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Loads on a Supersonic Wing Striking a Sharp-Edged Gust

M. A. BIOT*

Cornell Aeronautical Laboratory

ABSTRACT

This is a calculation of the chordwise lift distribution, total lift, and moment on a two-dimensional wing striking a sharp-edged gust at supersonic speed. A direct solution is established by considering a distribution of sources in a fluid at rest. An alternate method using Busemann's conical flow is also shown to be applicable. The time history of the total lift and mid-chord moment is discussed. It is shown that the total lift increases with time and reaches a maximum that corresponds to the steady-state phase of the flow. The mid-chord moment goes through a maximum independent of the Mach Number if the latter value is larger than $4/\pi$, while this maximum can become infinite for a range of Mach Numbers between $4/\pi$ and 1.

(1) INTRODUCTION

CONSIDERABLE ATTENTION has been given lately to nonstationary flow problems of wings flying at supersonic speeds. Most of the work, however, has been concerned with the aerodynamic forces on an oscillating airfoil from the standpoint of flutter analysis. The problem of the wing hitting a sharp-edged gust is of a different nature and turns out to be actually much simpler than the oscillating airfoil problems.

It is shown in section 2 that it may be treated by a distribution of sources of a simple type along the chord and that the pressure distribution may be derived by elementary methods. The procedure does not introduce a moving fluid but considers a fluid at rest in which nonstationary sources are distributed in a layer of variable extent. This point of view, which is closer to acoustics than to aerodynamics, is somewhat novel and seems to present advantages of simplicity and closeness to physical reality in certain categories of problems. The pressure distribution derived by this method is applied to the calculation of the time history of lift and moment on the wing in section 3. Particular attention is given to the value of the mid-chord moment, which starts from zero, rises to a maximum, and goes back to zero. The value of this maximum and related data is evaluated in section 4. These results are of particular interest to the designer.

The derivation of the pressure as given in section 1 is only one of the methods that may be used in this problem. As an independent check and as an illustration of the application of Busemann's method of conical flow to a nonstationary problem, an alternate derivation is given for the pressure distribution in section 4.

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* Member of Cornell Aeronautical Laboratory consulting staff. Also Professor of Applied Physical Sciences, Brown University.

In a paper by Schwarz[1] procedures used in oscillating airfoil theory are extended to the problem of a wing striking a sharp-edged gust at supersonic speed. Results for an oscillating down-wash lead to the gust problem by a Fourier integral representation. This method constitutes a considerable detour and introduces intermediate results of a transcendental nature which are actually not needed and are more complicated than the result. It may be verified that the expression derived in the present paper for the pressure distribution is equivalent to that derived by Schwarz. He does not, however, discuss the physical aspects of the problem or derive expressions for lift and moment.

(2) DERIVATION OF THE PRESSURE DISTRIBUTION

The wing of chord l enters a uniform gust of upward velocity v_0 at the supersonic velocity V (Fig. 1). The velocity component normal to the wing must remain zero, and this condition is equivalent to the generation of a velocity normal to the wing which cancels the gust velocity (Fig. 2). This may also be considered as a "reflection" of the gust on the wing. Because the velocity is supersonic, the pressure distribution on one side does not influence the pressure on the other, and therefore we need only consider the bottom side. The pressure distribution on top will be the same except for a reversal of sign. For the same reason the pressure distribution is not influenced by the trailing edge, and

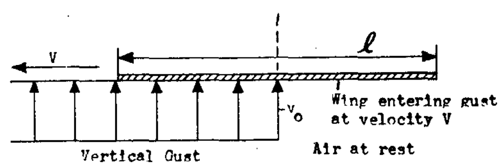

Figure 1

Wing Entering Vertical Gust

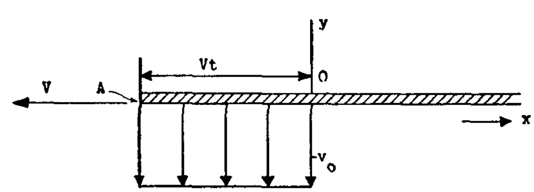

Figure 2

Reflection of the Gust on the Wing

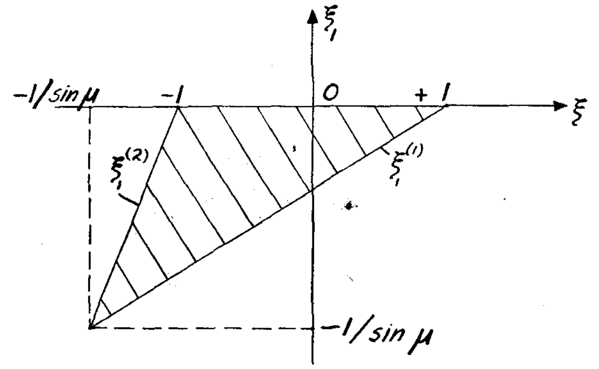

Figure 3

Limits of Integration for Integral (2.6)

everything is the same as if the wing were of infinite chord. The problem is then reduced to finding the pressure in a fluid, originally at rest, due to the presence of a uniform distribution of normal velocity v_0 along the chord from a point 0, corresponding to the edge of the gust, to the leading edge A. If the wing enters the gust at time zero the length 0A = Vt.

Now, such a distribution of normal velocity may be represented by a distribution of source singularities generated continuously at the leading edge and remaining constant thereafter. The velocity potential of such a source appearing suddenly with a constant intensity at time t_1 and location x_1 is

$$\phi = \frac{v_0}{\pi}\log r - \frac{v_0}{\pi}\log\left[c(t-t_1) + \sqrt{c^2(t-t_1)^2 - r^2}\right] \tag{2.1}$$

with

$$r^2 = (x - x_1)^2 + y^2$$

This expression satisfies the wave equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{2.2}$$

where c = velocity of sound. Moreover, for sufficiently small values of r with respect to $c(t - t_1)$, it represents a velocity field

$$\phi = (v_0/\pi)\log r \tag{2.3}$$

identical with that of a steady source in the incompressible flow. Hence, by analogy, it may be concluded that the uniform distribution of such sources will produce a uniform normal velocity component v_0.

The pressure generated by this source along the x axis ($y = 0$) is

$$p = -\rho\frac{\partial\phi}{\partial t} = \frac{\rho c v_0}{\pi\sqrt{c^2(t-t_1)^2 - (x-x_1)^2}} \tag{2.4}$$

where ρ = fluid density. Note that the source located at x_1 suddenly appears when the leading edge reaches that point, i.e., at a time $t_1 = -(x_1/V)$.

The local lift $2p$ at a point x and time t, due to the uniform distribution of such sources from $x_1 = -Vt$ to $x_1 = 0$, is given by

$$2p = \frac{2}{\pi}\rho c v_0 \int_{-Vt}^{0} \frac{dx_1}{\sqrt{c^2[t + (x_1/V)]^2 - (x - x_1)^2}} \tag{2.5}$$

or with the change of variables

$$x/ct = \xi, \qquad x_1/ct = \xi_1, \qquad c/V = \sin\mu$$

$$M = \frac{1}{\sin\mu} = \text{Mach Number}$$

$$2p = \frac{2}{\pi}\rho c v_0 \int_{-1/\sin\mu}^{0} \frac{d\xi_1}{\sqrt{(1 + \xi_1\sin\mu)^2 - (\xi - \xi_1)^2}} \tag{2.6}$$

In integrating this expression special attention must be given to the limits of integration. The function under the integral sign must be taken as vanishing whenever the radical is imaginary. The range of integration is therefore limited between the two roots of the equation,

$$(1 + \xi_1\sin\mu)^2 - (\xi - \xi_1)^2 = 0$$

These roots are

$$\xi_1^{(1)} = \frac{\xi - 1}{1 + \sin\mu}, \qquad \xi_1^{(2)} = \frac{\xi + 1}{1 - \sin\mu} \tag{2.7}$$

These quantities, plotted as functions of ξ, are represented by two straight lines (Fig. 3) which intersect at a point of abscissa

$$\xi = -(1/\sin\mu)$$

and ordinate

$$\xi_1 = -(1/\sin\mu)$$

The interval of integration is thus limited to the shaded area bordered by the two straight lines [Eq. (2.7)] and the axis ($\xi_1 = 0$). We must therefore distinguish between two ranges of values of ξ: for $-(1/\sin) < \xi < -1$

$$2p = \frac{2}{\pi}\rho c v_0 \int_{\xi_1^{(1)}}^{\xi_1^{(2)}} \times \frac{d\xi_1}{\sqrt{(1 + \xi_1\sin\mu)^2 - (\xi - \xi_1)^2}} = \frac{2\rho c v_0}{\cos\mu} \tag{2.8}$$

Hence, for this interval of ξ—i.e., between the leading edge and the abscissa $x = -ct$—the lift distribution is uniform and independent of x. It may be verified that it is identical with the lift on a wing in steady flow with an angle of attack v_0/V. This could have been concluded immediately, since that portion of the wing cannot be influenced by the presence of the gust edge because of the finite velocity of propagation of a signal originating at this edge.

The other range of integration $-1 < \xi < 1$ corresponds to points located at distances smaller than ct from the gust edge.

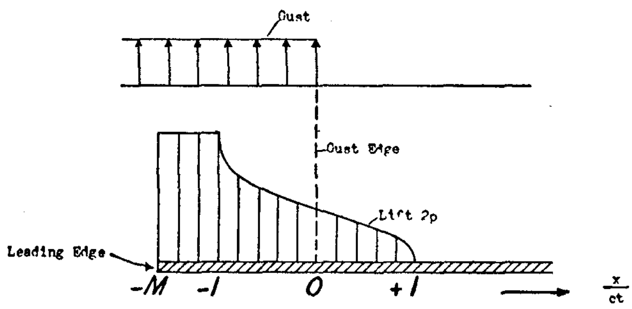

Figure 4
Lift Distribution Due to the Gust

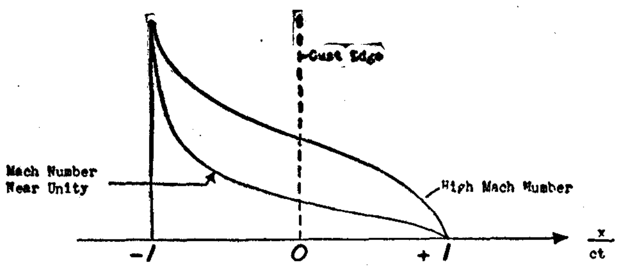

Figure 5
Lift Distribution For Various Mach Numbers

$$\left.\begin{aligned} 2p &= \frac{2}{\pi}\rho c v_0 \int_{\xi_1^{(1)}}^{0} \frac{d\xi_1}{\sqrt{(1+\xi_1\sin\mu)^2-(\xi-\xi_1)^2}} \\ &= \frac{2\rho c v_0}{\pi\cos\mu}\cos^{-1}\left(\frac{\xi+\sin\mu}{1+\xi\sin\mu}\right) \end{aligned}\right\} \quad (2.9)$$

In this region the lift distribution depends on the time. Substituting $\xi = x/ct$,

$$2p = \frac{2}{\pi}\frac{\rho c v_0}{\cos\mu}\cos^{-1}\left(\frac{x+ct\sin\mu}{ct+x\sin\mu}\right) \qquad (2.10)$$

The $\cos^{-1}$ branch is between zero and π. It will be observed that the lift distribution depends only on x/ct—i.e., there is a similarity law and we may draw a one parameter family of lift distribution curves with the variable x/ct and the Mach Number as a parameter. The appearance of the lift distribution is shown in Fig. 4. We see that the production of lift is not limited to the region of the gust. The lift propagates ahead of the edge of the gust with the velocity of sound. The lift distribution for various Mach Numbers within the region affected by the gust edge is shown qualitatively in Fig. 5.

(3) Lift and Moment

It is useful for practical purposes to obtain the values of total lift and moment as functions of time for a gust acting on a wing of chord l. Because of similarity properties, such quantities may be expressed by means of nondimensional functions of Vt/l only. In computing the time history of lift and moment, we must distinguish between three phases.

Phase 1.—Where the trailing edge is still outside the region where lift is produced, $(V + c)t < l$.

Phase 2.—Where the trailing edge is in the region influenced by the gust edge, $(V + c)t > l > (V - c)t$.

Phase 3.—Where the entire wing is outside the region influenced by the gust edge, $l < (V - c)t$.

Phase 1

We integrate the lift distribution $2p$ over the chord l and split the integration into two intervals, one in which the lift distribution is uniform and the other in which it is not. Thus the total lift is

$$L = 2\int_{-Vt}^{-ct} p\,dx + 2\int_{-ct}^{+ct} p\,dx \qquad (3.1)$$

or

$$L = 2ct\int_{-1/\sin\mu}^{-1} p\,d\xi + 2ct[p\xi]_{-1}^{+1} - 2ct\int_{-1}^{+1}\xi\frac{dp}{d\xi}\,d\xi \qquad (3.2)$$

This form is readily integrable. We find

$$L = 2\rho c v_0 l(Vt/l) \qquad (3.3)$$

Similarly, the pitching moment about the leading edge M_L is

$$M_L = 2\int_{-Vt}^{-ct}(Vt+x)p\,dx + 2\int_{-ct}^{+ct}(Vt+x)p\,dx \qquad (3.4)$$

or

$$M_L = 2(ct)^2\int_{-1/\sin\mu}^{-1}\left(\frac{1}{\sin\mu}+\xi\right)p\,d\xi + (ct)^2\left[\left(\frac{1}{\sin\mu}+\xi\right)^2 p\right]_{-1}^{+1} - (ct)^2\int_{-1}^{+1}\left(\frac{1}{\sin\mu}+\xi\right)^2\frac{dp}{d\xi}\,d\xi \qquad (3.5)$$

More often we are interested in the stalling moment $M_{1/2}$ about the mid-chord. We find, after integration,

$$M_{1/2} = \frac{1}{2}lL - M_L = \rho c v_0 l^2\frac{Vt}{l}\left(1-\frac{Vt}{l}\right) \qquad (3.6)$$

Phase 2

Similarly, the lift and moment during Phase 2 are

$$L = 2\int_{-Vt}^{-ct} p\,dx + 2\int_{-ct}^{l-Vt} p\,dx \qquad (3.7)$$

$$M_L = 2\int_{-Vt}^{-ct}(Vt+x)p\,dx + 2\int_{-ct}^{l-Vt}(Vt+x)p\,dx \qquad (3.8)$$

or

$$L = 2ct \int_{-1/\sin\mu}^{-1} p\,d\xi + 2ct[p\xi]_{-1}^{z} - 2ct\int_{-1}^{z} \xi \frac{dp}{d\xi}\,d\xi \quad (3.9)$$

$$M_L = 2(ct)^2 \int_{-1/\sin\mu}^{-1} \left(\frac{1}{\sin\mu} + \xi\right) p d\xi + (ct)^2 \left[\left(\frac{1}{\sin\mu} + \xi\right)^2 p\right]_{-1}^{z} - (ct)^2 \int_{-1}^{z} \left(\frac{1}{\sin\mu} + \xi\right)^2 \frac{dp}{d\xi}\,d\xi \quad (3.10)$$

with $z = (1/\sin\mu)[(l/Vt) - 1]$. We find

$$\frac{L}{L_s} = \frac{1}{\pi}\cos^{-1}\left[\frac{1}{\sin\mu}\left(1 - \frac{Vt}{l}\cos^2\mu\right)\right] + \frac{Vt}{\pi l}\cos\mu\left\{\sin^{-1}\left[\frac{1}{\sin\mu}\left(\frac{l}{Vt} - 1\right)\right] + \frac{\pi}{2}\right\} \quad (3.11)$$

with $L_s = 2\rho c v_0 l/\cos\mu$. The $\sin^{-1}$ branch is taken between $-(\pi/2)$ and $+(\pi/2)$. The stalling moment about the mid-chord $M_{1/2}$ is given by

$$\frac{M_{1/2}}{\rho c v_0 l^2} = \frac{Vt}{\pi l}\left(1 - \frac{Vt}{l}\right)\left\{\sin^{-1}\left[\left(\frac{l}{Vt} - 1\right)\frac{1}{\sin\mu}\right] + \frac{\pi}{2}\right\} + \frac{1}{\pi}\left(\frac{Vt}{l}\right)^2 \sqrt{\sin^2\mu - \left(\frac{l}{Vt} - 1\right)^2} \quad (3.12)$$

At the end of Phase 2, $l = (V - c)t$, and we may verify that the above formulas yield $L = L_s = 2\rho c v_0 l/\cos\mu$ and $M_{1/2} = 0$.

Phase 3

In this phase the lift and moment remain independent of time. We find $M_{1/2} = 0$ and $L = L_s = 2\rho c v_0 l/\cos\mu$, which is the lift in steady-state flight of a wing of angle of attack v_0/V. The lift increases all through Phases 1 and 2 and reaches its maximum in Phase 3. The time history of the mid-chord moment requires special attention, as shown in the next section.

(4) Maximum Value of the Mid-Chord Moment

With a nondimensional time variable $\tau = Vt/l$, the mid-chord moment during Phase 1 is [cf. Eq. (3.6)]

$$M_{1/2} = \rho c v_0 l^2 \tau(1 - \tau) \quad (4.1)$$

This curve is a parabola with a maximum at $\tau = 1/2$. The value of the maximum is

$$[M_{1/2}]_{max.} = (1/4)\rho c v_0 l^2 \quad (4.2)$$

It is interesting to note that during the first phase the mid-chord moment is independent of the Mach Number.

During Phase 2 the mid-chord moment is as given by Eq. (3.12)

$$\frac{M^{1/2}}{\rho c v_0 l^2} = \tau(1 - \tau)\frac{1}{\pi}\left[\sin^{-1}\left(\frac{1 - \tau}{\tau\sin\mu}\right) + \frac{\pi}{2}\right] + \frac{\tau^2}{\pi}\sqrt{\sin^2\mu - \left(\frac{1 - \tau}{\tau}\right)^2} \quad (4.3)$$

The second phase originates at $\tau = 1/(1 + \sin\mu)$ and terminates at $\tau = 1/(1 - \sin\mu)$. It is found that in this range the moment $M_{1/2}$ may go through another maximum and that this maximum may be larger than its value in the first phase. Value of this maximum and the value of τ at which it occurs are given in Table 1 for various Mach Numbers.

Table 1

Mach Number $(1/\sin\mu)$	τ	$\frac{[M_{1/2}]_{max.}}{\rho c v_0 l^2}$
1	∞	∞
1.11	2.40	0.331
1.17	1.55	0.281
1.25	1.10	0.255
$4/\pi$	1.00	$1/4$

The maximum of $M_{1/2}$ in the second phase is greater than the maximum $M_{1/2} = (1/4)\rho c v_0 l^2$ in the first phase, if $(1/\sin\mu) < (4/\pi) = 1.27$. The value of the absolute maximum of the mid-chord moment is plotted against the Mach Number in Fig. 6. This maximum is independent of the Mach Number if this Mach Number is greater than 1.27.

(5) Alternate Derivation of the Pressure Distribution by the Method of Conical Flow

The above results may be derived by an entirely different procedure. We may compute the velocity field due to the gust reflection on the wing. Because

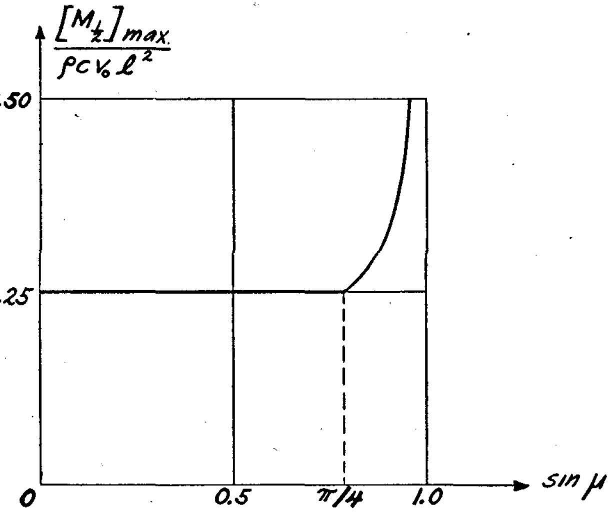

Figure 6

Absolute Maximum of the Mid-Chord Moment we deal with a supersonic wing velocity, the effect of the gust is the same as if the chord were infinite. Therefore, the principle of similarity applies, and the velocity field is similar at all instants except for a scale factor proportional with time. In other words, the velocity field depends only on the variables

$$\xi = x/ct, \qquad \eta = y/ct \tag{5.1}$$

The velocity field is disturbed by the reflection on the wing in a region bounded by the straight lines AFF′ and the circle FEF′ centered at the gust edge 0 (Fig. 7). In the shaded area AFDF′ the field is uniform and corresponds to the steady-state motion of a wing with constant angle of attack. The transient field where the effect of the gust edge is being felt is inside a circle of radius ct centered at 0. The field inside this circle may be computed by Busemann's conical flow method.

By the transformation

$$\left.\begin{matrix} \xi = \lambda \cos\theta, & \eta = \lambda \sin\theta, & \lambda = 2s/1 + s^2) \\ X = s\cos\theta, & Y = s\sin\theta & \end{matrix}\right\} \tag{5.2}$$

The wave equation [Eq. (2.2)] is transformed to Laplace's equation in the X, Y, plane

$$(\partial^2\phi/\partial X^2) + (\partial^2\phi/\partial Y^2) = 0 \tag{5.3}$$

Consider now the components of the velocity field

$$u = \partial\phi/\partial x, \qquad v = \partial\phi/\partial y$$

They also satisfy Eq. (5.3). It is convenient to introduce the complex variable $Z = X + iY$. Let us investigate the velocity field on the bottom side of the wing. The v component of the velocity field is

$$v = Re\,\frac{iv_0}{\pi}\,[\log(Z - Z_1) + \log(Z - Z_2) - \log Z] \tag{5.4}$$

where Re = real part of; $Z_1 = ie^{i\mu}$; $Z_2 = -ie^{-i\mu}$; $\sin\mu = c/V$. This expression satisfies Eq. (5.3) and the boundary value of v on the circle and the wing—i.e., $v = -v_0$ on 0DF′ and $v = 0$ on F′E0. In terms of real quantities,

$$v = \frac{-v_0}{\pi}\tan^{-1}\left\{\frac{[s - (1/s)]\sin\theta}{[s + (1/s)]\cos\theta + 2\sin\mu}\right\} \tag{5.5}$$

Now we are interested in the pressure distribution on the wing. This pressure distribution for $\eta = 0$ may be derived from the above expression for v by making use of the equations of motion and continuity

$$\rho\,(\partial u/\partial t) = -(\partial p/\partial x)$$

$$\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{-1}{c^2}\left(\frac{\partial p}{\partial t}\right) \tag{5.6}$$

By elimination of u and transformation of variables, we find that, on the x axis, $\cos\theta = \pm 1$. This implies

$$\cos\theta(\partial v/\partial\theta) = [(\lambda^2 - 1)/\rho c](\partial p/\partial\lambda) \tag{5.7}$$

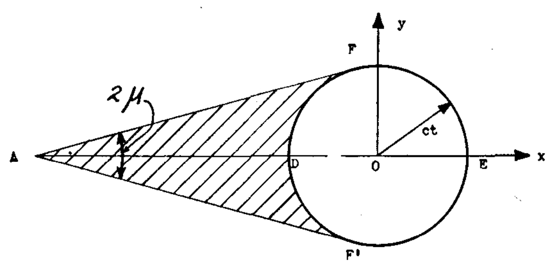

Figure 7

Region Affected by the Gust Reflection on the Wing

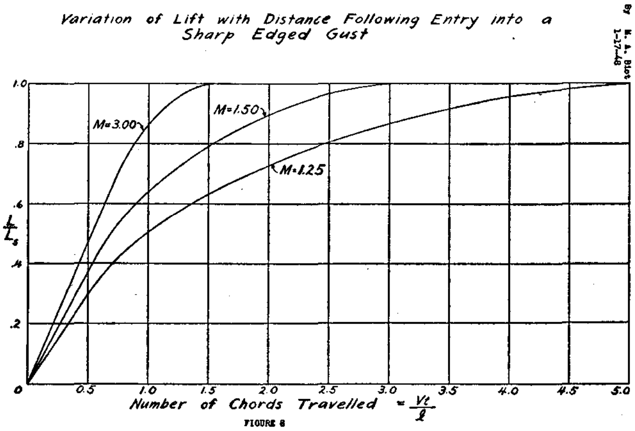

FIGURE 8

This may be written

$$-\frac{\rho c v_0}{\pi}\,\frac{1}{\sqrt{1 - \xi^2}}\,\frac{1}{1 + \xi\sin\mu} = \frac{\partial p}{\partial\xi} \tag{5.8}$$

By integration

$$p = \frac{\rho c v_0}{\pi\cos\mu}\cos^{-1}\left(\frac{\xi + \sin\mu}{1 + \xi\sin\mu}\right) \tag{5.9}$$

which coincides with expression (2.9) above.

Conclusions

It has been shown that the pressure distribution on a supersonic wing striking a sharp-edged gust may be expressed by a simple formula. This pressure distribution is obtained by direct integration of a variable distribution of sources in a fluid at rest. It is also shown that the same result is obtainable from Busemann's method of conical flow. From the time history of total lift and moment it is concluded that the largest value of the total lift is reached in the last phase—i.e., when a steady flow has been established—while for the mid-chord moment a maximum value $[M1/2]_{max.} = (1/4)\rho c v_0 l^2$ is reached if the Mach Number is larger than $4/\pi$. This maximum moment is independent of the Mach Number. However, for Mach Numbers between $4/\pi$ and 1, the maximum mid-chord moment varies with the Mach Number and becomes infinite at Mach Number 1. These conclusions are, of course, subject to the usual limitations of the linearized small perturbation theory.

Reference

[1] Schwarz, L., *Ebene Instationaere Theorie des Tragflaeche bei Ueberschallgeschurin digkeit AVA Goettingen,* B43/J/17, July, 1943, translation by AAF No. F-TS-934-Re.