modulus at A and B.

modulus k is added, and the torque applied at A, the total angular displacement θ at the driving point A is the displacement T/K at B plus the displacement T/k of A relative to B. Hence

$$\theta = T(1/k + 1/K)$$

The new dynamic modulus K_1 at point A is therefore given by the relation

$$1/K_1 = 1/k + 1/K \tag{5}$$

Similarly, if instead of a shaft, a disc of moment of inertia I is added at the driving point B the torque necessary to produce a displacement θ becomes

$$T = \theta(-I\omega^2 + K)$$

Hence the new dynamic modulus K_1 in this case is

$$K_1 = -I\omega^2 + K \tag{6}$$

Eqs. (5) and (6) give two fundamental rules by which it is possible to calculate the dynamic modulus of any chain of masses and springs.

Consider, for example, the system of two discs on a shaft represented in Fig. 3. The dynamic modulus of

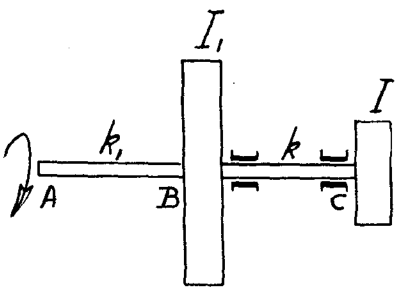

FIG. 3. Illustrating the dynamic modulus for a series of masses on a shaft.

the system at the driving point A can be found by means of the following steps. The dynamic modulus of the disc at C is $-I\omega^2$; adding the shaft k the dynamic modulus K of this system at B is given by the relation

$$\frac{1}{K} = \frac{1}{k} - \frac{1}{I\omega^2}$$

By adding the mass I_1 the dynamic modulus K_1 at B becomes

$$K_1 = -I_1\omega^2 + K$$

and the dynamic modulus K_2 at the driving point A is given by

$$\frac{1}{K_2} = \frac{1}{k_1} + \frac{1}{K_1}$$

or

$$K_2 = \frac{k_1\omega^2[I_1 I\omega^2 - k(I_1 + I)]}{I_1 I\omega^4 - (kI_1 + kI + k_1 I)\omega^2 + kk_1} \tag{7}$$

Semi-Empirical Evaluation of the Dynamic Modulus

Many practical cases involve the knowledge of the dynamic modulus of a complicated system, for which a direct computation would require too much time or simply lies beyond the power of mathematical analysis. On the other hand, a direct test involving the measurement of torque and rotation amplitude at the driving point throughout the frequency range is quite difficult and tedious. It will be shown here that it is possible to evade both difficulties by using a fundamental mathematical property of the dynamic modulus. This leads to a semi-empirical method of evaluation requiring only the measurement of a certain number of critical frequencies.

It is clear that those frequencies for which the dynamic modulus is zero correspond to a motion where no torque is applied at the driving point. Therefore the equation obtained by putting the dynamic modulus equal to zero yields the natural frequencies of the system when it is free at the driving point. For example, putting Eq. (7) equal to zero,

$$k_1\omega^2[I_1I\omega^2 - k(I_1 + I)] = 0 \tag{8}$$

The root $\omega = 0$ corresponds to the free rotation of the system in Fig. 3; the other

$$\omega_1^2 = k(I_1 + I)/I_1I \tag{9}$$

gives the natural frequency of the system when it is free at the driving point A. Similarly the equation obtained by putting the dynamic modulus equal to infinity corresponds to the case of zero amplitude at the driving point. Hence putting the denominator equal to zero in the expression for the dynamic modulus the natural frequencies of the system when it is clamped at the driving point is obtained. For example, equating to zero the denominator in Eq. (7),

$$I_1I\omega^4 - (kI_1 + kI + k_1I_1)\omega^2 + k_1k = 0 \tag{10}$$

which, as can be easily verified, is the equation giving the natural frequencies of the system in Fig. 3 when it is clamped at A.

Denote by $^*\omega_1^2$ and $^*\omega_2^2$ the two roots of Eq. (10) and call $^*\omega_1/2\pi$, $^*\omega_2/2\pi$ the "antiresonance frequencies" of the system, while $\omega_1/2\pi$ will be called the "resonance frequency." By factorizing the denominator in Eq. (7),

$$K_2 = \frac{k_1\omega^2(\omega^2 - \omega_1^2)}{(\omega^2 - {}^*\omega_1^2)(\omega^2 - {}^*\omega_2^2)} \tag{11}$$

This may be written in a more useful form by taking into account that

$$^*\omega_1^2\, {}^*\omega_2^2 = k_1k/I_1I$$

Combining this relation with the value (9) yields

$$k_1 = \frac{^*\omega_1^2\, {}^*\omega_2^2}{\omega_1^2}(I_1 + I)$$

and

$$K_2 = (I_1 + I)\omega^2 \frac{^*\omega_1^2\, {}^*\omega_2^2}{\omega_1^2} \frac{(\omega^2 - \omega_1^2)}{(\omega^2 - {}^*\omega_1^2)(\omega^2 - {}^*\omega_2^2)} \tag{12}$$

This expression shows that the dynamic modulus is completely determined by the resonance and antiresonance frequencies and by the total moment of inertia. Note that for vanishingly small frequencies

$$K_2 = -(I_1 + I)\omega^2 \tag{13}$$

In other words, the system behaves as a rigid system with the total moment of inertia $I_1 + I$.

These conclusions are quite general and hold for a system with any number of degrees of freedom. Consider such a general case in which the total moment of inertia of the system is I. Denote by ω_1, ω_2, ω_3 ... the angular frequencies of resonance, and by $^*\omega_1$, $^*\omega_2$, $^*\omega_3$ ... the angular frequencies of antiresonance. The numerator of the dynamic modulus is a polynomial in ω^2 whose roots are 0, ω_1^2, ω_2^2, ω_3^2, ..., while the denominator is a similar polynomial whose roots are $^*\omega_1^2$, $^*\omega_2^2$, $^*\omega_3^2$, ... Therefore the dynamic modulus may be written in factorized form

$$K = A\omega^2 \frac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)\ldots.}{(\omega^2 - {}^*\omega_1^2)(\omega^2 - {}^*\omega_2^2)(\omega^2 - {}^*\omega_3^2)\ldots} \tag{14}$$

The value of the constant A is determined by the condition that K reduces to $-I\omega^2$ for vanishing frequency, hence,

$$K = \pm I\omega^2 \times \frac{^*\omega_1^2\, {}^*\omega_2^2\, {}^*\omega_3^2\ldots(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)\ldots}{\omega_1^2\omega_2^2\omega_3^2\ldots(\omega^2 - {}^*\omega_1^2)(\omega^2 - {}^*\omega_2^2)(\omega^2 - {}^*\omega_3^2)\ldots} \tag{15}$$

with the plus or minus sign according to whether the total number of factors between brackets is odd or even.

This result shows that if the natural frequencies of a system for a free and clamped driving point have been measured, *i.e.*, if the resonance and antiresonance frequencies are known, the expression for the dynamic modulus may be written immediately. Experimentally, frequencies are easy to measure accurately by the excitation of resonance in the actual system or in a model. As for the moment of inertia appearing in Eq. (15) it can either be calculated or measured by the well known torsion pendulum method.

THE DYNAMIC MODULUS OF A PROPELLER

An alternating torque $T = T_0 \sin \omega t$ is applied to the shaft of a propeller, the driving point being located at the hub. The dynamic modulus of the propeller is then the ratio of the torque to the amplitude of oscillation at the hub (Fig. 4). In this case there are an infinite number of resonance and antiresonance frequencies. The resonance frequencies are the natural frequencies when the propeller is supported by a shaft free to rotate in fixed bearings (free-wheeling) (Fig. 5a), while the antiresonance frequencies are those for which the propeller is rigidly clamped at the hub (Fig. 5b).

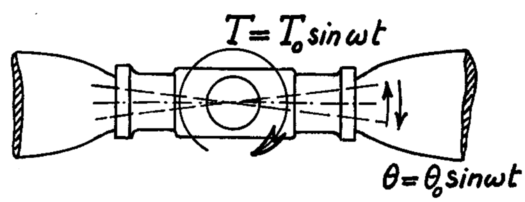

FIG. 4. Illustrating the dynamic modulus $K_p = T/\theta$ of a propeller. An alternating torque T is applied to the hub of a free-wheeling propeller.

The dynamic modulus may again be expressed in the form of Eq. (15) except that in this case there are an infinite number of factors in the numerator and the denominator. However, in practice the value of the

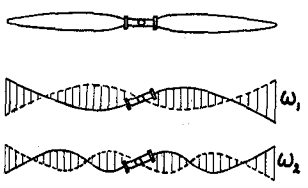

FIG. 5a. The resonance frequencies of a propeller.

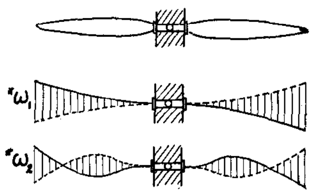

FIG. 5b. The antiresonance frequencies of a propeller.

dynamic modulus is required only in a range of frequencies below a certain upper limit and for this purpose only a finite number of factors have to be introduced in Eq. (15), namely those involving the frequencies of resonance and antiresonance located in the practical frequency range. That this is legitimate may be seen by writing the dynamic modulus in the form

$$K_p = -I\omega^2 \frac{(1 - \omega^2/\omega_1^2)(1 - \omega^2/\omega_2^2)\ldots}{(1 - \omega^2/{}^*\omega_1^2)(1 - \omega^2/{}^*\omega_2^2)\ldots} \qquad (16)$$

The factors involving the higher frequencies become equal to unity and may therefore be neglected in the infinite products. Also in the case of flexural vibrations, as is the case for a propeller, the sequence of frequencies tends to approach the sequence of the squares of the successive integers. Therefore the quantities ω^2/ω_1^2, ω^2/ω_2^2, ..., for instance, decrease as the inverse fourth powers of the successive integers and the higher factors become rapidly equal to unity.

This conclusion is further verified in the following numerical example. Consider the propeller to be a rectangular prismatic slab whose plane contains the axis of rotation. The dynamic modulus in this case is simple to compute and an expression for it may be picked out from the formulas derived by B. C. Carter[2] and J. Meyer.[3] Thus,

$$K_p = (2EJ/L)\psi(\alpha) \qquad (17)$$

with

$$\psi(\alpha) = \alpha\frac{\sin\alpha\cosh\alpha - \sinh\alpha\cos\alpha}{1 + \cos\alpha\cosh\alpha} \qquad (18)$$

$$\omega^2 = C\alpha^4 \qquad (19)$$

$$C = EJ/L^2\rho \qquad (20)$$

EJ is the flexural rigidity of the blade;
L the length of one blade;
ρ the mass of the blade per unit length.

The dimensionless variable α is proportional to the square root of the frequency. Plotting the function $\psi(\alpha)$ against α yields the curve in Fig. 6. It is found that the dynamic modulus is zero for the following values of α

$$\alpha_1 = 3.927;\ \alpha_2 = 7.069;\ \alpha_3 = 10.21;\ \text{etc.}$$

while it is infinite for the values

$$^*\alpha_1 = 1.875;\ ^*\alpha_2 = 4.694;\ ^*\alpha_3 = 7.855;\ ^*\alpha_4 = 10.99;\ \text{etc.}$$

From this the three lowest resonance frequencies and the four lowest antiresonance frequencies are derived.

$$\omega_1^2 = C\alpha_1^4;\ \omega_2^2 = C\alpha_2^4;\ \omega_3^2 = C\alpha_3^4;\ ^*\omega_1^2 = C^*\alpha_1^4;\ ^*\omega_2^2 = C^*\alpha_2^4;\ ^*\omega_3^2 = C^*\alpha_3^4;\ ^*\omega_4^2 = C^*\alpha_4^4$$

Hence the approximate value of the dynamic modulus can be computed from (15).

$$K_p = I\omega^2 \times \frac{^*\omega_1^2\,{}^*\omega_2^2\,{}^*\omega_3^2\,{}^*\omega_4^2\,(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)}{\omega_1^2\omega_2^2\,\omega_3^2(\omega^2 - {}^*\omega_1^2)(\omega^2 - {}^*\omega_2^2)(\omega^2 - {}^*\omega_3^2)(\omega^2 - {}^*\omega_4^2)} \qquad (21)$$

In this formula I is the moment of inertia of the propeller

$$I = {}^2/_3\rho L^3$$

This approximate expression for K_p can be written in a form directly comparable with Eq. (17) as follows

$$K_p = (2EJ/L)\psi'(\alpha) \qquad (23)$$

$$\psi'(\alpha) = {}^1/_3\alpha^4\left(\frac{^*\alpha_1\,{}^*\alpha_2\,{}^*\alpha_3\,{}^*\alpha_4}{\alpha_1\,\alpha_2\,\alpha_3}\right)^4 \times \frac{(\alpha^4 - \alpha_1^4)(\alpha^4 - \alpha_2^4)(\alpha^4 - \alpha_3^4)}{(\alpha^4 - {}^*\alpha_1^4)(\alpha^4 - {}^*\alpha_2^4)(\alpha^4 - {}^*\alpha_3^4)(\alpha^4 - {}^*\alpha_4^4)} \qquad (24)$$

The function $\psi'(\alpha)$ is an approximation for $\psi(\alpha)$ and is represented by the dotted line in Fig. 6. The approximate curve practically coincides with the exact value $\psi(\alpha)$ up to the value $\alpha = {}^*\alpha_3$. Beyond that point the approximate curve is still sufficiently accurate for practical purposes up to the highest root $^*\alpha_4$ used in the approximate expression.

The curve plotted in Fig. 6 is typical of a dynamic modulus function. There are a series of jumps from $+\infty$ to $-\infty$ and between these infinite values the curve crosses the horizontal axis only once. This is in accordance with a general theorem of dynamics which states that if the number of degrees of freedom of a system is decreased by one, each of the new natural frequencies lie between two of the original ones.[5] The infinite points correspond to the natural frequencies of the propeller when it is clamped at the hub, and

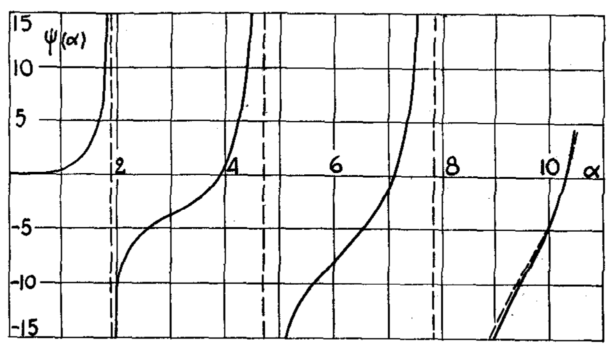

FIG. 6. Dimensionless plot for the dynamic modulus of a simplified propeller as a function of the frequency. The dotted line represents values derived from the approximate formula (24).

the points of zero modulus correspond to a freely rotating hub, therefore according to the above theorem each infinite point must lie between two zero points. It must be pointed out that curves have been computed by J. Meyer which do not satisfy this condition and show, for instance, two successive infinites of the same sign and two zero points which are not separated by any infinite point. The existence of such curves would lead to the paradoxical result that certain propeller frequencies would vanish due to coupling with the crankshaft. The contradiction is due to an error of a fundamental nature which affects Meyer's results in case the mass of the hub is taken into account.

The present method also yields the possibility of deriving the dynamic modulus of a rotating propeller and taking into account the stiffening effect of the centrifugal force on the flexural vibrations. It has been shown by W. Ramberg and S. Levy[6] that the natural frequencies of a rotating blade may be obtained accurately by Rayleigh's method using the modal shape of a non-rotating blade. It is shown moreover that the modal shapes of a rotating propeller differ very little from that of the fixed propeller in the practical range of speeds. Ramberg and Levy establish the following formula

$$\omega^2_{rot.} = \omega^2 + \alpha(\pi N/30)^2 \tag{25}$$

where $\omega_{rot.}$ and ω are, respectively, the angular frequencies of the rotating and non-rotating propeller, N is the speed of the propeller in r.p.m. and α is a coefficient

$$\alpha = \frac{\int_0^L [Ax\int_0^x (dw/dx)^2 dx]dx}{\int_0^L Aw^2 dx} \tag{26}$$

In this formula w is the deflection of the mode for the non-rotating propeller, A is the area of the blade's cross-section at the abscissa x along the blade, L is the blade length measured from the axis. Note that the value of α is independent of the amplitude of the mode and depends only on its shape.

Hence the important conclusion is reached that by measuring the shapes of the modes for a vibrating fixed propeller the natural frequencies of the rotating propeller can be derived, and thereby also the dynamic modulus of the rotating propeller through Eq. (15).

Coupled Oscillations of a 12-Cylinder Engine and a Propeller

Consider the 12-cylinder crankshaft-propeller system with the distribution of masses and elasticity indicated in Fig. 7.

$I = 0.415$ lb. in. sec.^{2}
$I_1 = 162$ lb. in. sec.^{2}
$k = 5.10\ (10^6)$ lb. in. per rad.
$k_1 = 2.05\ (10^6)$ lb. in. per rad.
$n = 6$

I_1 is the moment of inertia of the propeller and n is the number of cranks. It has been shown[1, 7, 8] that the natural frequencies $\omega/2\pi$ of this system are the roots of the equation

$$\mu n + \pi/2 + \varphi_d = \text{multiple of } \pi \tag{27}$$

with

$$\omega = 2\sqrt{k/I}\sin\mu/2 \tag{28}$$

and

$$\varphi_d = \tan^{-1}\left[\left(\frac{2k}{K_d} - 1\right)\tan\frac{\mu}{2}\right] \tag{29}$$

The quantity K_d in this expression is the dynamic modulus of the propeller and the driving shaft k_1. As mentioned in the introduction, in the previous paper[1] this quantity was called the mechanical impedance on the side of the propeller.

According to Eq. (5)

$$\frac{1}{K_d} = \frac{1}{k_1} + \frac{1}{K_p} \tag{30}$$

where K_p is the dynamic modulus of the propeller. If

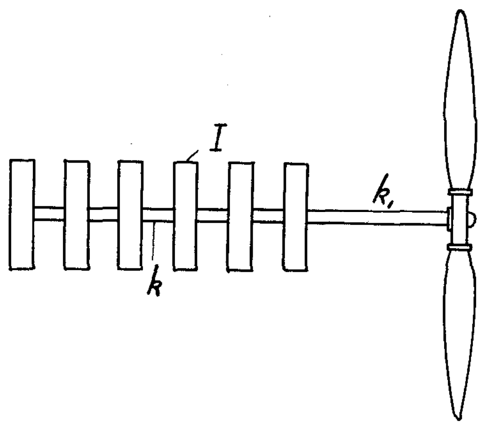

Fig. 7. Dynamical system equivalent to a 12-cylinder V-engine coupled to a flexible propeller.

the propeller is assumed rigid, $K_p = -I_1\omega^2$ is always large compared to k_1 and for practical purposes it may be assumed $K_d = k_1$. The frequencies for this case have been calculated in a previous paper.[1]

Actually the propeller is flexible and for K_p an expression of the type (15) has to be used. In the absence of experimental data on the resonance and antiresonance frequencies of an actual propeller, the frequencies are assumed to be distributed as in the case of the prismatic slab investigated above. Consider first the case of the non-rotating propeller and assume the lowest antiresonance frequency to be

$${}^*f_1 = 1800 \text{ c.p.m.}$$

Since the frequencies are proportional to α^2 the other resonance and antiresonance frequencies are derived

$$f_1 = 7{,}894 \text{ c.p.m.} \quad {}^*f_2 = 11{,}280 \text{ c.p.m.}$$
$$f_2 = 25{,}578 \text{ c.p.m.} \quad {}^*f_3 = 31{,}590 \text{ c.p.m.}$$
$$f_3 = 53{,}352 \text{ c.p.m.} \quad {}^*f_4 = 61{,}831 \text{ c.p.m.}$$

The angle ϕ_d as a function of μ is plotted in Fig. 8 using Eq. (29). The dotted line represents the function in the case of a rigid propeller.

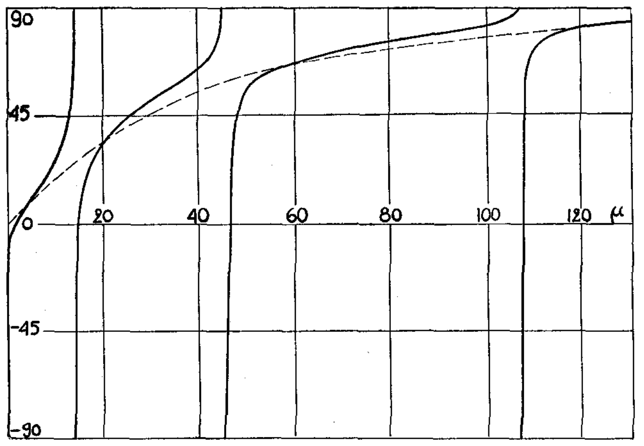

FIG. 8. The function ϕ_d of Eq. (29) plotted against μ.

Plotting now the left side of Eq. (27) the curves represented in Fig. 9 are obtained. The dotted line corresponds to the case in which the propeller is assumed rigid. It is seen that the effect of propeller flexibility is essentially to produce notches in the curve at points corresponding to the resonance frequency of the free-wheeling propeller. This curve has nine intersections with the horizontals drawn at the ordinates, π, 2π, 3π, etc. These points yield the nine roots μ of Eq. (27). The nine frequencies of the coupled crankshaft propeller system are then obtained from Eq. (28). Results are given in cycles per minute in the right hand column of the following table.

Free-Wheeling Propeller	Engine with Rigid Propeller	Engine with Flexible Propeller
	6,552	6,048
7,894		8,159
	20,965	20,169
25,578		26,158
	35,279	35,179
	48,321	48,078
53,352		53,817
	58,384	58,384
	64,668	64,668

In the left column are the resonance frequencies of the free-wheeling propeller (propeller frequencies), while the second column gives the natural frequencies of the engine with rigid propeller (engine frequencies). The propeller frequencies are increased by coupling with the engine, while the flexibility of the propeller decreases the engine frequencies. This effect is more marked with the lower than it is with the higher frequencies. Of course when the engine and the propeller are coupled it is not strictly correct to speak of the frequency of the propeller as something separate from the frequency of the engine since the system must be considered as a whole. In fact, it follows from the above method of derivation that if one of the propeller frequencies happens to coincide with one of the engine frequencies, the coupling splits this frequency into two components, or into a "doublet," to use the terminology of physics, the original frequency lying between the two new ones. In this case obviously neither of these two frequencies can be assigned to the propeller or the engine separately.

The same analysis may be repeated for various speeds of rotation of the engine. The dynamic modulus of the propeller at various speeds can be derived from the values of the resonance and antiresonance frequencies at various speeds, and these in turn may be evaluated through Eq. (25). By following the qualitative nature of the phenomenon it is possible to predict the aspect of the plot of the natural frequencies as a function of the speed of the engine. This is shown in Fig. 10. It will be noticed that when the speed increases, if one of the natural frequencies is followed

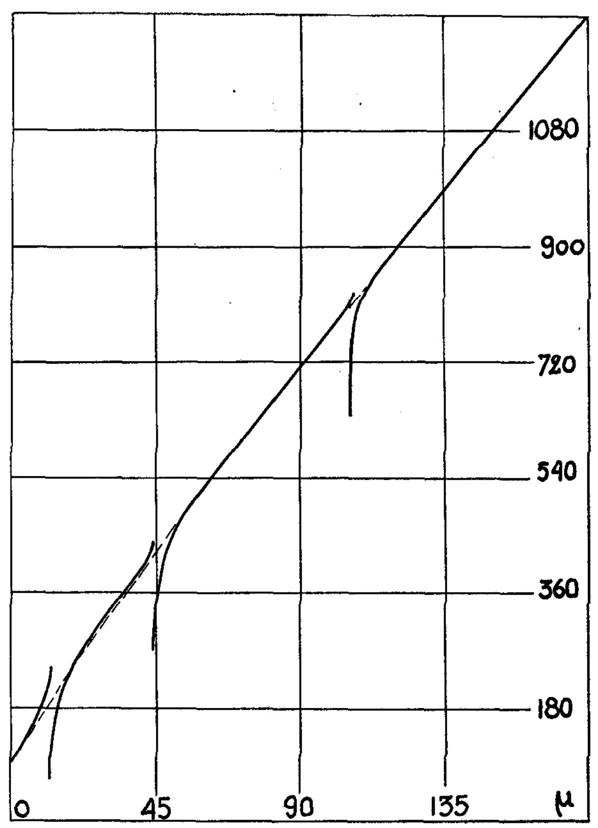

FIG. 9. The function $\mu n + \pi/2 + \varphi_d$ of Eq. (27) plotted against μ. Intersections with the horizontals determine the natural frequencies of the coupled engine propeller system.

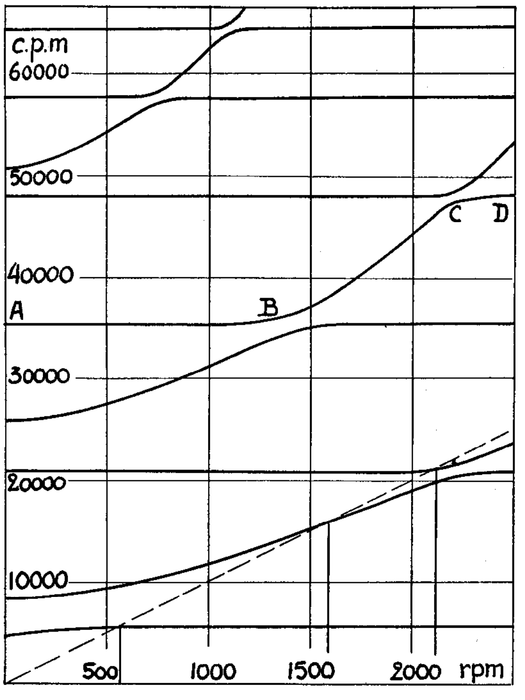

FIG. 10. Natural frequencies of the coupled V-12 engine propeller in Fig. 7 as a function of the engine speed.

through its variation, starting, for instance, with one of the engine frequencies at A, it will first stay relatively constant, then at B it will increase and become a propeller frequency, then along CD it will again become one of the engine frequencies.

The calculation of the shape of the crankshaft modes can be carried out as in the previous paper and used to calculate the energy input for each mode in terms of the firing order of the engine and the Fourier harmonics of the pressure cycle.[9] The frequency of the Fourier harmonic increases linearly with the engine speed and is represented in Fig. 10 by straight lines passing through the origin. The line for the Fourier harmonic of frequency equal to ten times the engine speed is drawn as a dotted line on the figure and the points of intersection correspond to resonance conditions. A similar line must be drawn for each Fourier harmonic of the torque cycle, hence the spectrum of critical speeds will look quite crowded. All of these speeds, however, are not necessarily dangerous because the energy input and the damping are very different from one to the other. It is believed that by computing the shapes of the vibratory modes along the lines indicated in this paper it is possible to compute approximately the energy input and the damping of each mode, hence to estimate with fair accuracy the stresses produced and their respective danger as regards shaft and propeller failures.

Conclusions

It is shown how the dynamical characteristics of a propeller are defined by its resonance and antiresonance frequencies, *i.e.*, by the natural frequencies of the free-wheeling propeller and that of the propeller rigidly clamped at the hub. This yields a practical method of evaluation of the dynamic modulus of a propeller by measuring a certain number of these frequencies within the practical range. The correctness of the method is checked on the numerical example of a vibrating slab. The dynamic modulus of a rotating propeller can be derived if the modal shapes for the non-rotating propeller are measured. It is then possible to carry out the exact calculation for the natural frequencies of a twelve-cylinder engine coupled with a flexible propeller. Denoting by engine frequencies those of the system when the propeller is considered rigid and propeller frequencies those of the free-wheeling propeller, the natural frequencies of the coupled system are close to the above values, the engine frequencies being slightly lowered and the propeller frequencies slightly increased. This effect is more marked for the lower frequencies. In the particular example above the lowest engine frequency is decreased by 8 per cent. If a propeller frequency coincides with an engine frequency, this frequency is split into two components. When the engine is rotating these frequencies vary with the speed. This variation is indicated qualitatively in Fig. 10. An interesting feature shown here is the fact that while the speed increases a particular frequency becomes alternately an engine or a propeller frequency by a continuous change.

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