AMERICAN SOCIETY OF CIVIL ENGINEERS
33 WEST 39TH STREET, NEW YORK, N. Y.

Analytical and Experimental Methods in Engineering Seismology

BY

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WITH DISCUSSION BY

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Paper No. 2183

Reprinted from TRANSACTIONS, Vol. 108 (1943), p. 365

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

TRANSACTIONS

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ANALYTICAL AND EXPERIMENTAL METHODS IN ENGINEERING SEISMOLOGY

By M. A. Biot,[1] Esq.

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Synopsis

The arithmetical steps in the computation of the spectrum are extremely lengthy. A mechanical analyzer was developed by the author at Columbia University, in New York, N. Y., to avoid this numerical work.[2] Some of the writer's earlier work with the earthquake spectrum is reviewed briefly in this paper. It also stresses engineering applications, and presents some new results, in particular regarding the effect of the foundation. Sections 1 and 2 introduce the definition of earthquake spectrum and show the results obtained for various earthquakes with the mechanical analyzer. Section 3 is a treatment of the spectrum curves obtained with the analyzer in relation to some observed facts and to the problem of stress prediction in actual structures. Section 4 considers examples of structures with more than one degree of freedom and shows how the stresses may be computed by means of the effectiveness factor. (The expression "efficiency factor" instead of "effectiveness factor" was used in the previous paper.[2]) The danger of a phenomenon referred to as the "whip effect" is also pointed out. Some attention has been given to another aspect of the problem in section 5; the effect of the foundation on the rocking motion of a rigid structure is taken into account. It is shown that in this case the same methods using a spectrum and effectiveness factors can still be applied by introducing an additional degree of freedom and a natural period corresponding to the rocking motion on the foundation.

Introduction

To a great extent the design of earthquake resistant structures is still an art based on observational facts and experience. Development of experimental

Note.—Published in January, 1942, *Proceedings*.

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[2] "A Mechanical Analyzer for the Prediction of Earthquake Stresses," by M. A. Biot, *Bulletin*, Seismological Soc. of America, Vol. 31, No. 2, April, 1941, pp. 151–171.

and analytical approaches has been very slow partly because of the lack of accurate information on the accelerations of strong-motion earthquakes and partly because of the great complexity of the phenomena involved. This gap between the empirical and scientific approaches is being reduced constantly, leading to improvement in codes and rules for the design of quake-resistant structures. Information on strong-motion earthquakes has been made available in recent years by the valuable work of the U. S. Coast and Geodetic Survey. Pooling the data furnished by the earthquake accelerograms with existing seismological knowledge only marks the beginning of analytical and experimental investigation. To picture the phenomenon in all its complexity one must imagine that the structure to be analyzed is floating on a medium in which highly irregular waves are propagating. The structure has the attributes of distributed elasticity and damping effects. Like a ship in the ocean it does not participate completely in the motion of the surrounding medium, its motion being dependent upon its own rigidity and mass and on its size relative to the waves. Internal friction and yield point in the surrounding soil must have an important effect on resonance phenomena. Also the properties of the surface layers of the earth vary greatly with location and depth so that complicated reflection, refraction, and diffraction of the waves must be expected. This is also true for agglomeration of buildings in which case the structures themselves must have a considerable influence on the intensity and direction of the waves.

Instead of approaching this problem as an entirely complex matter it must rather be expected that a solution will emerge gradually from the careful analysis of simplified cases in which the influence of each individual factor is clearly defined and checked critically against observation.

One of the simplifications usually introduced is the assumption that the ground behaves as a shaking table, the horizontal motion of which is taken to be the same as that derived from the horizontal accelerogram of an earthquake. A basic analytical approach to this problem was developed by the author in 1932,[3,4,5] in which the concept of the earthquake spectrum was introduced. This is a curve characteristic of a given earthquake which gives some kind of periodicity content by associating a certain acceleration intensity with a given period. It was shown in the earlier work how this curve could be computed and how use of it could be made for the acceleration of earthquake stresses. To this purpose, the motion of the structure is considered as the superposition of its various modes of vibration and the maximum stress produced in each mode is made to depend on a coefficient characteristic of the structure and of that particular mode. This coefficient in the present paper is referred to as the effectiveness factor. The procedure permits the easy comparison between various earthquakes and between various types of structures or modes within these structures as to the stresses produced by a given earthquake.

[3] "Transient Oscillations in Elastic Systems," by M. A. Biot, *Thesis No. 259*. Submitted to the Aeronautics Dept., California Inst. of Technology, Pasadena, Calif., in 1932 in partial fulfilment of the requirements for the Degree of Doctor of Philosophy.

[4] "Theory of Elastic Systems Vibrating Under Transient Impulse with an Application to Earthquake Proof Buildings," by M. A. Biot, *Proceedings*, National Academy of Science, Vol. 19 (1933), pp. 262–268.

[5] "Theory of Vibration of Buildings During Earthquakes," by M. A. Biot, *Zeitschrift für angewandte Mathematik und Mechanik*, Bd. 14, H. 4, (1934), pp. 213–223.

Applications of the concept of earthquake spectrum,[6] using the writer's analytical method, have been made by R. R. Martel, M. Am. Soc. C. E., and M. P. White, Assoc. M. Am. Soc. C. E., and experiments along the same line have been made by L. S. Jacobsen and N. J. Hoff.[7]

Notation.—The letter symbols in this paper are defined where they are first introduced and are assembled for reference in the Appendix.

I.—Effect of an Earthquake on a Structure with One Degree of Freedom

Consider a mass m connected to the ground by weightless springs (Fig. 1). The horizontal displacement of the mass relative to the ground is denoted by u, and the spring rigidity is such that a horizontal force F produces a displacement

$$u = \frac{F}{k} \quad (1)$$

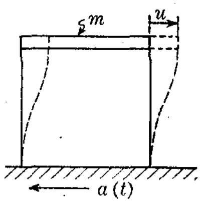

Fig. 1

The constant k is called the "spring constant."

If the ground is given a horizontal acceleration a_0 applied very gradually so that no transient oscillation occurs, the mass will assume a constant deflection:

$$u_0 = \frac{m\, a_0}{k} \quad (2)$$

The total shear in the springs is then

$$V = m\, a_0 \quad (3)$$

During an earthquake the horizontal acceleration is a function $a(t)$ of the time t. Denoting by v the displacement of the ground and neglecting the damping, the equation of motion of the mass m is

$$m \frac{d^2}{dt^2}(u + v) + k\, u = 0 \quad (4)$$

The displacements u and v are taken positive to the right and the acceleration is taken positive to the left; hence

$$\frac{d^2 v}{dt^2} = -a(t) \quad (5)$$

and Eq. 4 may be written

$$m \frac{d^2 u}{dt^2} + k\, u = m\, a(t) \quad (6)$$

Eq. 6 shows that the relative displacement u obeys the differential equation of motion of a simple oscillator under the force

$$m\, a(t) = F(t) \quad (7)$$

The earthquake is taken to start at the instant $t = 0$; the mass m being initially

[6] "Some Studies on Earthquakes and Their Effects on Structures," by R. R. Martel and M. P. White, Rept. on Earthquake Studies for Los Angeles County, Pt. I (1939) (unpublished).

[7] *Ibid.*, Pt. II.

at rest, the relative displacement u as a function of time is given by the well-known solution[8]

$$u = \frac{1}{\sqrt{k\,m}} \int_0^t F(\theta) \sin \sqrt{\frac{k}{m}}\,(t - \theta)\, d\theta \quad \text{(8a)}$$

or

$$u = \frac{T}{2\,\pi} \int_0^t a(\theta) \sin \frac{2\,\pi}{T}\,(t - \theta)\, d\theta \quad \text{(8b)}$$

in which: $F(\theta)$ = force; θ = time variable of integration; and, T, the natural period of the oscillator, equals

$$T = 2\,\pi \sqrt{\frac{m}{k}} \quad \text{(9)}$$

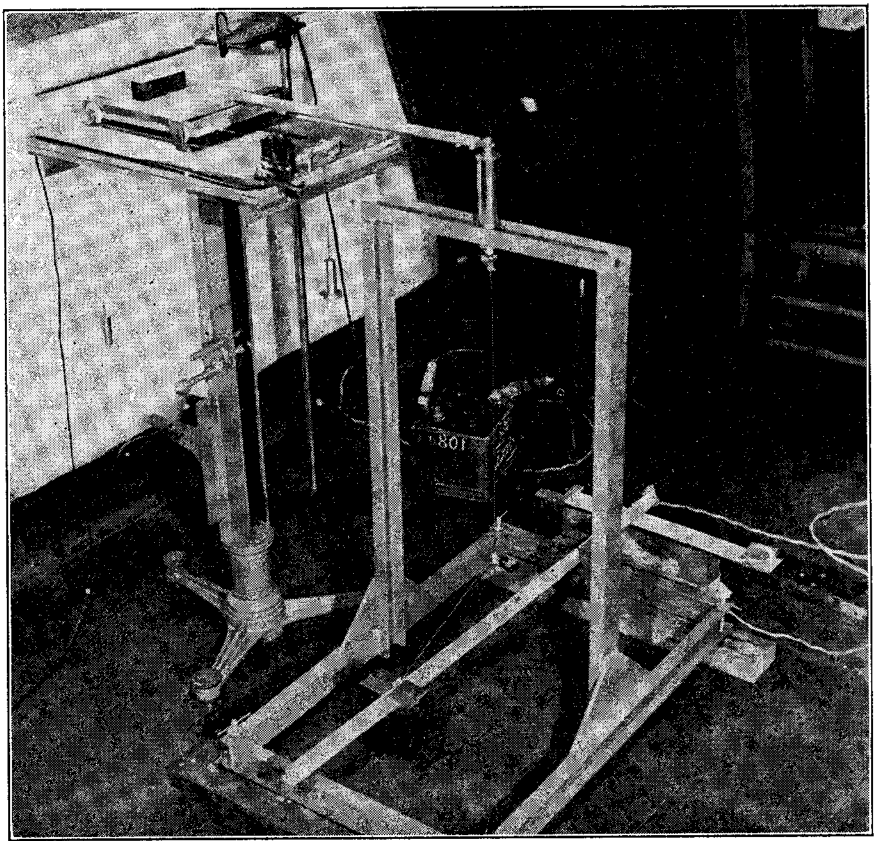

FIG. 2.—VIEW OF MECHANICAL ANALYZER

The total shear in the springs is

$$V = k\,u = m \frac{2\,\pi}{T} \int_0^t a(\theta) \sin \frac{2\,\pi}{T}\,(t - \theta)\, d\theta \quad \text{(10)}$$

[8] "Mathematical Methods in Engineering," by Theodor von Kármán and M. A. Biot, McGraw-Hill Book Co., Inc., New York, N. Y., 1940, p. 404.

The quantity:

$$f(t) = \frac{2\pi}{T} \int_0^t a(\theta) \sin \frac{2\pi}{T} (t - \theta)\, d\theta \quad \ldots\ldots\ldots\ldots\ldots (11)$$

is a function of time which gives the complete stress history of the oscillator. That is, if the integration is performed with respect to θ between the limits 0 and t, and repeated for all values of t, a function of time is obtained which, according to Eq. 10, will give the value of the total shear V at every instant t.

II.—Earthquake Spectrum Evaluated by Means of a Mechanical Analyzer

It is of special interest to find the maximum value of the stress produced by a given earthquake. Denoting by A the maximum value of Eq. 11, the maximum shear is written

$$V_{max} = m\, A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Comparing with Eq. 3, one may say that, as far as the maximum shear is concerned, the effect of the earthquake is equivalent to that of a constant acceleration A applied gradually at so slow a rate that only a statical deflection is produced without the occurrence of any transient oscillation.

Of course for a given earthquake the value of A depends on the parameter appearing in Eq. 11—that is, on the natural period T of the structure. The quantity A is referred to as the "effective acceleration" of the earthquake for the period T. It will be noticed that the effective acceleration for a particular earthquake depends only on the period of the oscillator. Therefore, one may evaluate this effective acceleration for various oscillator periods and consider it to be a characteristic function $A(T)$ of the period for each particular earthquake. This function is called the "acceleration spectrum" of the earthquake. (Use has sometimes been made of the expression "equivalent acceleration" for "effective acceleration," and "oscillator response curve" or "influence line for horizontal shear" instead of "spectrum.")

The engineering significance of this concept lies in the fact that, once the spectrum is known, it is possible to write immediately the value of the maximum shear produced by the earthquake on any undamped, one-degree-of-freedom, structure. To obtain the shear produced by an earthquake in such a structure of period T the mass of the structure is multiplied by the ordinate of the spectrum for the particular value T of the abscissa. Furthermore, as will be shown, it is possible to extend the usefulness of the spectrum to structures much more complicated than the one-degree-of-freedom oscillator considered herein.

It is relatively tedious to evaluate the spectrum by analytical methods, as this would involve the calculation of the integral (Eq. 11) from a graphically given accelerogram $a(t)$ and for a great number of values of both T and t. Fortunately there are simple experimental methods by which this can be done.

A mechanical analyzer shown in Fig. 2 has been developed for this purpose. A detailed description of the apparatus was given in a previous publication.[2] It is a torsion pendulum with variable tuning whose point of suspension can be made to turn proportionally to the acceleration of the earthquake. When the pendulum is tuned for the period T, it can be shown that its maximum amplitude yields the value of A. By varying the tuning it is then possible to plot the spectrum $A(T)$ point by point. The main advantages of this type of analyzer are its low cost, its simplicity of operation, and the fact that it takes an average of eight hours to plot one spectrum curve. The use of a torsion pendulum at the Bureau of Reclamation to evaluate earthquake stresses was mentioned by J. L. Savage, Hon. M. Am. Soc. C. E., in 1939.[9]

The following earthquakes have been analyzed (their spectrum curves are plotted in fractions of gravity g against period in seconds):

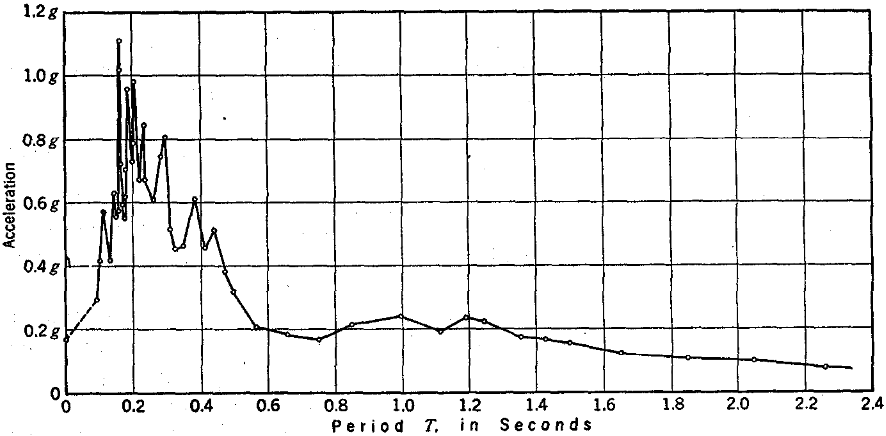

FIG. 3.—SPECTRUM OF EAST-WEST HORIZONTAL ACCELERATION OF THE EARTHQUAKE OF HELENA, MONT., OCTOBER 31, 1935

(A) Helena, Mont. (October 31, 1935) a horizontal east-west acceleration. The spectrum in Fig. 3 shows that a peak value of 1.05 g occurs for $T = 0.16$ sec. The maximum recorded acceleration of the earthquake is 0.16 g; and the amplification due to resonance is $\frac{1.05}{0.16} = 6.5$ times this value.

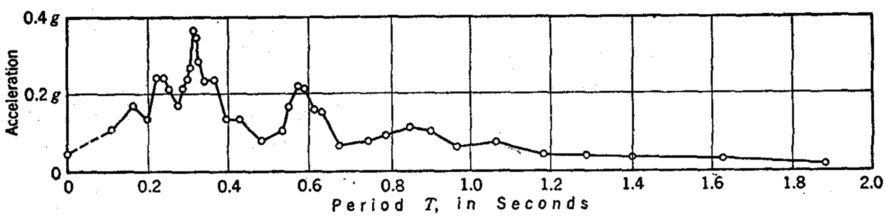

FIG. 4.—SPECTRUM OF THE NORTH-EAST HORIZONTAL ACCELERATION OF THE EARTHQUAKE OF FERNDALE, CALIF., FEBRUARY 6, 1937

(B) Ferndale, Calif. (February 6, 1937) a horizontal northeast acceleration. The spectrum in Fig. 4 shows that this is a minor earthquake. Its maximum intensity is 0.039 g, and the amplification is 9.5.

[9] "Earthquake Studies for Pit River Bridge," by J. L. Savage, *Civil Engineering*, August, 1939, pp. 470–472.

(*C*) Ferndale (September 11, 1938) horizontal accelerations in both northeast and southeast directions. The maximum recorded intensity is 0.17 g in the northeast direction and the amplification is 6. Both spectrums are found to be analogous to (*A*).

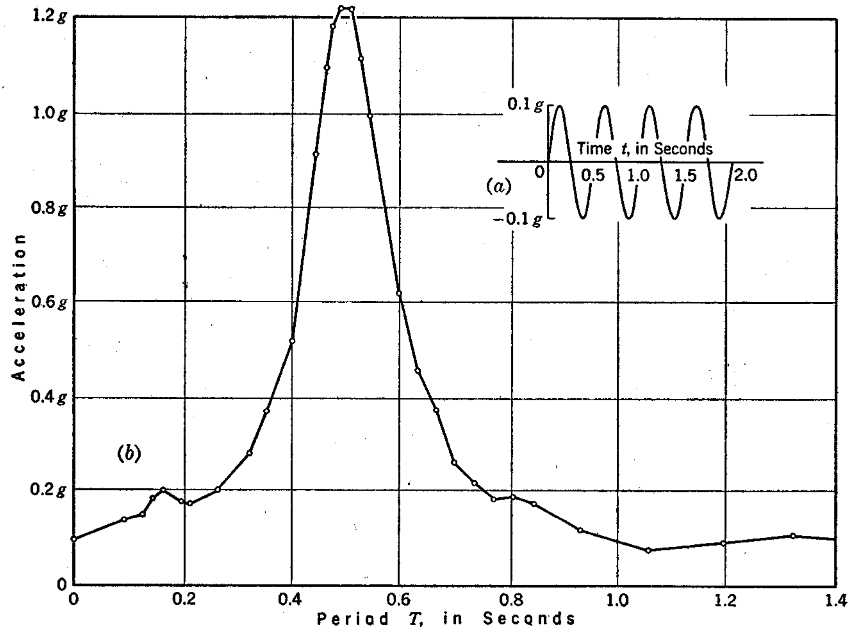

FIG. 5.—SPECTRUM OF AN ARTIFICIAL EARTHQUAKE CHARACTERIZED BY A SINUSOIDAL ACCELERATION

(*D*) A sinusoidal earthquake (Fig. 5(*a*)) of intensity 0.1 g and a duration of four cycles of 0.5 sec each was analyzed. Its spectrum is plotted in Fig. 5(*b*). The peak value of 1.23 g obtained is in good agreement with the theoretical value 1.25 g.

III.—SPECTRUM RELATED TO DESIGN AND ACTUAL BEHAVIOR OF STRUCTURES

It is apparent that, from the viewpoint of the designer, the individual sharp peaks in the spectrums are unimportant since their frequencies do not seem to be characteristic constants of the location as found by a comparison of the two Ferndale earthquakes (*B*) and (*C*) (see section II). The envelope of the spectrum, or better still, the envelope of a collection of spectrum curves obtained at the same location, constitutes the basic information for design purposes. A simple spectrum such as that plotted in Fig. 6 might well be taken to represent the Helena and Ferndale earthquakes ((*A*) and (*C*), section II). For $T > 0.2$ sec, the curve is chosen as the hyperbola

$$A = \frac{0.2\,g}{T} \qquad (13)$$

which emphasizes in quantitative form the fact that buildings of longer periods undergo smaller stresses. In general, therefore, high buildings will be less vulnerable to earthquakes than the smaller structures with shorter periods. The accelerograms from which these spectrum curves have been derived were recorded with the instrument designed by the U. S. Coast and Geodetic Survey. Its natural period is 0.1 sec. Although it is properly damped to function as an accelerograph, too much significance must not be attached to that part of the spectrum for periods smaller than 0.2 sec.

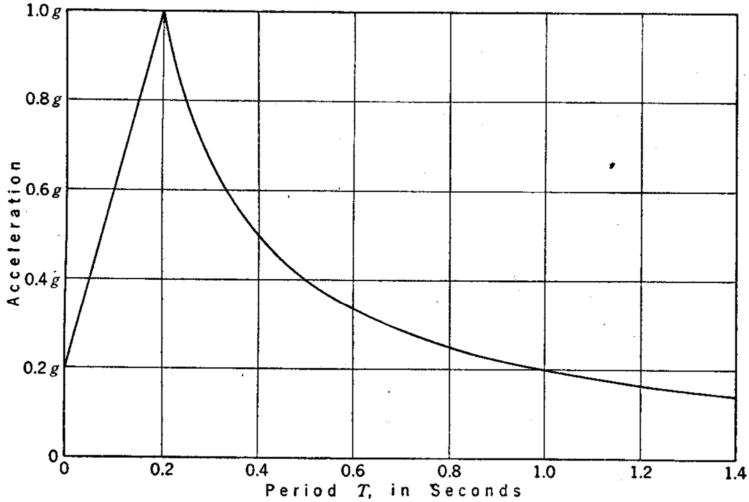

FIG. 6.—STANDARD SPECTRUM PROPOSED TO REPRESENT EARTHQUAKES (A) AND (C) FROM THE STANDPOINT OF DESIGN

Before proceeding any further it is important to give more careful consideration to the possible engineering interpretation of the foregoing results. The spectrums for earthquakes (A) and (C) indicate that an undamped structure with a period of approximately 0.2 sec would undergo a horizontal shear equal to its own weight. From observation of the effects of earthquakes this value seems rather high, but it must be remembered that it constitutes an upper limit and that actually a number of stress-reducing factors enter into play.

One of these factors, of course, is the damping influence of the structure. The magnitude of this damping effect must depend very much on the nature of the structure and the amplitude of the stresses. In fact, the damping observed with the aid of building vibrators or during minor earthquakes may very well be small, but one must be prepared to discover that in a strong earthquake the damping is considerably greater. When the amplitude of the stress reaches the yield point in some part of a structure, plastic deformation and friction will produce a high degree of energy dissipation. Assuming, for instance, that this type of damping occurs as soon as the spectrum ordinate is greater than 0.2 g, any further increase of stress by resonance will be strongly counteracted. That structural damping is an important factor agrees with the conclusion of studies by Professor White.[10]

[10] "Friction in Buildings: Its Magnitude and Its Importance in Limiting Earthquake Stresses," by M. P. White, *Bulletin*, Seismological Soc. of America, Vol. 31, No. 2, April, 1941, pp. 93–99.

Another stress reduction is that due to the influence of the foundation. This influence is threefold. When oscillations are set up in a building, strains are produced in the foundation and energy is dissipated, by internal friction in the soil. This effect depends on looseness, shear strength, internal damping, etc. A second cause of stress reduction is the radiation of elastic waves into the soil because of the motion of the building. This phenomenon was the object of a theoretical investigation by K. Sezawa and K. Kanai.[11] By this effect the energy of the oscillations is drained from the building through the foundation and radiated into the soil in the form of elastic waves. The magnitude of this effect depends on the size and natural period of the structure and on the elastic constants and density of the surrounding soil. The elasticity of the foundation will have an influence on the stresses because it increases the natural period of vibration of buildings. In other words, the building will not follow the horizontal motion of the ground but, due to the elasticity of the foundation, it will tend to rock about its center of percussion. This effect is examined in more detail in section V, and is shown to be considerable.

Finally, it must be noted that a variation of the period with amplitude is very effective against resonance effect. The stress limitation due to this factor can be important especially in strong earthquakes when large deformations and local failures occur.

From the choppy aspect of the spectrum it may be concluded that slight differences in building periods may cause great differences in earthquake stresses. This may very well be one of the reasons for the paradoxical observation that the destructiveness of an earthquake varies greatly from one building to another at the same location. The differences are not as great as the spectrums would indicate, however, but this could be explained by taking into account the influence of the damping.

IV.—Effect of an Earthquake on a Structure with Many Degrees of Freedom

The fact has been established that a number of modes of oscillation are excited by the earthquake, each of which contributes to the stress.[4, 5] It can be shown that each mode behaves like a system with a single degree of freedom and that the concept of spectrum is applicable to each mode separately. This will be demonstrated briefly by a simple example. Consider a building of height h, total mass m, and total rigidity k, both m and k being uniformly distributed (Fig. 7). The building is assumed to behave like a shear beam such that a unit horizontal displacement at the roof is produced by a force k. The shearing oscillations of the building obey the equation

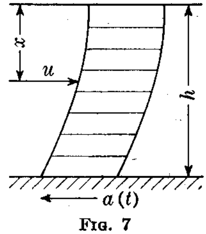

Fig. 7

$$h\,k\,\frac{\partial^2 u}{\partial x^2} + \frac{m}{h}\,a(t) = \frac{m}{h}\,\frac{\partial^2 u}{\partial t^2} \qquad (14)$$

in which u is the displacement relative to the ground at a distance x from the

[11] "Some New Problems of Forced Vibrations of a Structure," by K. Sezawa and K. Kanai, *Bulletin*, Earthquake Research Inst., Tokyo, Vol. XII, Pt. 4, December, 1934, p. 845; also "Decay in the Seismic Vibrations of a Simple or Tall Structure by Dissipation of Their Energy into the Ground," by K. Sezawa and K. Kanai, *ibid.*, Vol. XIII, Pt. 3, September, 1935, p 681.

roof, and $a(t)$ is the horizontal acceleration of the ground, taken positive in a direction opposite to u. Write the acceleration $a(t)$ as a Fourier series in x

$$a(t) = \frac{4}{\pi}\left(\cos\frac{\pi x}{2h} - \frac{1}{3}\cos\frac{3\pi x}{2h} + \frac{1}{5}\cos\frac{5\pi x}{2h} \cdots \right) a(t) \quad \ldots\ldots (15a)$$

and the solution $u(x,t)$ of Eq. 14 in the same form:

$$u(x,t) = U_1(t)\cos\frac{\pi x}{2h} + U_2(t)\cos\frac{3\pi x}{2h} + U_3(t)\cos\frac{5\pi x}{h} \quad \ldots\ldots (15b)$$

The quantities U_1, $U_2 \cdots$, represent the unknown roof deflections of each mode. Substituting Eqs. 15 into the differential Eq. 14 and equating the coefficients of $\cos\frac{\pi x}{2h}$, $\cos\frac{3\pi x}{2h}$, etc., to zero, the following equations are found:

$$\left.\begin{aligned} m\frac{d^2U_1}{dt^2} + \frac{k\pi^2}{4}U_1 &= \frac{4}{\pi}m\,a(t) \\ m\frac{d^2U_2}{dt^2} + \frac{9k\pi^2}{4}U_2 &= -\frac{4}{3\pi}m\,a(t) \\ &\cdots\cdots\cdots\cdots \\ m\frac{d^2U_n}{dt^2} + \frac{(2n-1)^2k\pi^2}{4}U_n &= \pm\frac{4m}{(2n-1)\pi}a(t) \end{aligned}\right\} \quad \ldots\ldots (16)$$

These equations are analogous to Eq. 6 for the structure with a single degree of freedom. According to Eq. 8a the roof deflection of the nth mode is:

$$U_n(t) = \frac{16m}{(2n-1)^3\pi^3k}\frac{2\pi}{T_n}\int_0^t a(\theta)\sin\frac{2\pi}{T_n}(t-\theta)\,d\theta \quad \ldots\ldots (17a)$$

in which T_n is the period of the nth mode. The largest value attained during the earthquake for the roof deflection in the nth mode can be written:

$$U_n = \frac{16m}{(2n-1)^3\pi^3k}A(T_n) \quad \ldots\ldots (17b)$$

in which $A(T_n)$ is the ordinate at $T = T_n$ of the aforementioned acceleration spectrum. The shear stress

$$V = hk\frac{\partial u}{\partial x} \quad \ldots\ldots (18a)$$

is maximum at the base, $h = x$; hence the maximum shear in the nth mode is

$$V_n = \frac{8}{(2n-1)^2\pi^2}m\,A(T_n) \quad \ldots\ldots (18b)$$

Denoting by V_g the stress which would be produced by a statical horizontal acceleration equal to gravity

$$V_n = C_n\frac{A(T_n)}{g}V_g \quad \ldots\ldots (18c)$$

in which

$$C_n = \frac{8}{(2n-1)^2 \pi^2} \quad \text{(19)}$$

The expression $\frac{A}{g} V_g$ represents the shear that would occur in a structure of one degree of freedom with the same mass and period as the particular mode considered. The factor C_n may be called an "effectiveness factor" because it represents the extent to which this shear appears in each mode of the structure with distributed mass and rigidity. A more direct method of calculating these factors is available. Assume that some restraining mechanism is used so that the building can only deflect in its fundamental mode, and apply a horizontal statical acceleration g. The deflection will be found by the energy principle. The external work, W, done by the inertia forces applied gradually is:

$$W = \frac{1}{2} \frac{m\, g\, U_1}{h} \int_0^h \cos \frac{\pi x}{2h}\, dx \quad \text{(20)}$$

The elastic potential energy E_p in this deformation is more conveniently evaluated as the kinetic energy in the fundamental mode of vibration; hence:

$$E_p = \frac{1}{2} \frac{m}{h} \left(\frac{2\pi}{T_1}\right)^2 U^2{}_1 \int_0^h \cos^2 \frac{\pi x}{2h}\, dx \quad \text{(21)}$$

Equating expressions 20 and 21 gives

$$U_1 = \frac{16}{\pi^3} \frac{m\, g}{k} \quad \text{(22a)}$$

and the shear

$$V_1 = \frac{8}{\pi^2} m\, g = \frac{8}{\pi^2} V_g \quad \text{(22b)}$$

The factor $C_1 = \frac{8}{\pi^2}$ appears in this expression, and coincides, for $n = 1$, with the value (19) already found by the more elaborate previous method. The same method is applicable to the higher modes.

The values of the coefficients for the three first excited modes are: $C_1 = 0.810$; $C_2 = 0.0900$; and $C_3 = 0.0324$. Similarly coefficients can be found for all kinds of structures provided the natural modes of vibration are known. In the earlier work[4, 5] the effectiveness factors were computed for a building with an elastic first story. Some of these values are given in Table 1 for different values of the elasticity ratio $\frac{k_1}{k}$. The rigidity k_1 of the first story is

TABLE 1.—CONSTANTS C FOR VARIOUS ELASTICITY RATIOS

$\frac{k_1}{k}$	C_1	C_2	C_3
0	1	0	0
0.556	0.993	0.0295	0.00436
2.50	0.947	0.0712	0.0155
∞	0.810	0.0900	0.0324

the force necessary to produce a unit deflection of the second floor with respect to the ground, and k is the rigidity between the second floor and the roof. Note that for a rigid first story $\left(\frac{k_1}{k} = \infty\right)$ the coefficients are the same as the foregoing for a uniform building. These coefficients make possible a comparison of the relative importance of the various modes. In a building with uniform distribution of mass and rigidity, if 100% denotes the shear that would act in a structure with one degree of freedom under an earthquake of constant spectrum the fundamental mode of the uniform building picks up 81% of the shear, whereas the second and third modes pick up only 9% and 3% of the shear. Table 1 shows that the effect of an elastic story is to decrease, further, the importance of the higher modes relatively to the fundamental.

For the case of a cable or simple truss the maximum shear stress is also given by Eq. 18c with appropriate values of C_n. The bending moment in a truss is expressed as

$$M_n = B_n \frac{A}{g} (T_n) M_g \qquad (23)$$

in which M_g denotes the maximum bending moment produced by a static horizontal force equal to gravity. The coefficients C_n and B_n depend on the type of structure and the particular mode of vibration. Values of these coefficients for a cable and a pin-ended uniform truss are given in Table 2. Only the symmetrical modes (n odd) are excited by the earthquake so that the coefficients for modes of even order are zero. It is seen that the importance of the higher modes for the bending moment tends to decrease more rapidly than for the shear. Comparing the importance of the various modes under the assumption that the spectrum follows a law of the hyperbolic type as in Eq. 13, the conclusion may be drawn that generally the higher modes are less dangerous than the fundamental. Exception must be made for the case of the shear in a flexural beam where it seems that each mode would carry about the same amount of shear. This is due to the fact that the effectiveness coefficients for shear in a mode of oscillation of the order n decreases as $1/n^2$ whereas the frequency of a flexural beam increases approximately as n^2. However, in practice, due to the increasing influence of damping in the higher modes, one would expect that their importance would always be less than that of the fundamental. The attention of the reader is called to the fact that the values of the effectiveness factors for the higher modes of a truss[2, 12] are erroneous.

TABLE 2.—COEFFICIENTS C_n AND B_n

Order of mode	$n = 1$	$n = 2$	$n = 3$
C_n for cable	0.810	0.090	0.032
C_n for pin-ended uniform truss	0.810	0.090	0.032
B_n for pin-ended uniform truss	1.03	0.038	0.008

Table 2 was applied to the evaluation of an upper limit for the stresses that would be produced by an earthquake of the Helena or Ferndale type in the San Francisco-Oakland Bay Bridge.[12] Conditions are found to be least

[12] *Transactions*, Am. Soc. C. E., Vol. 106 (1941), p. 1385.

favorable in the side-span truss which is treated as a pin-ended truss with a period of 3 sec. For this period the effective acceleration of the standard spectrum in Fig. 6 is $A = 0.066\ g$. The effectiveness factor B_1 for the bending moment in the fundamental mode is $B_1 = 1.03$; hence the bending moment is

$$M_1 = 0.068\ M_g \qquad (24)$$

This means that the maximum bending moment during the earthquake is not greater than that due to a static horizontal acceleration of 6.8% gravity. From the foregoing values of the effectiveness factors it seems as if the stresses in structures with more than one degree of freedom are of the same order as those in a structure with a single degree of freedom. This is not always true, however, and a particularly dangerous condition may arise from assuming that it is.

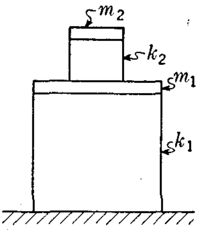

FIG. 8.—SYSTEM WITH TWO DEGREES OF FREEDOM, THE MAIN BUILDING AND THE ROOF STRUCTURE

Consider, for instance, that a mass m_1 is elastically restrained to the ground with a stiffness k_1, and that a much smaller mass m_2 is restrained to m_1 with a stiffness k_2 (Fig. 8). Denoting by u_1 and u_2 the horizontal displacement of m_1 and m_2, respectively, the equations for the amplitudes of the harmonic motion are

$$m_1 u_1 \omega^2 = k_1 u_1 + k_2 (u_1 - u_2) \qquad (25a)$$

and

$$m_2 u_2 \omega^2 = k_2 (u_2 - u_1) \qquad (25b)$$

in which ω is the circular frequency.

Assume that the small mass is "tuned" to the vibration of the large mass, by making

$$\frac{k_1 + k_2}{m_1} = \frac{k_2}{m_2} = \omega^2{}_1 \qquad (26)$$

Writing $\frac{\omega}{\omega_1} = \lambda$ the equations of motion become

$$(\lambda^2 - 1)\ u_1 + \frac{m_2}{m_1} u_2 = 0 \qquad (27a)$$

and

$$u_1 + (\lambda^2 - 1)\ u_2 = 0 \qquad (27b)$$

Hence, by elimination of u_1 and u_2,

$$\lambda^2 - 1 = \pm \sqrt{\frac{m_2}{m_1}} \qquad (28)$$

From Eqs. 27b and 28 the ratio of amplitudes is

$$\frac{u_2}{u_1} = \mp \sqrt{\frac{m_1}{m_2}} \qquad (29)$$

The lower sign refers to the fundamental mode. Now, in order to evaluate the effectiveness factor a restraint is assumed to force the structure to deflect in its fundamental mode while a static horizontal acceleration equal to gravity is applied.

The work done by the gravity force in deflecting the structure is

$$\frac{g}{2}(m_1 u_1 + m_2 u_2) = \frac{1}{2} m_1 g u_1 \left(1 + \sqrt{\frac{m_2}{m_1}}\right) \qquad (30a)$$

and the potential energy is

$$\frac{\omega^2}{2}(m_1 u^2{}_1 + m_2 u^2{}_2) = \frac{m_1}{m_2} k_2 u^2{}_1 \left(1 - \sqrt{\frac{m_2}{m_1}}\right) \qquad (30b)$$

Equating these two expressions yields

$$u_1 = \frac{1}{2} \frac{m_2 g \left(1 + \sqrt{\frac{m_2}{m_1}}\right)}{k_2 \left(1 - \sqrt{\frac{m_2}{m_1}}\right)} \qquad (31a)$$

and the shear in the spring k_2 is

$$V_1 = k_2 (u_2 - u_1) = \frac{1}{2}\left(1 + \sqrt{\frac{m_1}{m_2}}\right) m_2 g \qquad (31b)$$

Therefore the effectiveness factor is

$$C_1 = \frac{1}{2}\left(1 + \sqrt{\frac{m_1}{m_2}}\right) \qquad (32)$$

It is seen that for small values of the mass ratio $\frac{m_2}{m_1}$ this factor becomes quite large. For instance, if this mass ratio is $\frac{1}{100}$, then $C_1 = 5.5$. If an earthquake acts on such a system the stress in the spring k_2 will be

$$V_1 = 5.5 \frac{A}{g} (T_1) V_g \qquad (33)$$

or 5.5 times what it would be if the spring mass ($k_2 m_2$) were directly connected to the ground. Note that in Eq. 33, $V_g = m_2 g$ is the stress in the spring k_2 under a horizontal statical acceleration equal to gravity. This introduces an example in which the effectiveness factor C_n refers to a particular location in the structure. It is then possible to refer to the effectiveness factor "at a certain point" in the structure. The amplification effect revealed here belongs to a class of phenomena which might be designated as a "whip effect." Considered from the standpoint of wave propagation it is analogous to what happens in a whip, when a wave generated at the base is propagated to the tip. The wave may be considered as an energy lump moving from the heavy part of the whip to the lighter section at the tip. This produces at the tip a concentration of energy in a small mass and therefore gives rise to a high velocity. This effect is essentially associated with taper and explains many vibration failures in tapered beams, such as propeller tip failure or the fact that a tapered column receiving a blow at the base will fail at the tip. The case of two degrees of freedom, treated in the example, may well be considered as a simplified model of a tapered building. It points to the possibility of serious danger in roof structures and is in agreement with the observation of disproportionate damage in penthouses in the Long Beach earthquake of March, 1933. The California Building Code takes this danger into consideration by raising the lateral gravity force for which the penthouse must be designed.

The case of a structure in which the mass and stiffness both vary linearly with distance from the top was investigated by Professors Martel and White.[6] The effectiveness factor C_1 for total shear at various heights is:

Distance, $\frac{x}{h}$, from the roof	C_1
0	1.60
0.2	1.55
0.5	1.33
0.8	0.97
1.0 (base)	0.693

It is seen that these coefficients also point to the existence of a "whip" effect.

V.—Influence of Foundation on Motion of Blocks

The foregoing mathematical developments are based on the tacit assumption that the effect of an earthquake is the same as that of a shaking table. As will be shown now there is strong theoretical evidence that this concept must be modified to take care of the influence of the foundation. The elastic properties of the structure cannot be dissociated entirely from those of the ground and both must be studied simultaneously in order to predict the dynamic properties of the system.

The problem is extremely complex because it involves a complete knowledge of the propagation and properties of the seismic waves in the strongly heterogeneous surface layers of the earth as well as their diffraction and reflection by objects built on the surface. The influence of the soil on the damping of oscillations in structures was also discussed in the foregoing. An immediate answer to such a complex problem cannot be expected. It is believed, however, that the simplified problem noted in this section throws revealing light on the nature of foundation effects.

The question investigated is that of the influence of ground elasticity on the rocking motion of a block. How resistant is the surrounding soil to the rocking displacement of a foundation? What are the factors influencing this rigidity? Can this effect have a practical influence on the action of earthquakes on buildings? The problems are simplified by neglecting the mass of the soil, the internal friction in the soil, and the radiation of elastic waves due to the rocking.

Assume that the coordinate axes xy lie on the surface of the soil. A load-distribution $p(x)$-function only, of the distance x, is applied to the surface on a strip infinitely long in the y-direction and extending from $x = -L$ to $x = +L$. The distribution is asymmetric with respect to the y-axis so that

$$p(-x) = -p(x) \qquad (34)$$

It can be shown from the theory of elasticity that the soil deflection $w(x)$ is given by

$$w(x) = \frac{2}{\pi}\frac{(1-\mu^2)}{E}\int_0^L p(\xi)\log_e\left|\frac{x+\xi}{x-\xi}\right| d\xi \qquad (35)$$

in which E is Young's modulus of the soil and μ its Poisson ratio. Substitute in Eq. 35 the pressure distribution

$$p(x) = \frac{\alpha x}{\sqrt{L^2 - x^2}} \qquad (36)$$

Then, the deflection is a straight line:

$$w = 2\alpha\frac{(1-\mu^2)}{E}x \qquad (37)$$

In other words, the assumed pressure distribution (Eq. 36) is that under a rigid slab of width $2L$ rocking about the y-axis. The ratio between the elastic moment M of p about the axis and the slope $\frac{dw}{dx}$ is the elastic stiffness coefficient C_r for rocking motion

$$C_r = \frac{M}{\frac{dw}{dx}} = \frac{\pi}{4}\frac{E}{(1-\mu^2)}L^2 \qquad (38)$$

If a uniform pressure distribution equal to a constant p from $x = 0$ to $x = +L$ and $-p$ from $x = 0$ to $x = -L$ is applied, the deflection of the soil is

$$w = \frac{2}{\pi}\frac{(1-\mu^2)}{E}pL\left(\log_e\left|\frac{L+x}{L-x}\right| + \frac{x}{L}\log_e\left|\frac{L^2-x^2}{x^2}\right|\right) \qquad (39)$$

From this solution it is possible to derive, by superposition, the deflection due to uniform asymmetric load distribution extending from $x = \pm L$ to various distances from the axis. It is found that the average rocking stiffness is given by a formula similar to Eq. 38 except that the numerical coefficient is somewhat different from $\frac{\pi}{4}$. For instance, in the case in which the load extends from

$x = 1$ to $x = \frac{4}{5} L$ and $x = -1$ to $x = -\frac{4}{5} L$ the coefficient $\frac{\pi}{4}$ is replaced by $0.231\ \pi$. It may be concluded that Eq. 38 represents a reasonable approximation, independent of the load distribution.

Eq. 38 implies the knowledge of the elastic constants E and μ of the soil. These are obtainable from tests made by M. Ishimoto and K. Iida.[13] Samples from the underground of several regions within Tokyo, Japan, at depths to 60 ft, were made into the form of rectangular prisms and submitted to vibration tests. The average values of the Poisson ratio are found to be about $\mu = \frac{1}{3}$.

Some of the values found for Young's modulus are

Material	E, in lb per sq in.
Silty clay in natural state with 50% moisture	792
Same with 42% moisture	5,750
Loam	8,200

From measured velocities of waves in soils[14, 15] the following order of magnitudes may be derived:

Material	E, in lb per sq in.
Loose sand	4,000
Clay	6,000
Sandstone	100,000

D. D. Barkan[16] has experimented, under field conditions, with various foundations weighing as much as 30 tons and having various areas at the bottom as great as 90 sq ft. The foundation was tested dynamically for rocking oscillations and also statically for the rocking rigidity of the foundation slab on the soil. The results are in good agreement with Eq. 38. A length of 21 was taken to represent the size of the square foundation slab used in the test. The values of E derived from this test are

Material	E, in lb per sq in.
Loess	12,000
Water-soaked brown loam	5,000

The order of magnitude of these values is in satisfactory agreement with the aforementioned values cited from other sources.

It is now possible to derive the rocking period of oscillation of a block lying on the soil. The results established in the preceding sections have shown

[13] "Determination of Elastic Constants of Soils by Means of Vibration Methods," by M. Ishimoto and K. Iida, *The Bulletin of the Earthquake Research Institute*, Vol. XIV (1936), Pt. I; also *ibid.*, Vol. XV (1937), Pt. II.

[14] "Soil Mechanics," by D. P. Krynine, McGraw-Hill Book Co., Inc., New York, N. Y., 1941.

[15] "Praktische Anwendungen der Baugrund Untersuchungen," by W. Loos, Berlin, J. Springer, 1935, p. 53.

[16] "Field Investigations of the Theory of Vibrations of Massive Foundations Under Machines," by D. D. Barkan, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Vol. II, 1936, p. 285.

that the fundamental period of oscillation of a structure is one of the essential controlling factors in earthquake stresses. The rocking period of a block is

$$T = \frac{2\pi}{\sqrt{\frac{C_r}{m\,r^2} - \frac{g\,h}{r^2}}} \qquad (40)$$

in which C_r is the stiffness of the foundation according to Eq. 38, m the mass of the block, r the radius of gyration with respect to the rocking axis, and h the height of the center of gravity above the ground. The term $\frac{g\,h}{r^2}$ represents the destabilizing influence of gravity. The case in which

$$\frac{C_r}{m\,r^2} = \frac{g\,h}{r^2} \qquad (41)$$

corresponds to statical instability when the tipping moment due to gravity is equal to the restoring moment of the soil. Usually the term $\frac{g\,h}{r^2}$ may be neglected and if the weight $m\,g$ is written as $m\,g = 2\,p_s\,L$ the period becomes

$$T = 9.5\sqrt{\frac{p_s}{E}\frac{r^2}{g\,L}} \qquad (42)$$

The pressure p_s is an average load on the foundation due to the weight of the structure. Consider the case of hard clay, and use for p_s the bearing capacity given by the Boston Building Code—p_s = 84 lb per sq in. According to the aforecited data, a reasonable value for hard clay seems to be E = 15,000 lb per sq in. The periods given in Table 3 are then obtained. With these values of the periods the spectrum may be used to evaluate earthquake stresses as explained previously herein.

TABLE 3.—VALUES OF THE PERIOD T, IN SECONDS

2 L (ft)	Values of r, in Feet			
	5	10	50	100
5	0.39	0.78	3.9	7.8
10	0.28	0.55	2.8	5.5
50	0.12	0.24	1.2	2.4

According to the standard spectrum in Fig. 6 the stresses are inversely proportional to the period of the structure. Since the soil can have a marked effect on the period it is to be expected that it exerts also a proportional influence on the destructiveness of the earthquake. Actually, of course, structures do not behave as rigid blocks, so that their period will be influenced by both the building and the foundation rigidity. The combined periods must be used in applying the spectrum curves to evaluate earthquake stresses. According to the data in Table 3 and Eq. 42 the slenderness of a structure has a considerable effect on its vulnerability to earthquakes. This probably bears some relation to the observed fact that columns and towers sometimes show paradoxical resistance with respect to neighboring structures.

Conclusions

Experimental and analytical methods of approach have been developed for the evaluation of earthquake stresses. An earthquake may be characterized by a certain function of period, called a spectrum, which is derived from the accelerogram by means of a mechanical analyzer. By using the spectrum an upper limit can be evaluated simply for the stresses produced in undamped structures of given dynamic properties. The stresses in each mode are shown to be derived quite simply from the value of the period and a coefficient dependent on the nature of the structure and referred to as the "effectiveness factor."

The comparison of stresses calculated by this method with observed destructiveness of an earthquake on a building seems to indicate that, at least for short periods, the values of the stresses obtained are considerably higher than could be expected. The possible effect of certain stress-reducing factors is discussed, such as internal damping in the soil and in the structure, the radiation of elastic waves, etc. The influence of these factors can become very large in certain cases.

Particular attention is given to the influence of the elasticity of the foundation and it is shown that considerable stress reduction occurs through the lengthening of the natural period due to the foundation. This is equivalent to stating that rigid slender structures will have a tendency to rock about the center of percussion because of the elastic yielding of the foundation.

Attention is directed to a phenomenon referred to as the "whip" effect which increases the destructiveness of earthquakes on penthouses and the tip of tapered columns and buildings.

Spectrum curves presented herein indicate that the effectiveness of an earthquake is inversely proportional (roughly) to the period, so that increasing the period means an increase in safety. Application of the methods developed in this paper should be helpful as a guide in extending safety rules and interpreting earthquake data.

A considerable field lies open for further research, especially as regards the stress-reducing factors discussed in section III. The influence of damping, for instance, could be taken into account by using a controllable amount of damping in the mechanical analyzer. This would yield a set of "damped response curves" which would give directly the effective acceleration for a building with given period and damping. In this terminology the spectrum would correspond to the undamped response curve. Further research should also be directed toward a better knowledge of the dynamics of the foundation, the interference of structures with one another, and the influence of agglomerations on the earthquake waves themselves.

Acknowledgment

The investigation summarized in section V was undertaken as part of a research program directed by Professor Martel and sponsored by the Los Angeles County Department of Buildings and Safety.

APPENDIX

NOTATION

The following letter symbols, adopted for use in this paper, conform essentially to Standard Letter Symbols for Mechanics, Structural Engineering, and Testing Materials, prepared by a Committee of the American Standards Association with Society representation and approved by the Association in 1932:[17]

A = effective acceleration giving the maximum value of shear through Eq. 12: $A(T)$ = acceleration spectrum;

a = linear acceleration:

- a_0 = constant horizontal acceleration;
- $a(t)$ = earthquake acceleration as a function of time (accelerogram curve);
- $a(\theta)$ = acceleration as a function of the variable of integration θ (Eq. 8b);

B_n = effectiveness factor for the bending moment in the nth mode (see Table 2 and Eq. 23); extent to which bending moment appears in each mode of the structure;

C_n = effectiveness factor for the shear in the nth mode (see Tables 1 and 2 and Eq. 18c); extent to which shear appears in each mode of the structure;

C_r = coefficient defining the stiffness of a foundation (Eq. 38);

E = modulus of elasticity: E_p = potential energy;

F = force;

g = gravity constant;

h = height;

k = spring constant; rigidity: k_1 = the rigidity between ground and second floor;

L = length;

M = bending moment:

- M_n = maximum bending moment in the nth mode of a truss (Eq. 23);
- M_g = maximum bending moment produced by a static horizontal force equal to gravity;

m = mass;

p = unit pressure; uniform pressure distribution:

- p_s = average load on a foundation due to the weight of the structure;
- $p(x)$ = pressure distribution under a rocking foundation at a distance x from the axis;
- $p(\xi)$ = pressure distribution with the variable of integration ξ in the place of x;

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[17] ASA-Z10a-1932.

r = radius of gyration;
T = period of vibration; natural period of oscillation:
  $A(T)$ = acceleration spectrum;
  T_n = the period of the nth mode;
t = time;
U_n = $U_n(t)$ = largest value of roof deflection obtained during an earthquake for the nth mode, as a function of time;
u = displacement relative to the ground (u and v are taken positive to the right):
  u_0 = a constant deflection (Eq. 2);
  $u(x,t)$ = displacement at the instant t and distance x from the roof;
V = total shear in a structure:
  V_n = maximum shear in the nth mode of a cable or truss;
  V_g = maximum shear produced by a static horizontal force equal to gravity;
v = (see u);
W = work;
$w(x)$ = soil deflection;
x = distance from roof (Eq. 14); distance from rocking axis of foundation (Eq. 35);
α = (Eq. 36) a coefficient;
θ = variable of integration in Eqs. 8;
λ = ratio of natural frequency ω to ω_1 (Eq. 27a);
μ = Poisson's ratio;
ξ = variable of integration in place of x (Eq. 35);
ω = circular frequency: ω_1 = circular frequency defined by Eq. 26.

DISCUSSION

George R. Rich,[18] M. Am. Soc. C. E.—In modern seismic design, the use of arbitrary inertia loadings acting at the center of gravity of the structure has been superseded properly by the application of ground motions fairly representative of earthquakes to be expected at the site. Additional important economies and improved distribution of material still remain to be accomplished by increased recognition of the mitigating factors outlined by the author: First, the earthquake shock is transitory and the particular pattern required for resonance is generally repeated for only a few cycles; second, the effect of damping is controlling in cases approaching the resonant condition; and, third, a part of the seismic shock is dissipated in the propagation of elastic waves in the foundation due to the motion of the structure. Improved analytical technique is a distinct aid to progress along these lines.

Operational methods somewhat similar to the one used by the author in obtaining the seismic spectrum were evolved originally by Oliver Heaviside for studying the effect of transient electrical impulses and may be extended profitably, by means of the theory of functions of a complex variable,[19,20] in analyzing the effect of earthquake transients upon structures. These methods not only effect a great economy of time and labor by their directness and automatic elimination of integration constants, but they also afford a ready means of depicting the motion of the structure after the forced ground disturbance has been suppressed. In rigid structures with damping, this feature may be important since the response lags the transient driving impulse. Maximum stresses in such cases might occur after the ground motion ceases.

As an illustration, suppose it is desired to impress only n cycles of a sinusoidal ground acceleration upon Fig. 1 and to include the effect of damping. The basic differential equation is:

$$m \frac{d^2u}{dt^2} + \beta \frac{du}{dt} + k\,u = m\,a \qquad (43)$$

in which β is the coefficient of damping. For convenience, let $\beta/m = 2\,A$ and $k/m = B^2$. Then

$$\frac{d^2u}{dt^2} + 2\,A \frac{du}{dt} + B^2\,u = a \qquad (44)$$

The operational form for a sustained sinusoidal ground acceleration is $\frac{\omega\,p\,a_m}{p^2 + \omega^2}$, in which a_m is the maximum value of the ground acceleration (0.1 g in Fig. 5); ω is the angular velocity of the ground motion equal to $2\,\pi/T$, in

[18] Chf. Design Engr., TVA, Knoxville, Tenn.

[19] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939.

[20] "Operational Methods in Mathematical Physics," by Harold Jeffreys, Cambridge Tract No. 23, Cambridge Univ. Press, 2d Ed., 1931.

which T is the period in seconds; and p, in the best modern terminology, represents the transform parameter in the Mellin inversion theorem.[21, 22] Heaviside called p the operator d/dt.

The ground acceleration is suppressed after n cycles by incorporating the shift[23] operator e^{-q} (in which, to simplify typography, $q = 2\, n\, \pi\, p/\omega$), so that the operational form for the right-hand side of the differential equation becomes $(1 - e^{-q}) \dfrac{\omega\, p\, a_m}{p^2 + \omega^2}$. From the basic differential equation, the operational form of u is:

$$u \supset \frac{(1 - e^{-q})\, \omega\, p\, a_m}{(p^2 + \omega^2)\,(p^2 + 2\, A\, p + B^2)} \qquad (45)$$

For the case of interest in seismic design, the zeros of $(p^2 + 2\, A\, p + B^2)$ are complex numbers, so, for convenience, let $\phi^2 = B^2 - A^2$; and then placing $i = \sqrt{-1}$:

$$u \supset \frac{(1 - e^{-q})\, \omega\, p\, a_m}{(p + i\,\omega)\,(p - i\,\omega)\,(p + A - i\,\phi)\,(p + A + i\,\phi)} \qquad (46)$$

By the Mellin inversion theorem:

$$u = \frac{\omega\, a_m}{2\,\pi\, i} \int_{c-i\infty}^{c+i\infty} \frac{(e^{zt} - e^{zv})\, dz}{(z + i\,\omega)\,(z - i\,\omega)\,(z + A - i\,\phi)\,(z + A + i\,\phi)} \qquad (47)$$

or

$$u = \frac{\omega\, a_m}{2\,\pi\, i} \left[\int_{c-i\infty}^{c+i\infty} \frac{e^{zt}\, dz}{(z + i\,\omega)\,(z - i\,\omega)\,(z + A - i\,\phi)\,(z + A + i\,\phi)} - \int_{c-i\infty}^{c+i\infty} \frac{e^{zv}\, dz}{(z + i\,\omega)\,(z - i\,\omega)\,(z + A - i\,\phi)\,(z + A + i\,\phi)} \right] \qquad (48)$$

in which, to simplify typography, $v = t - \dfrac{2\, n\, \pi}{\omega}$.

Since the only singularities of the integrand are simple poles $z = -\, i\,\omega$; $+\, i\,\omega$; $-\,A + i\,\phi$; $-\,A - i\,\phi$; integration along the standard Bromwich contour is equivalent to $2\,\pi\, i$ times the summation of the residues[24, 25, 26, 27] at the poles. The value of the first integral inside the brackets, Eq. 48, accordingly is:

$$u = a_m \left\{ \frac{(B^2 - \omega^2)\sin\omega\, t - 2\, A\, \omega\cos\omega\, t}{(B^2 - \omega^2)^2 + (2\, A\, \omega)^2} + \frac{(2\, A^2 - B^2 + \omega^2)\,\omega\sin\phi\, t + 2\, A\, \phi\, \omega\cos\phi\, t}{e^{At}\,\phi\,[(B^2 - \omega^2)^2 + (2\, A\, \omega)^2]} \right\} \qquad (49)$$

for $t > 0$ and $< 2\, n\, \pi/\omega$.

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[21] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939, p. 117.

[22] "The Theory of Fourier Integrals," by E. C. Titchmarsh, Oxford Univ. Press, 1937, pp. 7, 46.

[23] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939, p. 129.

[24] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939, p. 53.

[25] "Functions of a Complex Variable," by E. T. Copson, Oxford Univ. Press, 1935 ,p. 117.

[26] "The Taylor Series," by P. Dienes, Oxford Univ. Press, 1931, p. 233.

[27] "The Theory of Functions," by E. C. Titchmarsh, Oxford Univ. Press, 2d Ed., 1939.

The value of u for all times between $t = 0$ and $t = 2\,n\,\pi/\omega$ is to be taken from this expression only, which, it will be noted, correctly gives $u = 0$ and $du/dt = 0$ when $t = 0$.

The values of u after suppression of the ground acceleration are given by the sum of both integrals within the brackets, Eq. 48:

$$u = a_m \left\{ \frac{(2\,A^2 - B^2 + \omega^2)\,\omega \sin \phi\, t + 2\,A\,\phi\,\omega \cos \phi\, t}{e^{A\,t}\,\phi\,[(B^2 - \omega^2)^2 + (2\,A\,\omega)^2]} \right.$$

$$\left. - \frac{(2\,A^2 - B^2 + \omega^2)\,\omega \sin \left[\phi\left(t - \frac{2\,n\,\pi}{\omega}\right)\right] + 2\,A\,\phi\,\omega \cos \left[\phi\left(t - \frac{2\,n\,\pi}{\omega}\right)\right]}{e^{A\,v}\,\phi\,[(B^2 - \omega^2)^2 + (2\,A\,\omega)\,]} \right\} \quad (50)$$

for $t > 2\,n\,\pi/\omega$.

The use of this combined value is valid only for times greater than $t = 2\,n\,\pi/\omega$. Emphasis is placed upon this important characteristic of the use of shift operators in general.[28] The time is referred to the time of quiescence as an origin, but the use of the sum of the two integrals is correct only subsequent to $t = 2\,n\,\pi/\omega$.

The operational method is not limited to impressing sinusoidal transients. Operational forms for a wide variety of impulses are available.[29] Among these may be noted as possible components of actual accelerograms the diminishing Bessel wave $p/\sqrt{p^2 + 1}$ and the Morse dash or hammer blow $(1 - e^{-p\tau})$. Operational methods also make it possible to combine these various elementary impulse forms; for example, the Bessel wave represented by the foregoing operator may be impressed immediately following a sine wave of $n + 1/4$ cycles, fairly approximating the characteristic graph observed on typical accelerograms. In snythesizing ground motions in this manner, care should be taken to incorporate in the operational derivation the fact that the system is quiescent at the start of the first impulse, but in motion at the start of the second component impulse.

N. J. Hoff,[30] Esq.—Predicting the stresses in buildings due to earthquakes is probably the most complicated problem a civil engineer may encounter. In problems of stresses caused by static or steady dynamic loads the use of refined mathematical methods may be required to obtain a rigorous solution, but, as a rule, an approximate answer satisfactory for practical design purposes can be obtained by the use of only elementary mathematics. Such is not the case with earthquake vibrations. Without an involved mathematical analysis not even the order of magnitude of earthquake stresses can be predicted.

The first part of this paper presents Professor Biot's earlier complicated theoretical contributions to the solution of problems of earthquake stresses in

[28] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939, p. 130.

[29] *Ibid.*, p. 155.

[30] Asst. Prof., Aeronautical Eng., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

a form easily understandable to structural engineers. The explanation of the concept of the "earthquake spectrum" is of special interest since it may lead eventually to a simple routine engineering approach to this involved problem.

Advantages of the mechanical analyzer in Fig. 2 are obvious for the determination of the "spectrum." It may be of interest to compare this device to the "shaking table" used in the experiments conducted in the Earthquake and Vibration Laboratory of Stanford University, Stanford University, Calif., where the writer had the opportunity to work under the direction of Professor Jacobsen. This shaking table is mounted on balls and is actuated by a cam and follower arrangement in such a way that its motion is a replica of the recorded horizontal motion of the ground during an earthquake. On the shaking table is mounted a single-degree-of-freedom vibrating system, the forced vibration of which is recorded. The spectrums of several earthquakes were established with the aid of the shaking table.

The general appearance of the spectrums obtained at Stanford University is very much the same as that of the curves in Professor Biot's paper. The maximum oscillator acceleration, however, was found to be about 0.5 g in the Stanford tests whereas Fig. 3 of the paper shows a peak value of about 1.1 g. It is suggested that the reason for this discrepancy is the intrinsically higher frictional damping of the shaking table. Systematic investigations at Stanford University showed that small changes in the friction have little effect upon the motion of the oscillator if the friction is comparatively high, but the effect is very marked when the friction is low. Of course, even the accelerations found at Stanford are much higher than those experienced by actual buildings during the same earthquake. This must be so, since the great majority of buildings designed to withstand a horizontal acceleration of 0.1 g only was not damaged. The reason may be found in the internal friction of the buildings and in the effects of the foundation as explained by Professor Biot. Nevertheless, a mechanical analyzer with low friction has theoretical advantages and a damping can always be applied to it if required.

A great advantage of Professor Biot's mechanical analyzer is its ease of operation. This is due mainly to the fact that it makes direct use of the accelerogram of the earthquake. For the investigations on the shaking table the accelerogram is integrated twice and the displacement curve so obtained is used for constructing the cam. The numerical integration is very laborious as is evident to all who ever saw an earthquake accelerogram. For this reason, Professor Martel of the California Institute of Technology suggested the use of the accelerogram for the construction of the cam. Such a procedure, however, was not found practicable since it would require a cam of prohibitively large diameter.

In the investigation of the vibrations of buildings with distributed mass and elasticity, Professor Biot made the tacit assumption that the maximum shear in the building is the greatest of the maximum shears of the different natural frequencies. In the Stanford tests it was found that in many cases the time of the occurrence of the maximum shear was approximately the same for single-degree-of-freedom models of different natural frequencies when subjected to the same ground motion. If this holds true for buildings of many degrees of freedom, the cumulative effect may cause a maximum shear greater than indicated by the spectrum. Furthermore, it is even conceivable that in such a case the maximum shear would occur somewhere else than at the base of the building. It is thought that this problem may merit some further consideration.

The calculations relative to the "whip effect" and the influence of the foundation are of great importance to practical design. It is desirable that the investigation of these problems be continued with a view to establishing a reliable and easily applicable procedure of calculation such as was achieved in the simpler cases with the aid of the concept of the earthquake spectrum. It is hoped that, in due season, Professor Biot again will be able to devote some of his time to problems of earthquake stresses in buildings.

MERIT P. WHITE,[31] ASSOC. M. AM. SOC. C. E.—Professor Biot's paper is probably one of the most significant of those which have appeared in the field of engineering seismology. Although many of the ideas presented are not new, nevertheless the fact that this is the first published attempt to present a complete picture of the earthquake problem makes it important.

As in nearly every other branch of engineering, there are two possible ways to attack the problem of earthquake resistant design. One of these represents the empirical, trial-and-error school of thought which, in general, is responsible for the present design methods. Actually, this approach can give, and has given, excellent results, partly on account of what must be called "engineering intuition." Nevertheless, the writer prefers the other approach, which may be characterized as the rational approach, which attempts to isolate and to understand the significance of the different factors involved in a problem. The rational method may be based on experiment, or on a combination of experiment and theory. Certainly, Professor Biot's paper is representative of this approach.

Mechanical Analysis of Accelerograms.—The use of the mechanical analyzer described by the author certainly will result in a saving of time when compared with the arithmetical solution for the response of an oscillator to an earthquake. However, the writer believes that the difference is not always as great as the author implies. In 1938 a comparatively rapid tabular method for solving the equation of motion was developed at the California Institute of Technology.[6] This method has the advantage that the effects of various amounts of damping can be found with little additional labor. Of course, a further advantage is the fact that such a method requires no mechanical equipment. So far as the writer knows, the first use of the mechanical analyzer for finding oscillator response to an earthquake motion was by Frank Neumann of the U. S. Coast and Geodetic Survey in 1936. In this work the earthquake displacement curve, obtained by double integration of an accelerogram, was used to govern the motion of a torsional pendulum. In 1939, Ralph E. Byrne, Jr., Jun. Am. Soc.

[31] Palmer Physical Laboratory, Princeton Univ., Princeton, N. J.

C. E., and the writer suggested a method by which an accelerogram might be used directly to actuate a mechanical analyzer.[32] This is the principle of the author's analyzer.

Peaks Appearing in Spectrum Curve.—The physical explanation for the peaks that always appear in an earthquake spectrum is a matter of importance. If these peaks truly represent characteristics of the basic earthquake motion, then, as suggested by the author, they may account for some of the paradoxical occurrences in earthquakes. However, to the writer, a more satisfying explanation is one suggested by Hugo Benioff of the California Institute of Technology; namely, that the peaks appearing in an earthquake spectrum are due to the influence of the motion of the building housing the recorder on the record made by the recorder. To test this hypothesis, extensive forced vibration tests, in which a shaking machine was used, were made by the U. S. Coast and Geodetic Survey with the assistance of Mr. Byrne and the writer in the summer of 1938.[6] It was found: (*a*) That there was definite correlation between the natural building periods and the periods of spectrum peaks for the earthquake records made by recorders in the buildings in question; and (*b*) that forced oscillation of a building gave measurable motion not only in the basement of the structure, where recording instruments may be located, but over a surrounding area, 1,000 ft or more in extent. Hence, spectrum peaks may even be caused by neighboring structures. The building coupling effect, in which a faint but regular motion is superimposed on the basic earth motion, might not be in evidence on the seismogram, but on account of its regularity it would have a large effect on the spectrum.

Change of Building Period with Amplitude.—The author states that a change of building period with amplitude of motion is effective against resonance effect. This is generally true, although it is conceivable that the opposite may occur—that is, an oscillator which is slightly off resonance may acquire enough amplitude to cause a period change, putting it in better resonance. However, if the spectrum peaks are not characteristic of the basic earth motion, true resonance cannot exist, and a small change of building period will have no particular importance.

Effect of Foundation Yielding and of Non-Uniformity of Mass Distribution or Stiffness of Building.—The general case, in which the structure is not uniform and rests on a yielding foundation (but has vertical planes of symmetry, or near symmetry), can be treated quite simply as follows: To determine the maximum shearing force at the base of the structure, the frequencies and shapes of the lower modes of vibration and the distribution of weight along the height of the building are needed. No other information regarding the foundation or the characteristics of the structure is required (damping is neglected). The natural frequencies of any building are found easily by the use of vibration meters or by comparison with similar structures in similar locations. The mode shapes (usually only the fundamental is really important) ordinarily can

[32] "Model Studies of the Vibrations of Structures During Earthquakes," by Merit P. White and Ralph E. Byrne, Jr., *Bulletin*, Seismological Soc. of America, Vol. 29, No. 2, April, 1939, pp. 327–332.

be assumed with sufficient accuracy. The distribution of weight is easily found, of course.

As was stated by the author, the total response will be the sum of the responses of the various modes, the fundamental mode predominating. Each mode will be excited in much the same way as is a simple oscillator of the same period.

Letting $Y_n(x)$ be the shape, T_n the period, and V_n the maximum shear at the base of the building, all for the nth mode of vibration, and $\rho(x)$ the mass per unit height of structure at the height x, it can be shown that

$$V_n = A(T_n) \frac{\left[\int_0^h Y_n(x)\,\rho(x)\,dx\right]^2}{\int_0^h Y^2_n(x)\,\rho(x)\,dx} = m\,A(T_n)\,R_n \qquad (51)$$

in which

$$R_n = \frac{1}{m} \frac{\left[\int_0^h Y_n(x)\,\rho(x)\,dx\right]^2}{\int_0^h Y^2_n(x)\,\rho(x)\,dx} \qquad (52)$$

In Eq. 51, $A(T_n)$ is the ordinate of the author's acceleration spectrum at $T = T_n$, h is the height of structure, and m is the total mass of the building. Thus, R_n is the multiplying factor that gives the ratio between the maximum shear at the base, caused by the nth mode, and the maximum shear in a simple oscillator of equal mass and the same period. (Note that only the shape of $Y_n(x)$ is significant, and that the scale assumed has no effect.) Eq. 51 can be shown to reduce to Eq. 18*b* of the paper for the particular case in which weight distribution and stiffness are both constant.

Now, for the fundamental mode, consider the importance of foundation yielding and of mode shape on the numerical value of the multiplying ratio R, defined by Eq. 52, as demonstrated for the following cases:

(1) Weight concentrated at a point (the simple oscillator). Ratio $R = 1.0$.

(2) Structure has uniform stiffness and weight distribution; base is rigid. Ratio $R_1 = 0.81$.

(3) Weight is uniformly distributed, and the fundamental mode is linear as in Fig. 9 (this might represent a very stiff structure on a yielding base, a structure having a particular variation of stiffness, or one in which bending and shearing deflections are about equally important). Ratio $R_1 = 0.75$.

(4) Weight uniformly distributed; fundamental mode is parabolic as in Fig. 10 (this is approximately the shape for bending deflection). Ratio $R_1 = 0.555$.

The effect of non-uniform distribution of weight will be similar to the effect of variation of mode shape. It appears that for the fundamental mode the maximum shear generally will vary between the value corresponding to the ordinate of the acceleration spectrum and about one half this amount. It can be shown that the sum of all the R_n-values for any symmetric structure must equal unity. Hence, R_1 cannot exceed unity, which is its value in the case of the simple oscillator. Also, the smaller the value of R_1, the greater must be the remaining R-values and the more important will be the shears due to the higher modes.

Relative Importance of the Fundamental and the Higher Modes.—The relative importance of the fundamental and the higher modes in a particular case will depend on: (*a*) At what point of the structure shear is determined; (*b*) the values of the different R_n-values of the structure; (*c*) the periods and shapes of the different modes; and (*d*) the shape of the acceleration spectrum.

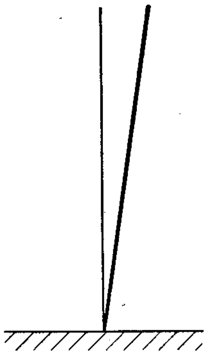

FIG. 9.—LINEAR FUNDAMENTAL MODE

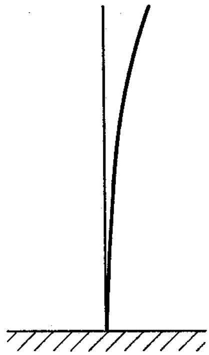

FIG. 10.—PARABOLIC FUNDAMENTAL MODE

As an illustration, consider a uniform structure on a rigid base. Then, $R_1 = 0.81$; $R_2 = 0.09$; $R_3 = 0.032$ (from Table 1); and $T_1 : T_2 : T_3 = 1 : \frac{1}{3} : \frac{1}{5}$. Assume that the spectrum shape is given by Eq. 13—that is, $A(T) = \frac{0.2\,g}{T}$. Then, $A_1 : A_2 : A_3 = 1 : 3 : 5$; and, at the base of a uniform structure, $V_1 : V_2 : V_3 = 0.81 : 0.27 : 0.16$.

Variation of Shear with Elevation.—Away from the base, the situation will be somewhat different from that at the base. For example, using the same spectrum as before, at a point one third down from the roof of a uniform building, the maximum shears due to the different modes are in the ratios $V'_1 : V'_2 : V'_3 = 0.40 : 0.27 : 0.08$. Here, the second mode shear is about two thirds as great as the fundamental shear.

The effective acceleration for a section of a building, or the ratio of the maximum shear at that point to the total mass above it, is not a constant, as is generally implied in building codes, but increases with elevation. Considering only the fundamental mode of a uniform building, the following effectiveness factors are found:

Relative distance $\frac{x}{h}$ below the roof	Effectiveness factor C'_1
0.0	1.275
0.2	1.255
0.5	1.145
0.8	0.965
1.0	0.810

Thus, the maximum acceleration at the roof of a uniform building, caused by its fundamental mode, is 1.275 times the maximum acceleration of a simple oscillator of the same period, and is $\frac{1.275}{0.81} = 1.57$ times the effective acceleration for the entire structure. This is the "whip effect" mentioned by the author. It is present in any flexible structure.

In conclusion, the writer suggests that much of the information and many of the conclusions which apply to earthquakes are also applicable to what are now even more important problems in dynamics—namely, the effects on structures of explosions, and the impact of projectiles.

N. H. HECK,[33] M. AM. SOC. C. E.—With the development of the program for recording strong earth motions, the Coast and Geodetic Survey has recognized the necessity of full utilization of the results. Its principal contribution has been the determination of velocity and displacement (oscillatory) from the accelerograms so that all the elements of motion are known. The Massachusetts Institute of Technology, Cambridge, Mass., has cooperated in testing the validity of the results and sufficiency of their accuracy for engineering purposes. Little has been done in the application of the results to engineering design. However, the collection of engineering information after a destructive earthquake and the determination of building periods and the periods of other structures, in some cases at intervals during construction, constitute a close relation with practical engineering problems.

For these reasons the paper and previously reported work of Mr. Biot have been especially welcome. Its essential feature is the use of the earthquake spectrum obtained from strong-motion accelerograms with his mechanical analyzer. The spectrum gives a direct measurement of the maximum shear in a building due to a given earthquake. One thing is disconcerting—the high maxima in the relatively weak Ferndale (Calif.) shocks. The motions at El Centro, Calif., due to the Imperial Valley earthquake of March 18, 1940, were considerably greater than had been expected and accordingly one might expect very high maximum values of the spectrum for a major earthquake.

Mr. Biot states that there are a number of reasons for believing that the internal damping of structures is appreciable, and that since he has used an undamped pendulum, high resonance values are due in some measure to lack of damping in the torsion pendulum. The analysis obviously should be extended to include some forms of damped pendulums.

In general, it has been held that the tower type of building has special advantages as an earthquake resistant type, but the whip effect on the upper stories cannot be neglected.

The effect of the building on the ground is important from the viewpoint of the Coast and Geodetic Survey strong-motion program. By engineering advice all but about six of the sites adopted were in buildings and the instruments installed in the basements probably are affected by the motion of the building in the earthquake. The fact that the building at El Centro is small has enhanced the value of the conclusions from the El Centro record.

The paper emphasizes the importance of the determinations which have been made of the vibration periods of buildings, since without this knowledge the

[33] Asst. to the Director, U. S. Coast and Geodetic Survey, Washington, D. C.

maximum stress cannot be determined for a given building. The conclusion that change of period damps out resonance is undoubtedly correct, but further investigation of the possibility that prolonged resonance in special cases may result in serious damage is needed.

The paper is stimulating to thought and opens up important lines of investigation. Interest in the subject has not been diverted by the war effort, but few persons are left to work on the problems. Interest in high-acceleration high-frequency vibrations has greatly increased but progress should be made in the entire field of vibrations including the relatively low-frequency long-period vibrations characteristic of destructive earthquakes.

FRANK NEUMANN,[34] AFFILIATE AM. SOC. C. E.—Equivalent acceleration spectrums, as well as those for velocity and displacement, open up a new and vitally important avenue of attack on the engineering aspects of the earthquake problem. The way now seems to be clear to furnish the engineer seismic data in a form that he can put to direct use. The development of spectrums by practical methods has been considered a seismological as well as an engineering problem, and workers in both fields have made important contributions toward current accomplishments.

An interesting feature of the "standardized" spectrum is that, if reduced to "equivalent maximum velocity," the curve will show a constant maximum velocity for all periods greater than 0.2 sec. This would seem to provide an "intensity factor" in standard units for any earthquake motion that has been thus analyzed. This factor, in the case of the Helena north-south component, is 2.4 times the maximum velocity of the ground motion; but the Helena "standardized" curve cannot, as Professor Biot infers,[35] be considered typical of all curves in view of the effect of epicentral distance, for one thing, on ground periods. A notable instance of this distance effect occurred in 1933 when an earthquake originating 183 miles from the San Jose (Calif.) Bank of America Building registered an acceleration, on the thirteenth floor, seven times greater than the recorded ground acceleration of 0.005 g. The approximate resonance period in this case was 1.6 sec. The need for families of spectrums is thus obvious if earthquake motions are to be adapted to standard patterns. It appears that, for any one earthquake, the acceleration spectrums at different epicentral distances may be restricted in magnitude to the limits of a spectrum obtained very close to the epicenter, barring all geological influences.

It is thought that the apparent multiplicity of periods in the short-period end of the spectrum may be due, in some measure at least, to difficulty in following the acceleration curve during the analyzer work with the necessary precision, especially since resonance is likely to occur between the motion of the torsion-head lever arm (which is manually operated) and the motion of the pendulum. In a similar operation in 1936[36] (to determine the equivalent re-

[34] Chf., Section of Seismology, Div. of Geomagnetism and Seismology, U. S. Coast and Geodetic Survey, Washington, D. C.

[35] "A Mechanical Analyzer for the Prediction of Earthquake Stresses," by M. A. Biot, *Bulletin*, Seismological Soc. of America, Vol. 31, No. 2, April, 1941, pp. 163–164.

[36] "A Mechanical Method of Analyzing Accelerograms," by Frank Neumann, *Proceedings*, Am. Geophysical Union, Washington, D. C., May 1 and 2, 1936; also "The Simple Torsion Pendulum as an Accelerogram Analyzer," by Frank Neumann, *Publications du Bureau Central Seismologique International*, Serie A: Travaux Scientifiques, Fascicule 15, 1937.

sponse of a 5-sec pendulum with a torsion pendulum analyzer) the writer found that it required considerable skill to follow the east-west component of the Helena acceleration curve, moving with the equivalent of the middle speed used in Professor Biot's work, even though the curve was magnified seven times. It is believed that the use of acceleration curves expanded many times, as now used in integration processes in the Coast and Geodetic Survey, would work for greater precision at these critical periods. As to the general form of the spectrum, recent ground period studies[37] indicate that the period pattern changes somewhat with the orientation of the seismograph pendulums, a condition which may be expected to be reflected in the computed spectrums.

With reference to that part of the spectrum below 0.2 sec, the error of Coast and Geodetic Survey accelerometers very seldom exceeds 15% or 20% between 0.1 and 0.2 sec. An experiment is now (1942) under way to test the practicability of shortening accelerometer pendulum periods with a view to increasing the range (downward) for recording true acceleration, and increasing the general recording capacity by decreasing the sensitivity.

It is unfortunate that some of the most valuable acceleration records are difficult to read because of overlapping of curves and occasional overdevelopment of the photographic paper. Overdevelopment is sometimes necessary to bring out the faint records of light spots moving at higher speeds and through larger amplitudes than had been anticipated in the original recorder design. Although these records do not always offer the sharpness of detail and cleanliness which Professor Biot desired in his work, it is nevertheless a fact that only one record to date has been lost to the extent that the data are not available for engineering studies. Available records include two of the Long Beach (Calif.) earthquake of 1933 yielding maximum accelerations of 0.06 g and 0.21 g, and maximum displacements of 20 cm double amplitude; also one of the Imperial Valley earthquake with maximum acceleration of 0.35 g and double amplitude of 40 cm. Comparing these values with the maximum of 0.16 g and displacement of 6 cm involved in Professor Biot's Helena spectrum, some idea may be obtained of the magnitude of the spectrums to come. Moreover, in spite of the magnitude of the recorded motion in the case of the Imperial Valley earthquake, the damage at the recording point, El Centro,[38] was "confined to walls that were not reinforced or tied, and to projecting balconies." The amplitude to be expected in an earthquake of catastrophic proportions can only be conjectured.

Concerning the inference that the torsion pendulum is[39] "a less accurate but simpler analyzer" than the proposed electrical apparatus, there will always be a doubt in the writer's mind until actual experience with both types proves the point. Any lack of accuracy in torsion pendulum analyzers would seem to lie in the manner in which the variable acceleration is transferred to the torsion head of the pendulum rather than in the functioning of the pendulum itself; and again, with reference to the statement in Section II of the paper

[37] "Analysis of the El Centro Accelerograph Record of the Imperial Valley Earthquake of May 18, 1940," *Manuscript 9*, U. S. Coast and Geodetic Survey, Washington, D. C.

[38] "Analysis of the El Centro Accelerograph Record of the Imperial Valley Earthquake of May 18, 1940," *Manuscript 9*, U. S. Coast and Geodetic Survey, Washington, D. C., p. 3.

[39] "A Mechanical Analyzer for the Prediction of Earthquake Stresses," by M. A. Biot, *Bulletin*, Seismological Soc. of America, Vol. 31, No. 2, April, 1941, p. 154.

that "it takes an average of eight hours to plot one spectrum curve," engineers are likely to be misled on the magnitude of the task which future workers in this field will face, remembering that even records of intensity VIII earthquakes present a much more formidable problem in processing than one of the Helena type which does not represent an intensity greater than VI. The Helena instrument was mounted on a limestone foundation and not on valley fill where most of the damage occurred in the 1935 earthquake.

JACOB FELD,[40] M. AM. SOC. C. E.—In characteristic fashion the author prepares a complete and logical derivation of various factors concerning the effect of earthquake vibration on physical structures. The writer is chiefly interested in that section concerning the influence of the foundation on the motion of blocks. In the approach to a solution of that factor, Professor Biot makes the assumption that the soil characteristics do not change during the vibration. Based also upon the assumption that the soil displacements within the limits of the deformations or deflections resulting from the vibrations of the block do not exceed the elastic limit strain, he deduces a formula for the rocking motion of the block, as restrained by the resisting characteristics of the soil.

Practical experience with vibrating structures embedded in soil does not confirm the assumptions made. It is well known in construction practice that, when vibrating machinery such as air compressors are placed on concrete foundations embedded in soil that has been made very heavy to dampen vibrations and thereby reduce their transference to adjacent structures, there is a change in the damping effectiveness with time. The only explanation for the change must be an alteration of the characteristics of the soil. It seems that continuous vibration of this nature increases the density of the surrounding soil. In some instances the writer has noted that a definite gap develops between the faces of the embedded concrete foundation and the adjacent soil. In such instances the vibration transference is considerably reduced, suddenly appearing again when, due to rain or other causes, the gaps are filled in. In one contract the writer designed a support that was kept entirely free from the adjacent soil and was bedded on a layer of cork. A gap was maintained on the four sides of the supporting base over a period of three years, and measurements made of the transference of vibration showed a loss of at least 90% of the compressor vibration wave. The period of the compressor vibration was 0.25 sec. Measurements were made by a vibrograph instrument which reported both horizontal and vertical components on a celluloid sheet.

Investigations of the effect of earthquakes and similar vibrations on the lateral pressure of earth have been made by Nagaho Mononobe and Haruo Matsuo.[41] These men found that the pressure of earth against a vertical retaining wall increased quite suddenly to a maximum upon the application of a vibration similar to an earthquake. The amount of increase depended upon the severity of the shock; but in each case the total pressure was found to be maximum immediately upon the application of the vibration, with a slight de-

[40] Cons. Engr., New York, N. Y.

[41] "Experimental Investigation of Lateral Earth Pressure During Earthquakes," by Nagaho Mononobe and Haruo Matsuo, *Bulletin*, Earthquake Research Inst., Vol. X, 1932, Pt. 4.

crease thereafter. However, the measured pressure after vibration was larger than that before vibration, and the only explanation seemed to be that the material changed in character. This was somewhat substantiated by a series of tests with more compact fills, which did not show as large an increase in pressure from vibration. It also was found that the pressure against rigid walls was increased to a larger extent than pressure against deformable walls. This again seems to indicate that the change in pressure was caused by a change in the characteristics of the fill. Incidentally, the author concludes that the maximum earth pressure under earthquake conditions can be calculated from the usual earth-pressure formula by the use of a density equal to the resultant acceleration obtained by graphic summation of the acceleration of gravity and the maximum seismic acceleration.

GEORGE W. HOUSNER,[42] JUN. AM. SOC. C. E.—In so far as it treats the earthquake problem from the viewpoint of vibration theory, this paper should prove of interest, particularly to engineers working in regions of seismic activity. That the effect of an earthquake is dependent on the physical properties of structures has long been known, and it was early recognized that the period of vibration of a structure was a significant index of its properties. One of the first attempts to measure the intensity of strong-motion earthquakes was made by Robert Mallet (1810–1881), a famous British engineer who made important contributions to seismology. The method used by Mallet[43] was to set up a group of six cylindrical wood blocks of graduated heights. By observing which of the blocks were overturned, he hoped to be able to classify the destructiveness of the earthquake. This wood-block approach has continued to interest engineers to the present time, but the rocking motion of a block is so complex that it has not been possible to draw any conclusions from the behavior of such a set of blocks.

A more satisfactory approach was taken by the late K. Suyehiro (1877–1932), noted Japanese engineer and one-time head of the Earthquake Research Institute of Japan. Suyehiro constructed a seismic vibration analyzer[44] to measure the intensity of earthquakes for different periods. This instrument[45] consisted of thirteen oscillators of different periods of vibration ranging from 0.22 to 1.81 sec. Such an instrument determines thirteen points on an earthquake spectrum such as that presented by the author. A defect of the instrument was the insufficient number of oscillators—that is, the gaps between the periods recorded were too large. Acting on a suggestion from Professor Martel, the San Francisco office of the U. S. Coast and Geodetic Survey constructed an instrument on this principle with a larger number of oscillators whose maximum displacement is recorded.

The strong-motion earthquake records furnished by the U. S. Coast and Geodetic Survey since 1932 have made it possible to compute the response of

[42] U. S. Engr. Office, Los Angeles, Calif.

[43] "Dynamics of a System of Rigid Bodies," by E. J. Routh, Macmillan Co., 1913, p. 175.

[44] "A Seismic Vibration Analyzer," by K. Suyehiro, *Proceedings*, Imperial Academy II, 1926 (Tokyo), p. 286.

[45] A brief description of the instrument may be found in "Engineering Seismology: Notes on American Lectures," by K. Suyehiro, *Proceedings*, Am. Soc. C. E., May, 1932, Pt. 2.

oscillators of different periods, thus greatly increasing the scope of this phase of seismological research. This computation can be done in several ways—namely, by numerical integration as was done by M. P. White[6] at the California Institute of Technology in Pasadena, and by F. Neumann[46] of the U. S. Coast and Geodetic Survey; by actuating a shaking table and measuring the response of oscillators as was done by Professor Jacobsen[7] at Stanford University in Stanford University, Calif.; or by utilizing the torsion pendulum as was done by Mr. Savage[9] at the Bureau of Reclamation and also by Professor Biot.

A defect in this approach, as pointed out by the author, is that the computations are based on records of unknown accuracy. (A testing program to determine the accuracy of these strong-motion seismographs is being conducted at the Massachusetts Institute of Technology, in Cambridge, for the U. S. Coast and Geodetic Survey. The results of this investigation have not yet [June, 1942] been made public.) Any inaccuracy in the accelerogram will be reflected in the computed oscillator response curve or spectrum. It would be well for the author to call attention to the fact that the most reliable point on the computed spectrum is that for a period equal to zero. As stated by Professor Biot, the value of the computed oscillator response curve or spectrum is much increased by virtue of its application to structures more complex than a simple vibrating mass. This is particularly true since the vibration theory required has already been developed in the theory of acoustics. It is interesting to note that a building with shearing deformations is the mathematical analogue of a vibrating string, and the building with flexural deformations is the analogue of a tuning fork. This, then, is another instance in which the mathematics required by engineers has already been developed by physicists in the solution of analogous problems; even the analogue of the building with the flexible first story has been treated by them.

Since 1936, a program of earthquake research, sponsored by the Los Angeles County (California) Department of Building and Safety, has been conducted at the California Institute of Technology under the direction of Professor Martel. During this time spectra or oscillator response curves have been evaluated for the following earthquake records: Los Angeles Subway Terminal, March, 1933; Vernon, Calif., March, 1933; Los Angeles Subway Terminal, October, 1933; El Centro, Calif., December, 1934; Helena, Mont., October, 1935; and El Centro, May, 1940. In engineering terminology, these so-called spectra are in reality influence lines for maximum shear. The influence lines or spectra for the aforementioned earthquakes were found to be of two general types. These are illustrated in Fig. 11 where it will be noted that the records used were obtained from the same seismograph but for different earthquakes. The curve in Fig. 11(*a*) is similar to those presented by the author—that is, the ordinates decrease rapidly for increasing periods of vibration. In Fig. 11(*b*) this rapid decrease does not occur. It is seen that in this case the shear is not reduced, in general, appreciably by lengthening the period of vibration.

[46] Mr. Neumann also called attention to the use of the torsion pendulum for this problem in "Special Publication 201, Earthquake Investigations in California 1934–1935," Dept. of Commerce publication.

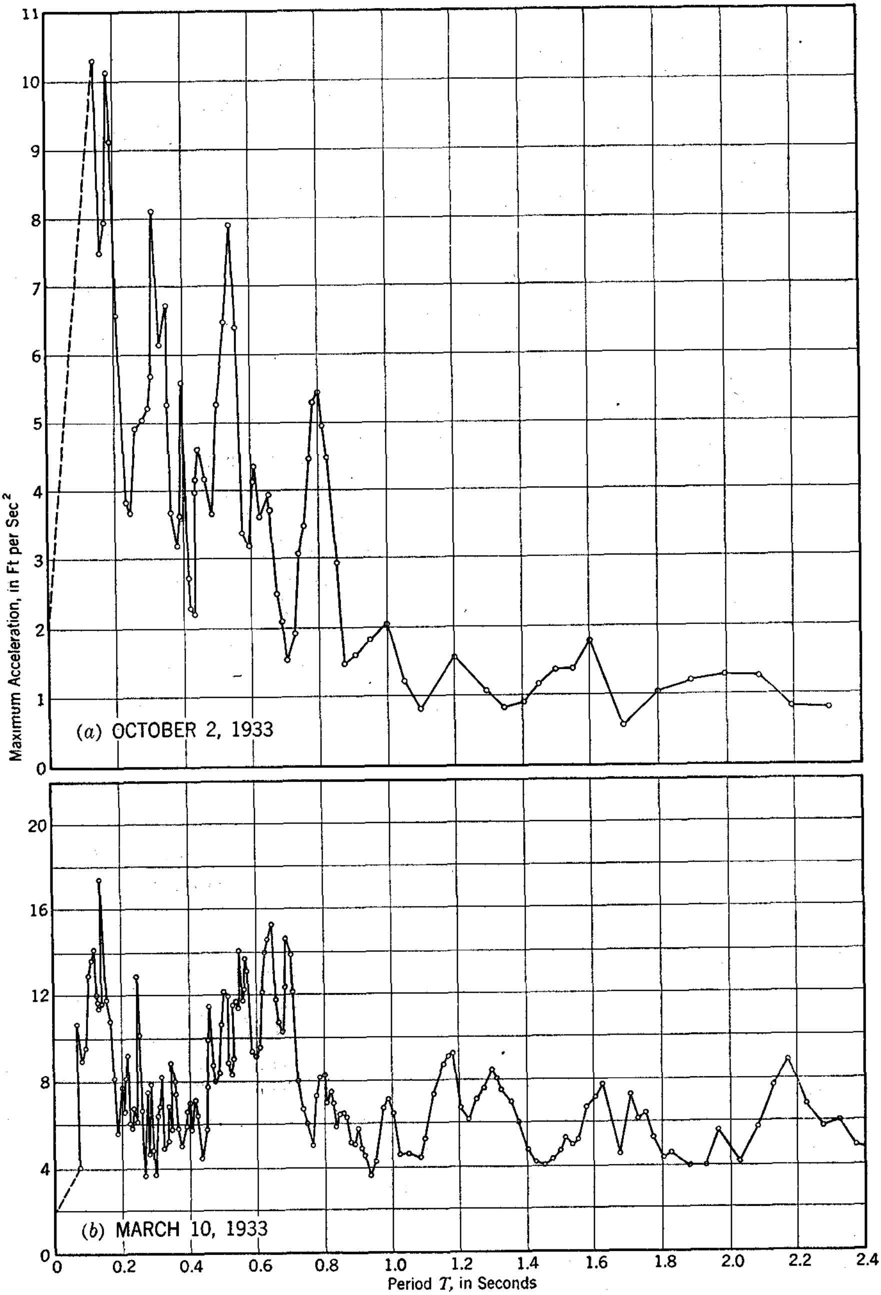

FIG. 11.—SPECTRA OF THE N 141° W COMPONENT OF THE LOS ANGELES SUBWAY TERMINAL RECORD

The problem of foundation yielding discussed by the author is a phenomenon well worth studying. It was first investigated by Suyehiro and Ishimoto[47] in 1926. These investigators constructed a special vibrograph and made measurements on the rocking oscillations of buildings.[45] Some results of their measurements are shown in Table 4. These were made on reinforced

TABLE 4.—OBSERVED PERIODS OF VIBRATION

Laboratory	Stories	Meters			Ground	Period (sec)
		Length	Breadth	Height		
Aeronautics; main building	2[a]	56	15	11	Reclaimed	0.50
Mitsubishi Laboratory; main building	2[b]	60	14	10	Loam	0.35
Mitsubishi Laboratory; metallurgical room	1[a]	11	9	5	Loam	0.20

[a] No basement. [b] With basement.

concrete buildings of very rigid construction. To compare values with the results of the author's analysis, it is necessary to recompute Table 3. In his calculations, Professor Biot has used the value p_s = 84 lb per sq in., apparently confusing allowable bearing pressure with actual building weight. On the basis of the dead load of typical buildings, it would be proper to take p_s = 1.04 lb per sq in. for each 12 ft of height. Recomputing on this basis gives the following values:

Values of r (ft)	For $2L$ = 50 ft, period T, in seconds, equals:
16	0.04
50	0.35
100	0.99

Only the structure 50 ft wide has been computed and the value r = 16 represents a height of 12 ft. As recomputed, it is seen that the calculated periods of the rocking oscillations are much shorter than the corresponding periods of the first modes of vibration would be. These computations would indicate that foundation yielding does not cause an appreciable stress reduction due to period lengthening.

A very extensive program of vibration measurements is reported in "Special Publication 201."[46] The periods of 212 buildings were measured and, of these, four showed evidence of possible foundation yielding. These four were low, rigid buildings on soft ground. The building vibrations (due to microseisms) measured by Suyehiro and Ishimoto[47] would also indicate that foundation yielding may be of importance for low, rigid structures on soft ground. It is of interest to note that Suyehiro concluded that the rocking oscillations were nonlinear since the period was dependent on the amplitude. The approach used by the author would not be applicable to nonlinear vibrations, although in itself nonlinearity would seem to be beneficial in reducing maximum shears.

[47] "Vibrations of Buildings," by K. Suyehiro and M. Ishimoto, *Proceedings*, Third Pan-Pacific Science Cong., Tokyo, 1926, p. 1482.

It appears that more experimental research is required to determine the fundamental characteristics of rocking oscillations of buildings.

HOMER M. HADLEY,[48] ASSOC. M. AM. SOC. C. E.—The gist of Professor Biot's scientific approach to the problem of earthquake resistant construction and to the improvement of codes appears to lie in his advocacy that recognition be given to the varying responses which different structures having different periods of vibration give to the same earthquake and, further, that recognition be given to varying types of foundations with their differing elastic properties and consequent differing effects upon structures built upon them. That such differences exist cannot be questioned, and it would seem entirely reasonable and logical to include consideration of these matters in design. When specific application is attempted, however, the engineer finds himself embarrassingly deficient in reliable data upon which to proceed. A sage saying which the writer heard years ago in Alaska was this: "Now, if we only had some ham, we'd have some nice ham and eggs for breakfast—if we only had some eggs." In short, for ham and eggs one must have both ham and eggs. Even so, for the application of the scientific method, the engineer must have reliable data for its basis. What is definitely known about earthquakes, about foundations, about the structures themselves?

One scientific fact about an earthquake always faithfully reported is the location of the epicenter. Not much else may be learned of it, but the location of the point on the earth's surface above the subterranean origin is readily determined. From the accumulation of these data engineers have come to know what regions are most subject to earthquakes, but beyond this there is no particular application of the knowledge made to earthquake resistant construction. Even the position of the epicenter is subject to question at times. At Long Beach, Calif., on March 10, 1933, the epicenter was first reported in the ocean bed some 15 miles off Newport Beach, Calif., where only slight damage occurred.[49] It seemed somewhat anomalous that the severe damage should have occurred at places much farther removed from the epicenter than Newport Beach. Later observations[50] led to the conclusion that there had been a series of epicenters at various points along the Inglewood, Calif., fault, there being a series of successive origins. Since these later epicenters were adjacent to the districts of severe damage, that much empirical confirmation of their existence is found.

As to the earthquakes themselves, the various records show certain common characteristics: An initial period of slight movement; then the period of severe, violent movement; and then the final period of diminishing, dying movement. As to their intensities, it is known that they vary widely. The accelerograms of the two Ferndale, Calif., earthquakes of February 6, 1937, and September 11, 1938,[2] are quite different. Who, on March 1, 1937, for instance, having seen the accelerogram and spectrum of the February 6, 1937, earthquake and of all

[48] Regional Structural Engr., Portland Cement Assn., Seattle, Wash.

[49] "Preliminary Report on the Long Beach Earthquake," by Harry O. Wood, *Bulletin*, Seismological Soc. of America, April, 1933.

[50] "The Determination of the Extent of Faulting with Application to the Long Beach Earthquake," by Hugo Benioff, *ibid.*, April, 1938.

preceding earthquakes at Ferndale, could have said that a more severe earthquake would occur on September 11, 1938, or, on any particular date, could have foretold what its accelerogram or spectrum would look like? Of necessity, such a prophet would have to employ "art based on observational facts and experience" rather than "science" as the basis of his prediction; so it is, with Ferndale, Calif., or with any Ferndale, large or small, upon this earth. Somewhere in his writings, Omori discussed the earthquake peculiarities of certain places and stated that there were individual characteristics and features to be observed; but except for a certain directional similarity there was nothing strikingly individualistic in these characteristics. Now, and probably for a long time in the future, engineers must be empirical in judging the earthquake to be assumed in design.

As to the influence of the foundations on the motion of the structure, Professor Biot clearly states the complexity of the problem (see heading "V.—Influence of Foundation on Motion of Blocks"): "* * * it involves a complete knowledge of the propagation and properties of the seismic waves in the strongly heterogeneous surface layers of the earth as well as their diffraction and reflection by objects built on the surface." This undoubtedly is true and undoubtedly presents an obstacle to "science." It is stated that Eq. 38 implies the knowledge of the elastic constants E and μ of the soil but that "These are obtainable from tests made by M. Ishimoto and K. Iida.[13]" This is a happy solution of the difficulty: See what Messrs. Ishimoto and Iida give for these values. Nevertheless, it is to be noted in their quoted findings that a change of moisture content of silty clay from 42% to 50%—that is, a change of 8%—changes E from 792 lb per sq in. to 5,750 lb per sq in. Who is to know whether what he terms silty clay is the same as the silty clay defined by Messrs. Ishimoto and Iida, or what the moisture content of that silty clay may be when a foundation is subjected to an earthquake? Again, what manner of bedrock underlies the silty clay? Those who have driven the highways of Southern California and looked upon the confused, contorted, heterogeneous geologic structure revealed in cuts and excavations must feel mistrust and misgivings in whatever assumptions they may make as to the values of E and μ for scientific calculations.

As to the elastic properties of the structure that is being designed, it should be possible from measurements, past and future, of existing comparable structures to assume values fairly approximate to those that will develop. With values of T decided on, with what spectrum shall they then be used? Apparently there is only a very limited number in existence. Fig. 6 is a "standard" spectrum of two earthquakes only—that of Helena, Mont., on October 31, 1935, and the Ferndale earthquake of September 11, 1938. It shows that a natural period of 0.5 sec gives an acceleration of 0.4 g; a natural period of 1.0 sec gives an acceleration of 0.2 g. From the text one learns (see heading "III.—Spectrum Related to Design and Actual Behavior of Structures") that "The differences [in the behavior of buildings of different periods] are not as great as the spectrums would indicate." Apparently "Science" must be empirically modified, at times.

There is a further matter of engineering seismology to be considered, as well as the matter of intensity of stresses, which can be modified and altered by change of sections. This is the matter of greater deformations that accompany longer periods. The Marunouchi Building in Tokyo, Japan, the largest steel-frame building in that city, was under construction in 1922 when a considerable earthquake occurred. As a result of the distortions it sustained at that time, it was felt that the building needed stiffening. Consequently, a number of panels of diagonal bracing were introduced, extending from basement to roof, and these were covered with metal lath and plaster. When the great earthquake of September 1, 1923, occurred, these bracing panels were loaded terrifically at the outset and either stretched amazingly or ruptured. The natural period of the building then reverted to its original value. A number of different people who were in the building told the writer that they did not enjoy the experience. Partitions were cracked and shattered, marble trim in the corridors loosened and fell, and pendant lighting fixtures swung so violently that their reflectors knocked plaster off the ceilings. After it was all over the owners—perhaps not scientifically, yet empirically—rebuilt the exterior and interior walls and introduced bracing walls, for the definite purpose of making the building stiff—that is, giving it a low natural period of vibration. States a well-known advertisement: "Ask the man who owns one."

It must not be thought that making a building stiff will prevent an earthquake from being felt in it. On the contrary, whatever movement occurs at the foundations will be transmitted with little damping to the roof, and the building's contents will be shifted and perhaps thrown to the floor quite as if they stood upon the ground. On the other hand, after the earthquake has passed, a well-designed, rigid-type building will be found very satisfactory to all concerned. It may be added that all the comment made herein is limited to buildings of moderate height. As to skyscrapers, the writer has no knowledge other than that such buildings successfully passed through the San Francisco, Calif., earthquake of 1906.

Yet another aspect of the earthquake problem is the matter of occurrence. Fortunately, severe earthquakes are not frequent. It may well be questioned whether provision against them warrants highly elaborate and refined engineering design. The severe damages which have repeatedly occurred in the past have not resulted from failure to design for this or that value of shearing stresses in the structures involved but from completely ignoring earthquakes and all horizontal forces they entail. A definite observed deficiency in present design or proof that current practise is unwarrantedly too severe should constitute the basis of change and progress.

There is a widespread feeling among American engineers that some special virtue is imparted to the solution of highly complex, not to mention uncertain and unknowable, problems by going through mathematical processes in reaching a decision regarding them. They prefer to assume E and μ and all the various other variables that may be involved and to "figure" themselves to an answer rather than to accumulate empirical data and proceed simply and directly from them. There is much to be said for the mathematical process, but it contributes no more truth, no more science, than is put into its basic assumptions. The writer would conclude with the words of the late David A. Molitor, M. Am. Soc. C. E., used in the "Synopsis "of his paper "Wave Pressures on Sea-Walls and Breakwaters":[51] "Since all factors contributing to the solution of the problems herein considered are only approximately knowable, no theoretical approach was deemed advisable."

ROBERT E. GLOVER,[52] ESQ.—The author develops a method of using a torsion pendulum to determine the amount of resonance to be expected when a structure of known frequency is subjected to an earthquake. This ingenious procedure makes use of an accelerometer record obtained during the quake. A torsion pendulum operation based on the use of a displacement curve, derived by double integration of an accelerometer record, has been used by the writer for a similar purpose. A comparison of results obtained by the two methods is shown in Table 5, which is based on the East-West component of the Helena quake of October 31, 1935.

TABLE 5.—COMPARISON OF ACCELERATION RESULTS

Structure period (sec)	Maximum Acceleration	
	Author's data[a]	Writer's data[b]
0.20	0.73 g	0.79 g
0.25	0.67 g	0.69 g
0.40	0.50 g	0.52 g
0.50	0.32 g	0.27 g
0.80	0.19 g	0.20 g
1.00	0.24 g	0.26 g
2.00	0.10 g	0.06 g

[a] Scaled from author's spectrum curve. [b] Based on an integration made by the U. S. Coast and Geodetic Survey.

The slight differences are probably due to the choice of values for the constants of integration used in computing the displacement curve. Since the accelerometer instruments are generally arranged to be put in motion by a starter, which responds to ground movement, a small part of the record near the beginning is lost, and there is, therefore, some room for exercise of judgment when the values of these constants must be selected. The results obtained by the two methods appear, however, to be in essential agreement.

M. A. BIOT,[53] ESQ.—Mr. Rich presents an application of the operational calculus to a simplified case. Since the response of structures to earthquakes is essentially of a transient character, it is natural to apply the methods specifically designed for the treatment of such phenomena. However, the difficulty in applying the Heaviside methods to the full extent is the non-analytical and random character of the seismogram. The simplified procedure derived by the writer avoids the computation of the actual motion and makes possible the direct evaluation of the maximum stress by the use of a standardized spectrum. However, the operational method can be used, as such, in combination with the spectrum, as a convenient way of deriving the effectiveness coefficients or any desired characteristic in the structure. This is especially true when the information needed is restricted to a particular variable or location in the structure. The operational method is also useful in investigating

[51] *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 984.

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the response to simplified theoretical earthquakes, as shown in the examples set forth by Mr. Rich. The latter method acquires special value if a great number of cases are solved and if general principles are uncovered as to the comparative behavior of structures under standardized earthquakes. This approach might truly be called a mathematical, experimental method.

Professor Hoff mentions that the results obtained at Stanford University show the influence of the friction on the oscillator response. It is interesting to note the experimental fact that the response is very sensitive to changes in friction when the latter is small. It is probable that the differences in peak spectrum accelerations between the Stanford results and those of the writer are partly due to errors on the small-period components of the displacement cam. The question of the possible accumulation of the stresses in each mode by the simultaneous occurrence of their maximum values at a given point is well worth investigating. A conservative estimate can be made theoretically by adding the maximum values associated with each mode at various locations in the structure.

As stressed by Professor White, the idea of an automatic analyzer to avoid the numerical work in evaluating the response of a structure is not new. In the writer's earlier work in 1932, the mathematical groundwork was laid for showing the possibility of using the accelerogram directly without the process of evaluating the displacement. An electric model was suggested, of which the present torsional pendulum is the mechanical analogue. An electric model has been developed by Arthur C. Ruge, Assoc. M. Am. Soc. C. E., at the Massachusetts Institute of Technology in Cambridge, Mass.

Professor White raises an interesting and important point regarding the nature of the peaks in the spectrum. The writer called attention to the lack of correlation between the peaks in the two Ferndale earthquakes, (*B*) and (*C*), as supporting evidence that these peaks are not characteristic of the location. It is possible that some peaks might be characteristic of the location, or of a group of buildings, and that these would stand out particularly in earthquakes of small intensity where the effects of resonance seem to be more acute. More probably the hatched appearance of the spectrum is due to random transients superposed on a basic periodicity distribution. Professor White makes the statement that the whip effect, although less pronounced, also appears in structures that are not necessarily tapered. He also suggests the application of the present results to the effect of explosions and projectile impact on structures. The writer wishes to mention another possible application—determining the dynamic effect of wind on structures by introducing the spectrum of atmospheric turbulence.

Mr. Heck mentions the importance attached by the U. S. Coast and Geodetic Survey to the recording of high accelerations throughout the frequency range of earthquakes. The results already accomplished in this direction were an absolute prerequisite to the writer's work.

Mr. Neumann rightly warns against using a particular spectrum as a basis for final standardization. The writer uses the standardized Helena spectrum merely as an example. The spectrum peak seems to be correlated both in intensity and frequency to the distance of the epicenter. As stated by Mr. Neumann, it is probably true that the torsion pendulum analyzer is not necessarily less accurate than a more intricate electrical apparatus. The writer's statement on this matter was perhaps somewhat conservative, since the spectrum curves could be duplicated closely in spite of the rather crude design of the analyzer and the use of non-magnified seismographic records.

Mr. Feld recalls the change of damping characteristics with time in a vibrating soil. As stated by the writer, the treatment of a foundation is restricted to its elastic properties. The results are in good agreement with tests on actual foundations made by Mr. Barkan.[16] The magnitude of structure and soil damping is a factor left to future experimental investigation. The methods of the paper are applicable if it is desired to introduce the effect of this damping on the amplitudes of oscillation. However, in seeking to establish experimental data, it will be well to keep in mind that the vibrator itself can have a considerable effect on the properties of the soil.

The two spectra computed by Mr. Housner are interesting, in that they show the same general appearance as those obtained by the writer. The ratio of the spectrum peak to the maximum acceleration is also of the same order. The spectrum of the March 10, 1933, earthquake shows relatively high equivalent accelerations (of the order of 20% gravity) for the large periods. Mr. Housner presents an application of the writer's simplified formula for the rocking period of blocks. His results are based on a much lower bearing pressure than that used by the writer, and the rocking periods in this case are relatively short. Taking into account the possibility of the soil being considerably more spongy than assumed in the computations, it should be concluded that the effect of the foundation is important or not, depending on the magnitude of such factors as soil elasticity, foundation size, and height and rigidity of the building. It would seem, however, that in the case of towers or chimneys the rocking effect should be preponderant. In regard to the relations between acoustics and engineering seismology, a theorem[4,54] was originally developed by the writer which states that the energy accumulated by an oscillator depends only on the intensity of the impulse spectrum for the particular frequency. This theorem forms the basis of the present method and justifies the use of the spectrum.

Mr. Hadley's pimented discussion is enjoyable reading. His argument reduces to a confession of man's total ignorance of the subject, and the defeatist viewpoint that nothing can be done about it "Now, and probably for a long time in the future." Mr. Hadley, however, is anxious not to discredit the esoteric art of earthquake-proof design by contributing the following statement:

> "* * * after the earthquake has passed, a well-designed, rigid-type building will be found very satisfactory to all concerned. * * * As to skyscrapers, the writer has no knowledge other than that such buildings successfully passed through the San Francisco, Calif., earthquake of 1906."

[54] "Acoustic Spectrum of an Elastic Body Submitted to a Shock," by M. A. Biot, *Journal*, Acoustical Soc. of America, Vol. V, January, 1934, p. 207.

This, said the medieval doctor, is a sleeping potion; it makes you sleep because it contains a soporiferous virtue. It has been the writer's experience that a feeling of uneasiness usually develops when proselytes of the empirical method are brought in contact with symbols such as E and μ, even if they do not represent an essential part of the argument. This is entirely unfortunate and unjustified, as there is no fundamental conflict between empiricism and science. Mr. Hadley's objections might well have been raised three hundred years ago against the basic principles of Newtonian mechanics with their abstract and elusive concepts of force and mass.

Mr. Glover's results are in good agreement with those of the writer. They point to the comforting fact that satisfactory duplication of results is possible even when different methods and instruments are used.

In conclusion, the writer wishes to thank all those who have taken time to contribute to the present discussion for their constructive criticism.