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Some Simplified Methods in Airfoil Theory

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INTRODUCTION

THE USE OF the acceleration potential in thin airfoil theory was introduced by L. Prandtl.[1,2,6] The method has since been applied by several authors[3,4,5,8] to problems of stationary and non-stationary flow in both compressible and incompressible fluids. A striking feature of this method is that no use has to be made of vortex wakes because the acceleration potential is everywhere continuous in the fluid. Such concepts as the trailing vortices and the induced downwash play no direct part in this theory.

The purpose of the present article is the treatment of two dimensional airfoil theory in an incompressible fluid by the combined use of conformal transformation and the acceleration potential. This method of approach greatly simplifies the solution of certain airfoil problems.

First the reader is introduced to the fundamental properties of the acceleration potential in an incompressible fluid. Then it is shown how the problem of finding the acceleration potential can be solved by conformal representation of the airfoil on a circle and a simple relation is derived between the velocity and the "stream function" corresponding to this potential. The method is next applied to the stationary airfoil with and without flap, and to the determination of the airfoil camber line and thickness function for a given pressure distribution along the chord. The simplicity of the method is especially apparent for the case of the oscillating airfoil which is the subject of the last treatment. For the sake of brevity only translatory oscillations are considered.

PROPERTIES OF THE ACCELERATION POTENTIAL

Newton's law applied to fluid motion leads directly to the equation

$$\rho\bar{a} = -\text{grad}\, p \tag{1.1}$$

where ρ = the mass per unit volume, $\bar{a}$ = the acceleration vector, and p = the pressure.

For an incompressible fluid ρ = const., and Eq. (1.1) implies the existence of a scalar function φ such that

$$\begin{aligned} \bar{a} &= \text{grad}\,\varphi \\ \rho\varphi &= -p\dagger \end{aligned} \tag{1.2}$$

φ is called the *acceleration potential.*

The velocity $\bar{u}$ considered as a vector field function of the coordinates x, y, z and the time t is related to the acceleration by Euler's kinematic equations

$$\bar{a} = (\partial\bar{u}/\partial t) + (\bar{u}.\nabla)\bar{u} \tag{1.3}$$

Assume the velocity field to be composed of a uniform velocity $\bar{U}$ along the x direction and a *small perturbation* $\bar{u}'$.

$$\bar{u} = \bar{U} + \bar{u}' \tag{1.4}$$

Introducing the so-called *linearized theory* which neglects all quantities of higher order in $\bar{u}'$, Eq. (1.3) becomes

$$\bar{a} = (\partial\bar{u}'/\partial t) + U(\partial\bar{u}'/\partial x) \tag{1.5}$$

For an incompressible fluid the velocity $\bar{u}'$ also satisfies the continuity equation

$$\text{div}\,\bar{u}' = 0 \tag{1.6}$$

Taking the divergence on both sides of Eq. (1.5)

$$\text{div}\, a = 0 \tag{1.7}$$

hence from (1.2)

$$\nabla^2\varphi = 0 \tag{1.8}$$

The important result is thus obtained that in the linearized theory of an incompressible fluid the acceleration potential satisfies Laplace's equation. If there is a velocity potential ϕ,

$$\bar{u}' = \text{grad}\,\phi \tag{1.9}$$

and from Eq. (1.5) the following relation is derived between the acceleration and velocity potential.

$$\varphi = (\partial\phi/\partial t) + U(\partial\phi/\partial x) \tag{1.10}$$

In this method of approach flow problems are solved by considering the field of accelerations and using for the acceleration potential solutions of Laplace's equation. The velocity potential and the velocity field may then be derived by integrating Eq. (1.10) with ϕ as an unknown function. The above equations are applicable to thin airfoils where the velocity perturbation $\bar{u}'$ introduced by the airfoil is small compared to the main stream velocity U. It will be shown below

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† To be absolutely general this equation should contain an additional term which is an arbitrary function of time independent of the coordinates. This term can be omitted for our purpose because $\bar{a}$ and ϕ are taken to vanish at infinite distance in all applications treated in this article.

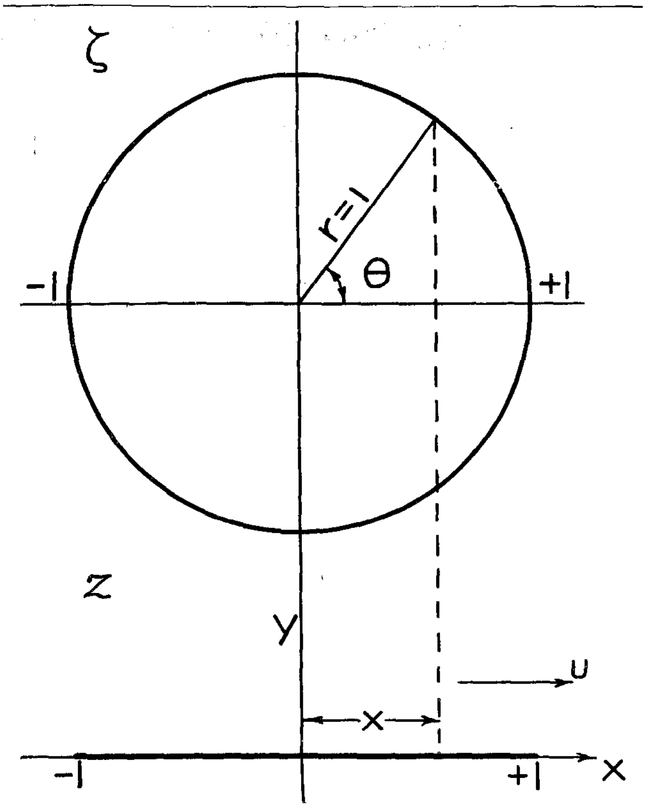

FIG. 1.

how this can be applied in the case of two dimensional airfoil theory.

Two Dimensional Airfoil Theory

According to the results derived in the previous paragraph the solution of thin airfoil problems depends on finding a solution of Laplace's equation for the acceleration potential. This potential will be determined by the condition that the acceleration normal to the airfoil has a given value depending on the shape and motion of the airfoil. The case of the airfoil of infinite span will be treated by this method. The flow is then two dimensional. Take x and y to be the coordinates in the plane of flow and the airfoil section to lie approximately along the x axis with the leading edge at $x = -1$ and the trailing edge at $x = +1$. The equation of the airfoil boundary for the general case of a movable or deformable airfoil is

$$y = y(x,t) \tag{2.1}$$

For a thin airfoil, neglecting higher order terms, the coordinate x of a fluid particle moving along the boundary is of the form

$$x = x_0 + Ut \tag{2.2}$$

Hence its velocity normal to the boundary is

$$\dot{y} = U(\partial y/\partial x) + (\partial y/\partial t) \tag{2.3}$$

and its normal acceleration

$$\ddot{y} = U(^2\partial^2 y/\partial x^2) + 2U(\partial^2 y/\partial x\partial t) + (\partial^2 y/\partial t^2) \tag{2.4}$$

Two dimensional potential problems are conveniently handled by conformal transformation. The equation

$$x + iy = z = [\zeta + (1/\zeta)]/2 \tag{2.5}$$

transforms the circle of unit radius in the ζ plane to a straight segment from $x = -1$ to $x = +1$ on the real axis of the z plane (Fig. 1). The point of polar coordinates $r = 1$ and θ on the circle corresponds to the point of coordinate $x = \cos\theta$ on the airfoil. The problem is then to find a complex function

$$f(\zeta) = \varphi + i\psi \tag{2.6}$$

such that the normal derivatives of the acceleration potential φ on the circle satisfy the values derived from the kinematic equation (2.4) for the *normal accelerations* on the airfoil.

The two functions φ and ψ are conjugate harmonic and satisfy the so-called Cauchy relations

$$\begin{aligned}(\partial\varphi/\partial x) &= (\partial\psi/\partial y)\\ (\partial\varphi/\partial y) &= -(\partial\psi/\partial x)\end{aligned} \tag{2.7}$$

Use is made of these relations for the following reason. The condition that the normal acceleration components satisfy the kinematic boundary condition on the wing does not necessarily imply that the kinematic condition is satisfied for the velocity. Therefore it is necessary to derive the value of the y component of the velocity,

$$v = (\partial\phi/\partial y) \tag{2.8}$$

Taking the y derivative of both sides of Eq. (1.10)

$$(\partial\varphi/\partial y) = (\partial v/\partial t) + U(\partial v/\partial x) \tag{2.9}$$

and from (2.7)

$$-(\partial\psi/\partial x) = (\partial v/\partial t) + U(\partial v/\partial x) \tag{2.10}$$

There are two important cases to consider. For a stationary airfoil the term $(\partial v/\partial t)$ vanishes. In this case, integrating both sides of Eq. (2.10) with respect to x and assuming that both ψ and v are zero at infinity, gives

$$v = -\psi/U \tag{2.11}$$

Hence the simple result that v is proportional to the conjugate function of the acceleration potential.

In the case of non-stationary flow for an airfoil performing harmonic oscillations the functions φ, ψ and v are replaced, respectively, by $\varphi e^{i\omega t}$, $\psi e^{i\omega t}$ and $ve^{i\omega t}$. Eq. (2.10) becomes

$$-(\partial\psi/\partial x) = i\omega v + U(\partial v/\partial x) \tag{2.12}$$

Solving for v as an ordinary linear differential equation and assuming that $v = 0$ at $x = -\infty$ gives

$$v = -(e^{-i\omega x/U}U)\int_{-\infty}^{x} e^{i\omega x/U}(\partial\psi/\partial x)dx \tag{2.13}$$

It must be remarked that the imaginary number i used in connection with harmonic functions of time in Eqs. (2.12) and (2.13) is different from the imaginary number appearing in the complex potential, Eq. (2.6).

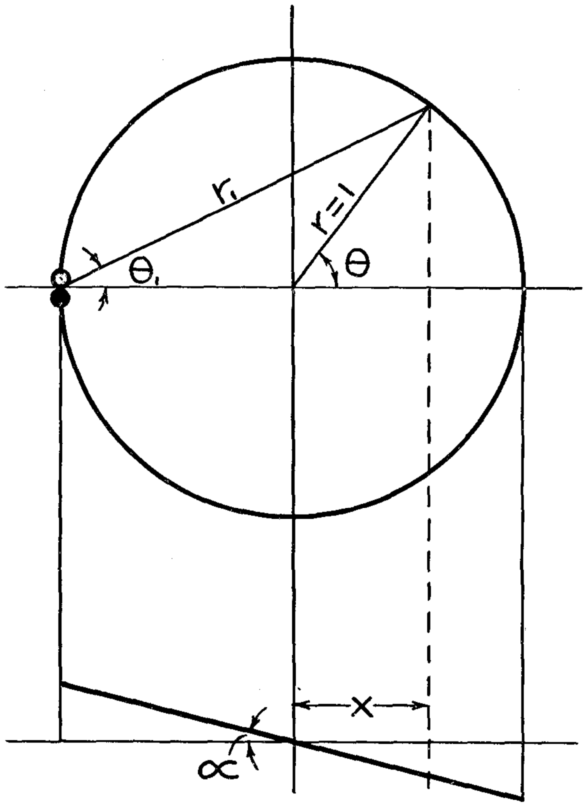

FIG. 2.

Actually different notations should be introduced to distinguish between them. However in the following applications they will not be used simultaneously so that confusion is easily avoided.

The Kutta-Joukowski condition in the classical theory states that the velocity is finite at the trailing edge. An equivalent form of this condition is that there is no pressure discontinuity at the trailing edge. The Kutta-Joukowski condition is therefore taken care of in the present theory by choosing an acceleration potential which is continuous at the trailing edge.

THE STATIONARY AIRFOIL

Symmetric Airfoil

Consider a thin symmetric airfoil at an angle of attack α (Fig. 2). Eq. (2.1) becomes

$$y = -\alpha x \tag{3.1}$$

From Eqs. (2.3) and (2.4)

$$\dot{y} = -\alpha U, \ \ddot{y} = 0 \tag{3.2}$$

The normal acceleration on the airfoil must be zero; therefore in the z plane the vertical component of the acceleration potential gradient must be zero on the circle. A potential satisfying this condition is that due to a source-sink doublet at the point of coordinate $\zeta = -1$. The conjugate harmonic functions for this case are

$$\left.\begin{aligned} \varphi &= A(\sin \theta_1/r_1) \\ \psi &= A(\cos \theta_1/r_1) \end{aligned}\right\} \tag{3.3}$$

where θ_1 and r_1 are the polar coordinates shown in Fig. 2. It is clear that this solution satisfies the Kutta-Joukowski condition that φ is continuous at the trailing edge. On the circle ψ is a constant

$$\psi = A/2 \tag{3.4}$$

The circle is a streamline of the acceleration potential but the constant A which determines the intensity of the doublet must be evaluated by introducing the kinematic condition for the velocity. From Eqs. (2.11), (3.2) and (3.4) at the airfoil

$$v = -\alpha U = -(A/2U) \tag{3.5}$$

Hence

$$A = 2\alpha U^2 \tag{3.6}$$

Hence from Eqs. (1.2) and (3.3) the pressure distribution on the airfoil

$$p = -2\alpha\rho U^2 (\sin \theta_1/r_1) \tag{3.7}$$

The lift distribution l is the pressure difference on both sides; introducing the values $\theta_1 = \theta/2$, $r_1 = 2 \cos (\theta/2)$ gives

$$l = 2\alpha\rho U^2 \tan (\theta/2) \tag{3.8}$$

which is the well known lift distribution on a thin airfoil (Fig. 3). The zero lift at the trailing edge is due to the Kutta-Joukowski condition. The total lift is

$$L = \int_{-1}^{+1} l dx = \int_0^\pi l \sin \theta d\theta = 2\pi\alpha\rho U^2 \tag{3.10}$$

The resultant of this lift distribution is located at the quarter chord point.

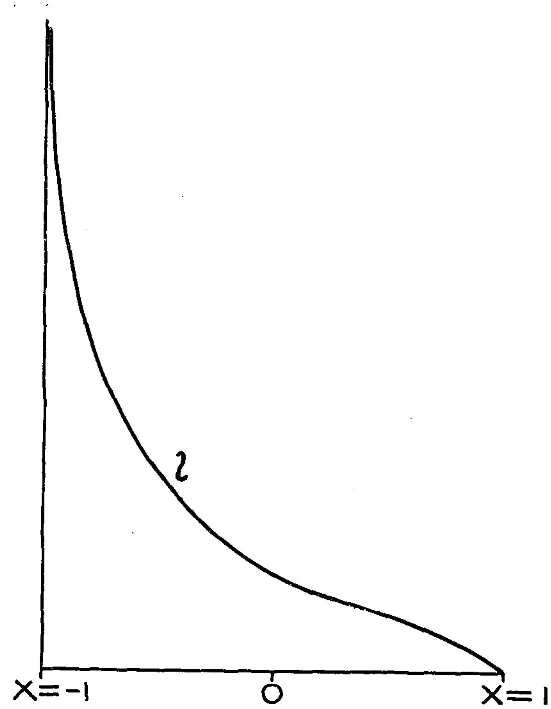

FIG. 3.

Airfoil with Flap

The case of an airfoil with flap is handled similarly. The same doublet as in the previous case is located at the point corresponding to the leading edge. In addition a source and sink are located on the circle at the points θ_0 and $-\theta_0$, respectively (Fig. 4). These points correspond to a flap hinge of coordinate

$$x = \cos\theta_0$$

The total acceleration potential is

$$\varphi = A \sin\theta_1/r_1 + B\log(r_2/r_3) \tag{3.11}$$

and the corresponding conjugate function is

$$\psi = A\cos\theta_1/r_1 + B\theta_2 \tag{3.12}$$

It is immediately verified that the circle belongs to the streamlines of this source-sink system. The constants A and B determine the strength of these sources and sinks. Their values are fixed by the kinematic condition of the velocity. The y component of the velocity forward of the hinge is obtained from Eqs. (2.11) and (3.12).

$$v = -(A/2U) - (B\theta_0/U) \tag{3.13}$$

This velocity must be zero because the angle of attack of the airfoil is assumed to be zero forward of the hinge, hence the equation

$$-A - 2B\theta_0 = 0 \tag{3.14}$$

Similarly aft of the hinge the vertical velocity is $v = -\beta U$ where β denotes the flap angle, hence the equation

$$-\beta U = -(A/2U) - (B/U)(\theta_0 - \pi)^* \tag{3.15}$$

Solving Eqs. (3.14) and (3.15) for A and B

$$A = 2\beta U^2(\theta_0/\pi) \qquad B = -(\beta U^2/\pi)$$

The lift distribution derived from the value of the acceleration potential Eq. (3.11) is,

$$l = 2\beta\rho U^2\left[\frac{\theta_0}{\pi}\tan\frac{\theta}{2} + \frac{1}{\pi}\log\frac{r_3}{r_2}\right]$$

or

$$l = 2\beta\rho U^2\left[\frac{\theta_0}{\pi}\tan\frac{\theta}{2} + \frac{1}{\pi}\log\left|\frac{\sin{}^1/_2(\theta+\theta_0)}{\sin{}^1/_2(\theta-\theta_0)}\right|\right] \tag{3.16}$$

The total lift is

$$L = \int_{-1}^{+1} l\,dx = \int_0^{\pi} l\sin\theta\,d\theta = 2\beta\rho U^2(\theta_0 + \sin\theta_0) \tag{3.17}$$

AIRFOIL WITH GIVEN PRESSURE DISTRIBUTION

The derivation of the camber line and the thickness function for a given pressure distribution along the airfoil by the use of the acceleration potential is shown for the particular case of an airfoil with uniform pressure.

* With reference to Fig. 4 it must be kept in mind that the vertex of the angle θ_2 is to be considered as lying in the region outside of the circle. Therefore when passing from the forward to the aft side of the hinge, θ_2 decreases by the amount π.

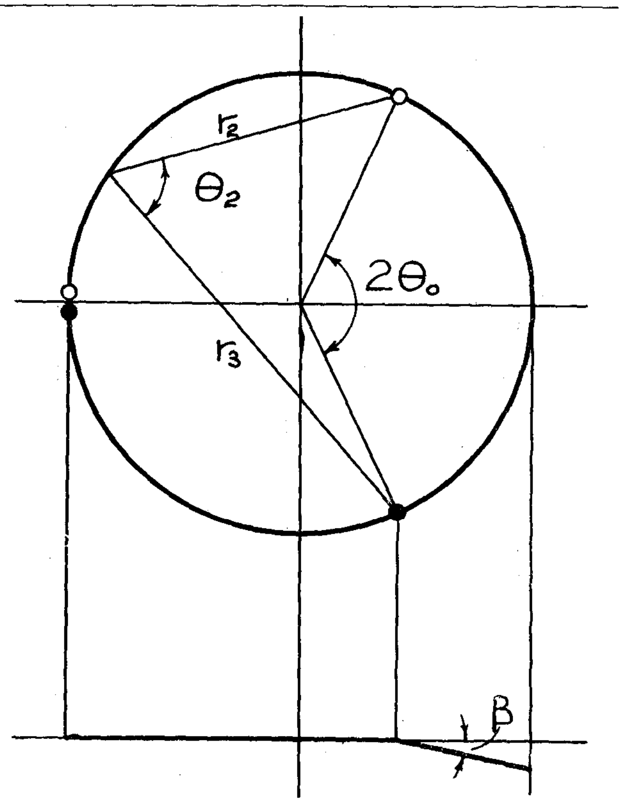

FIG. 4.

Camber Line of Airfoil with Uniform Pressure

The potential due to a pair of equal and opposite vortices at points $+1$ and -1 on the circle (Fig. 5) is

$$\varphi = A(\theta_2 - \theta_1) \tag{4.1}$$

and the conjugate harmonic function

$$\psi = A\log(r_1/r_2) \tag{4.2}$$

On the upper half circle the acceleration potential φ is obviously equal to

$$\varphi = (\pi/2)A \tag{4.3}$$

and on the lower half circle

$$\varphi = -(\pi/2)A \tag{4.4}$$

This gives a uniform lift distribution along the chord.

$$l = \pi\rho A \tag{4.5}$$

The normal velocity is given by Eq. (2.11). Using Eqs. (2.3), (4.2) and (4.5) the following equation is obtained

$$U(dy/dx) = -(l/\pi\rho U)\log(r_1/r_2) \tag{4.6}$$

Substituting the values $r_1^2 = 2(1+x)$, $r_2^2 = 2(1-x)$ which are simple to derive geometrically, Eq. (4.6) becomes

$$(dy/dx) = (l/2\pi\rho U^2)[\log(1-x) - \log(1+x)] \tag{4.7}$$

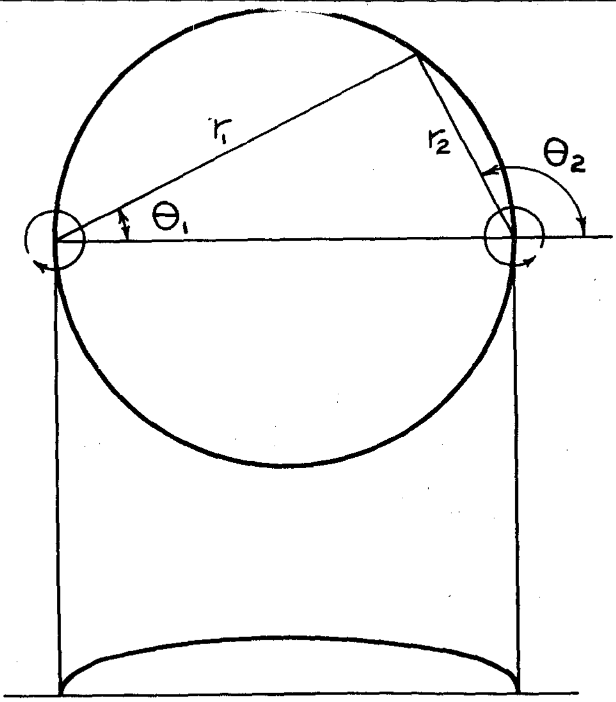

FIG. 5.

Integrating with the condition that $y = 0$ at the leading and the trailing edge the following expression is obtained for the camber line

$$y = \frac{l}{2\pi\rho U^2}\Big[2 \log 2 - (1 + x) \log (1 + x) - (1 - x) \log (1 - x)\Big] \quad (4.8)$$

This camber line is represented in Fig. 5.

Thickness Function of Airfoil with Uniform Pressure

In a similar way the thickness function for an airfoil with uniform pressure distribution can be found. The harmonic conjugate functions for this case are obtained from two doublets at points -1 and $+1$ as indicated in Fig. 6.

$$\begin{aligned} \varphi &= A[(\cos\theta_1/r_1) - (\cos\theta_2/r_2)] \\ \psi &= -A[(\sin\theta_1/r_1) - (\sin\theta_2/r_2)] \end{aligned} \quad (4.9)$$

The pressure on both sides of the airfoil is uniform and equal to

$$p = -\rho\varphi = -\rho A$$

Proceeding as above leads to the equation

$$U\frac{dy}{dx} = \frac{A}{U}\left[\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right] \quad (4.10)$$

This may be written

$$(dy/dx) = -(p/2\rho U^2)[\tan(\theta/2) - \cot(\theta/2)]$$

or since $dx = -\sin\theta d\theta$

$$dy = (p/\rho U^2)\cos\theta d\theta \quad (4.11)$$

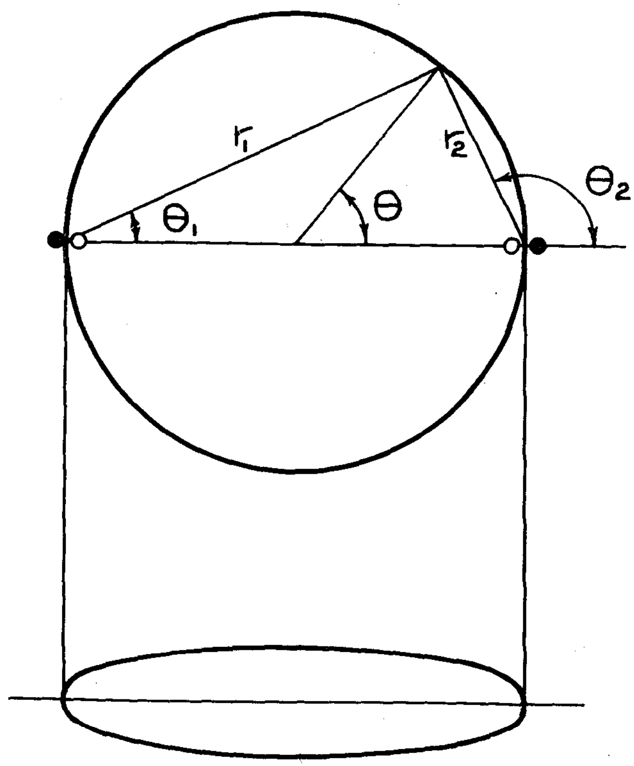

FIG. 6.

The thickness function is found by integration

$$y = -(p/\rho U^2)\sin\theta \quad (4.12)$$

The cross-section given by this thickness function is an ellipse.*

OSCILLATING AIRFOIL

The case of an oscillating airfoil is amenable to a particularly simple treatment by the acceleration potential. As an example the case of vertical translatory oscillations will be considered. The translation

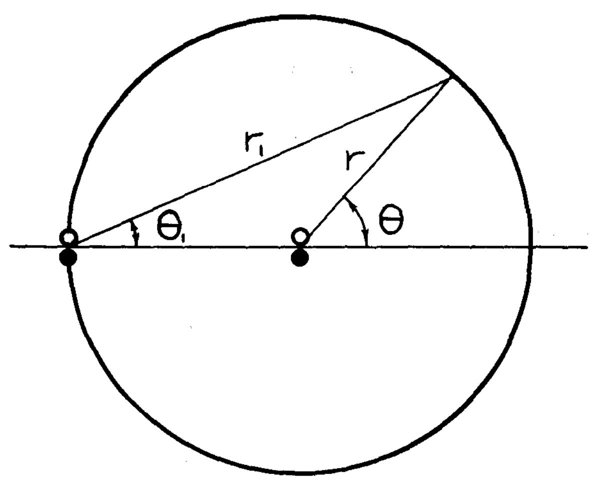

FIG. 7.

* It is clear that the theory breaks down at the leading edge and the trailing edge. This is due to essential limitations of the so-called thin wing theory which is only a first order approximation.

of the airfoil is represented by

$$y = y_0 e^{i\omega t} \tag{5.1}$$

There is now a vertical acceleration $\ddot{y} = -\omega^2 y_0 e^{i\omega}$ which is the same at each point of the airfoil. This is also true for the velocity $\dot{y} = i\omega y_0 e^{i\omega t}$. For the case of the stationary theory in the section on The Stationary Airfoil a doublet was located at the point -1. In the present case an acceleration potential must be added to take care of the vertical acceleration at the wing; this extra term is provided by introducing a doublet at the center of the circle as shown in Fig. 7.

The harmonic conjugate functions are

$$\varphi = \varphi_1 + \varphi_2$$
$$\psi = \psi_1 + \psi_2$$

with

$$\begin{aligned} \varphi_1 &= A(\sin\theta_1/r_1) \qquad \varphi_2 = B(\sin\theta/r) \\ \psi_1 &= A(\cos\theta_1/r_1) \qquad \psi_2 = B(\cos\theta/r) \end{aligned} \tag{5.2}$$

In the following it is assumed that all functions are multiplied by $e^{i\omega t}$. This factor cancels out of the equations and is therefore dropped out of all expressions below. The acceleration of the wing must be equal to $-\omega^2 y_0$. Hence the value $B = \omega^2 y_0$. The term with B gives rise to the well known apparent mass.

The value of A must be determined by the kinematic condition for the velocity. Using Eq. (2.13) gives for the vertical velocity at a point $x = -1 + \epsilon$ on the airfoil,

$$i\omega y_0 = \frac{-e^{-(i\omega/U)(-1+\epsilon)}}{U}\int_{-\infty}^{-1+\epsilon} e^{i\omega x/U}\left(\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial x}\right)dx \tag{5.3}$$

This value is independent of the coordinate ϵ on the airfoil so that ϵ can be made infinitely small. There is a singularity at the leading edge and in order to make the expression integrable it is convenient to integrate by parts the term containing $\partial\psi_1/\partial x$. The limiting value for $\epsilon \to 0$ becomes

$$i\omega y_0 = -\frac{\psi_1}{U} + \frac{e^{i\omega/U}}{U}\int_{-\infty}^{-1} e^{i\omega x/U}\left(\frac{i\omega}{U}\psi_1 - \frac{\partial\psi_2}{\partial x}\right)dx \tag{5.4}$$

The value of ψ_1 on the airfoil is

$$\psi_1 = A/2$$

Noting from the conformal transformation (2.5) that

$$r_1 = -x - 1 + \sqrt{x^2 - 1}$$
$$r = -x + \sqrt{x^2 - 1}$$

and substituting $\theta_1 = \theta = \pi$ in Eq. (5.2) it is found that outside the airfoil (for $x < -1$)

$$\psi_1 = -A/(-x - 1 + \sqrt{x^2 - 1})$$
$$\psi_2 = -B/(-x + \sqrt{x^2 - 1})$$

To integrate $\int_{-\infty}^{-1} e^{i\omega x/U}\,\psi_1 dx$ the variable of integration x is replaced by $-x$. In reducing the integrals to the form below care must be exercised to give the radical the proper sign.

$$\int_{-\infty}^{-1} e^{i\omega x/U}\,\psi_1 dx = \frac{A}{2}\int_1^{\infty} e^{-i\omega x/U}\left[1 - \sqrt{\frac{x+1}{x-1}}\right]dx = -\frac{A}{2}\left[K_1\left(\frac{i\omega}{U}\right) + K_0\left(\frac{i\omega}{U}\right) + \frac{Ue^{-i\omega/U}}{i\omega}\right] \tag{5.5}$$

Similarly,

$$\int_{-\infty}^{-1} e^{i\omega x/U}\frac{\partial\psi_2}{\partial x}dx = \omega^2 y_0\int_1^{\infty} e^{-i\omega x/U}\left[1 - \frac{x}{\sqrt{x^2-1}}\right]dx = -\omega^2 y_0\left[K_1\left(\frac{i\omega}{U}\right) - \frac{U}{i\omega}e^{-i\omega/U}\right] \tag{5.6}$$

K_0 and K_1 are modified Bessel functions of the second kind.* Eq. (5.4) is an equation for A; after some simple cancellations it becomes

$$-\frac{A}{2U}\left[K_1\left(\frac{i\omega}{U}\right) + K_0\left(\frac{i\omega}{U}\right)\right] - i\omega y_0 K_1\left(\frac{i\omega}{U}\right) = 0 \tag{5.7}$$

Hence

$$A = -2i\omega y_0 U\frac{K_1(i\omega/U)}{K_1(i\omega/U) + K_0(i\omega/U)} \tag{5.8}$$

The lift distribution is obtained from the acceleration potential (5.2),

$$l = \rho A\tan(\theta/2) + 2\rho\omega^2 y\sin\theta \tag{5.9}$$

where A is the complex function of the frequency (5.8).

The second term in Eq. (5.9) represents the apparent mass due to the surrounding fluid. The first term shows that the non-inertia part of the lift distribution is the same as for the stationary case and has its resultant at the quarter chord point.

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* For an evaluation of the integrals (5.5) and (5.6) see reference (7).