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THE INTERACTION OF RAYLEIGH AND STONELEY WAVES IN THE OCEAN BOTTOM*

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ABSTRACT

A theory is developed for the propagation of two-dimensional unattenuated waves in a system consisting of a liquid layer overlying an infinitely thick solid. Special attention is given to the interaction between the Stoneley type of wave and the Rayleigh wave. It is shown that the type of wave discussed corresponds to a dispersion branch for which the velocity varies continuously from a value lower than the velocity of sound in the liquid to that of the Rayleigh waves. The possible importance of this fact is pointed out in connection with the interpretation of the T phase of shallow-focus submarine earthquakes. The physical nature of these waves is illustrated by showing that they exist at the interface of a massless solid and an incompressible fluid.

Introduction. The problem of the propagation of elastic waves in the ocean when the elasticity of the ocean bottom is taken into account has been treated previously. Pekeris[1] developed a very thorough mathematical theory which assumed the ocean bottom to be an elastic liquid. Stoneley[2] had already considered the effect of an elastic bottom, but with special reference to the correction necessitated by the effect of the ocean on the propagation of Rayleigh waves; that is, for the case when the wave length is large as compared with the ocean depth. Press and Ewing[3] treated the general problem of fluid and solid bottom interaction. However, they did not derive the existence of waves which belong to the general category known as Stoneley waves and which propagate with a phase velocity lower than that of sound in the water.

These waves appear at small wave lengths, and it is shown further on that their velocity increases continuously with the wave length, finally corresponding to that of Rayleigh waves for large wave lengths. When the dispersion curve for the phase velocity is plotted, there is a point at which these waves reach the velocity of sound in water, but which does not correspond to any singular property of the propagation. It is also pointed out that there are an infinite number of branches for the dispersion curves. In the present paper, the branch of lowest velocity is the one which is studied in most detail and which shows the continuous transition between Stoneley waves and Rayleigh waves.

In the following numbered sections the problem of wave propagation in a liquid layer overlying an infinitely deep elastic solid is treated for the case of two-dimensional unattenuated waves. Section 1 studies waves in the fluid independently of the bottom, and section 2 considers waves in the bottom independently of the overlying fluid. In section 3, the two systems are coupled together by equating the ratio of normal stress to displacement at the interface for both media. This procedure has

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[1] C. L. Pekeris, "Theory of Propagation of Explosive Sound in Shallow Water," in "Sound Transmission in the Ocean," Geol. Soc. Am., *Memoir 27* (1948).

[2] R. Stoneley, "The Effect of the Ocean on Rayleigh Waves," *Mon. Not. Roy. Astron. Soc., Geophys. Suppl.*, 1: 349–356 (1946).

[3] E. Press and M. Ewing, "A Theory of Microseisms, with Geologic Applications," *Trans. Am. Geophys. Union*, 29: 163–174 (1948).

the advantage of bringing out more clearly certain physical aspects of the phenomenon. Phase and group velocity dispersion curves are computed for a number of typical values of the physical constants. In section 4 it is shown that Stoneley waves exist at the interface of a massless solid and an incompressible fluid, demonstrating that they are by nature independent of the existence of body waves. While the phase velocities of the Stoneley waves are found to be only slightly lower than the velocity of sound in water, the importance of this fact arises from the possibility that they may be strongly coupled with waves having the same velocity in the SOFAR channel as explained in section 5.

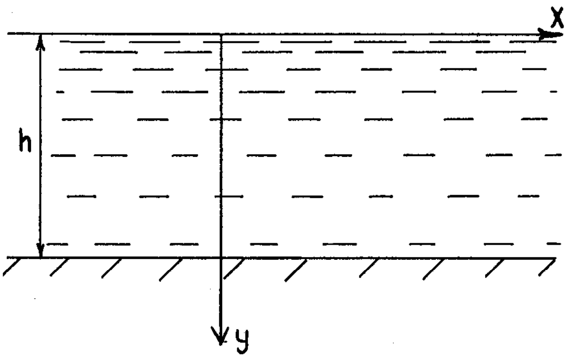

Fig. 1.

1. Motion and pressure in the fluid. Consider an ocean of finite depth h (fig. 1). The displacement components of the fluid are written as

$$(U', V') = \operatorname{grad} \varphi \,, \tag{1.1}$$

where ϕ satisfies the equation

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} \,, \tag{1.2}$$

c is the velocity of sound in the ocean, and t is the time variable. Equation (1.2) is verified by the solutions

$$\varphi = \frac{A}{\sqrt{\dfrac{\alpha^2}{c^2} - l^2}} \sin y \sqrt{\frac{\alpha^2}{c^2} - l^2} \cos (lx - \alpha t) \quad \left(\text{for } \frac{\alpha^2}{c^2} > l^2\right) \tag{1.3}$$

or

$$\varphi = \frac{A}{\sqrt{l^2 - \dfrac{\alpha^2}{c^2}}} \sinh y \sqrt{l^2 - \frac{\alpha^2}{c^2}} \cos (lx - \alpha t) \quad \left(\text{for } \frac{\alpha^2}{c^2} < l^2\right) . \tag{1.4}$$

A is a constant with the dimension of a length.

The pressure in the fluid is

$$p = -\rho \frac{\partial^2 \varphi}{\partial t^2}$$

where ρ is the fluid mass density. The pressures at the bottom $y = h$ corresponding to solutions (1.3) and (1.4) are, respectively

$$p = \frac{A\rho\alpha^2}{\sqrt{\frac{\alpha^2}{c^2} - l^2}} \sin y \sqrt{\frac{\alpha^2}{c^2} - l^2} \cos (lx - \alpha t) \tag{1.5}$$

$$p = \frac{A\rho\alpha^2}{\sqrt{l^2 - \frac{\alpha^2}{c^2}}} \sinh y \sqrt{l^2 - \frac{\alpha^2}{c^2}} \cos (lx - \alpha t) \ . \tag{1.6}$$

The particular solutions (1.3) and (1.4) are chosen in such a way that they satisfy the boundary condition $p = 0$ at the surface ($y = 0$) of the ocean.

The vertical displacement component at the bottom is

$$V' = \frac{\partial \varphi}{\partial y} \qquad \text{for } y = h \ ;$$

hence for each solution (1.3) and (1.4), respectively,

$$V' = A \cos \left(lh\sqrt{\frac{\alpha^2}{c^2 l^2} - 1} \right) \cos (lx - \alpha t) \tag{1.7}$$

$$V' = A \cosh \left(lh \sqrt{1 - \frac{\alpha^2}{c^2 l^2}} \right) \cos (lx - \alpha t) \ . \tag{1.8}$$

For the purpose of the present theory we shall need the ratio of the pressure to the displacement at the bottom. Putting $\zeta = \alpha/cl$, we may write

$$\frac{p}{V'} = \frac{\alpha^2 \rho}{l\sqrt{\zeta^2 - 1}} \tan (lh \sqrt{\zeta^2 - 1}) \qquad (\zeta > 1) \tag{1.9}$$

$$\frac{p}{V'} = \frac{\alpha^2 \rho}{l\sqrt{1 - \zeta^2}} \tanh (lh\sqrt{1 - \zeta^2}) \qquad (\zeta < 1) \ . \tag{1.10}$$

We shall make use of these expressions in section 3 below.

The interest in the ratio p/V' lies in its nature of a mechanical impedance. As such, it adds physical significance to the boundary conditions at the fluid-solid interface which can be fulfilled by a process of "impedance matching."

2. Motion and stresses in the solid. We consider now the motion of the solid bottom. We take the x axis along the solid boundary with the y axis directed downward (fig. 2). The displacement components of the solid are written

$$\begin{aligned} U &= \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \\ V &= \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \ , \end{aligned} \tag{2.1}$$

where ϕ and ψ satisfy the equations

$$\begin{aligned}\nabla^2\phi &= \frac{1}{v_c^2}\frac{\partial^2\phi}{\partial t^2} \\ \nabla^2\psi &= \frac{1}{v_s^2}\frac{\partial^2\psi}{\partial t^2}\,.\end{aligned} \tag{2.2}$$

v_c = velocity of dilatational waves.
v_s = velocity of rotational waves.

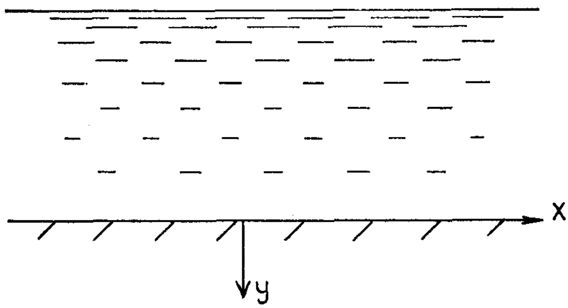

Fig. 2.

The stress components σ_y and τ are given in terms of ϕ and ψ by the expressions

$$\begin{aligned}\sigma_y &= \rho_1 b\,\frac{\partial^2\phi}{\partial t^2} + 2v_s^2\rho_1\left(\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\psi}{\partial x\partial y}\right) \\ \tau &= \rho_1\frac{\partial^2\psi}{\partial t^2} + 2v_s^2\rho_1\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x^2}\right),\end{aligned} \tag{2.3}$$

with $b = \nu/(1-\nu)$, ν = Poisson ratio of the solid, and ρ_1 = solid density. Solutions of equations (2.2) are

$$\begin{aligned}\phi &= \phi_0 e^{-my}\cos(lx - \alpha t) \\ \psi &= \psi_0 e^{-ky}\sin(lx - \alpha t)\end{aligned} \tag{2.4}$$

where ϕ_0 and ψ_0 are constants to be determined and

$$m = \sqrt{l^2 - \frac{\alpha^2}{v_c^2}}, \qquad k = \sqrt{l^2 - \frac{\alpha^2}{v_s^2}}.$$

We introduce the notation

$$\zeta_1 = \frac{\alpha}{v_s l}, \qquad \zeta_2 = \frac{\alpha}{v_c l}; \tag{2.5}$$

hence

$$m = l\sqrt{1-\zeta_2^2}, \qquad k = l\sqrt{1-\zeta_1^2}. \tag{2.6}$$

We note that in the expressions we must have $\zeta_1 < 1$ and $\zeta_2 < 1$ in order for the solutions (2.4) to vanish at $y = \infty$ in the solid. The boundary condition that the shear stress τ vanish at the water-solid interface is expressed by substituting solutions (2.4) in the second equation (2.3) and putting $\tau = 0$ for $y = 0$. This leads to the following relation between ϕ_0 and ψ_0

$$\frac{\phi_0}{\psi_0} = -\frac{1}{2}\frac{(2 - \zeta_1^2)}{\sqrt{1 - \zeta_2^2}} . \tag{2.7}$$

The vertical displacement V of the solid at the boundary $y = 0$ is

$$V = -(m\phi_0 + l\psi_0) \cos (lx - \alpha t) . \tag{2.8}$$

Using relation (2.7), this may be simplified to

$$V = -\tfrac{1}{2}\, l\psi_0\zeta_1^2 \cos (lx - \alpha t) . \tag{2.9}$$

The vertical stress component σ_y at $y = 0$ is given by the first relation (2.3)

$$\frac{\sigma_y}{\rho_1 v_s^2} = \left[-\frac{\alpha^2}{v_s^2}\, b\phi_0 + 2m^2\phi_0 + 2kl\psi_0\right] \cos (lx - \alpha t) . \tag{2.10}$$

The expression which is finally needed in the present theory is the ratio of the stress σ_y to the displacement V at the boundary. From (2.9) and (2.10), by substituting the value (2.7) for the ratio ϕ_0/ψ_0, we derive

$$\frac{\sigma_y}{\rho_1 v_s^2 l V} = -\frac{b(2 - \zeta_1^2)}{\sqrt{1 - \zeta_2^2}} + \frac{2}{\zeta_1^2}\sqrt{1 - \zeta_2^2}\,(2 - \zeta_1^2) - \frac{4}{\zeta_1^2}\sqrt{1 - \zeta_1^2} . \tag{2.11}$$

One may eliminate the parameter b by expressing it in terms of ζ_1 and ζ_2. We note that the relation between the dilatational and rotational wave velocities is

$$\left(\frac{\zeta_2}{\zeta_1}\right)^2 = \left(\frac{v_s}{v_c}\right)^2 = \frac{1 - 2\nu}{2(1 - \nu)} = \frac{1}{2}(1 - b) ;$$

hence

$$b = 1 - 2\left(\frac{\zeta_1}{\zeta_2}\right)^2 , \tag{2.12}$$

and we may write

$$\frac{\sigma_y}{\rho_1 v_s^2 l V} = \frac{1}{\zeta_1^2}\left[\frac{(2 - \zeta_1^2)^2}{\sqrt{1 - \zeta_2^2}} - 4\sqrt{1 - \zeta_1^2}\right] . \tag{2.13}$$

This last expression will be used in the following section. It represents the ratio of normal stress to displacement and corresponds to a mechanical impedance similar to expressions (1.9) and (1.10) for the fluid.

3. The propagation of waves in the coupled fluid-solid system. We now consider the interaction of the fluid and the solid by introducing the boundary condition at the interface

$$-p = \sigma_y$$

$$V' = V$$

or

$$\frac{p}{V'} = -\frac{\sigma_y}{V} . \tag{3.1}$$

It should be noted that (3.1) is a sufficient boundary condition since it implies that if $V = V'$ we also have $-p = \sigma_y$. On the other hand, $V = V'$ does not have to be introduced explicitly since the amplitude of the solutions obtained above contains the arbitrary factors A and ψ_0. The condition (3.1) corresponds to an "impedance matching."

Using expressions (1.9), (1.10), and (2.13) for these ratios, we have the equations

$$4\sqrt{1-\zeta_1^2} - \frac{(2-\zeta_1^2)^2}{\sqrt{1-\zeta_2^2}} = \frac{\rho}{\rho_1}\frac{\zeta_1^4}{\sqrt{\zeta^2-1}}\tan\left[lh\sqrt{\zeta^2-1}\right] \qquad \text{for } \zeta > 1 \tag{3.2}$$

$$4\sqrt{1-\zeta_1^2} - \frac{(2-\zeta_1^2)^2}{\sqrt{1-\zeta_2^2}} = \frac{\rho}{\rho_1}\frac{\zeta_1^4}{\sqrt{1-\zeta^2}}\tanh\left[lh\sqrt{1-\zeta^2}\right] \qquad \text{for } \zeta < 1 . \tag{3.3}$$

These equations yield the phase velocity

$$v = \frac{\alpha}{l} .$$

Since $\zeta_1 = (c/v_s)\zeta$ and $\zeta_2 = (c/v_c)\zeta$, we may consider ρ/ρ_1, c/v_s, c/v_c as given parameters and plot

$$\zeta = \frac{v}{c}$$

as a function of the variable

$$lh = 2\pi\frac{h}{\lambda} .$$

The variable λ/h is the nondimensional ratio of the wave length λ (in the x direction) to the ocean depth h.

The curves below are all plotted with lh or λ/h as the independent variable. Similar curves are also obtained in terms of the frequency α by following a representative point on the curve and using $(v/c)lh = \alpha h/c$ as the independent variable.

We shall first discuss the nature of the dispersion curve for the particular case

$$\frac{\rho}{\rho_1} = 1 , \qquad \frac{v_s}{c} = 1.5 , \qquad \nu = \frac{1}{2} .$$

This case corresponds to $v_c = \infty$, that is, an incompressible solid. For these values of the parameters we solve equations (3.2) and (3.3) and plot $\zeta = v/c$ as a function of

λ/h. The curves are made up of an infinite number of branches. Three of these branches, marked (1), (2), (3), are shown in figure 3. The lowest branch (1) corresponds to the interacting Rayleigh and Stoneley waves.

Let us discuss the nature of the lowest branch (1) of figure 3. For infinitely small wave lengths it starts at the ordinate $\zeta = 0.956$. That portion of the lowest branch

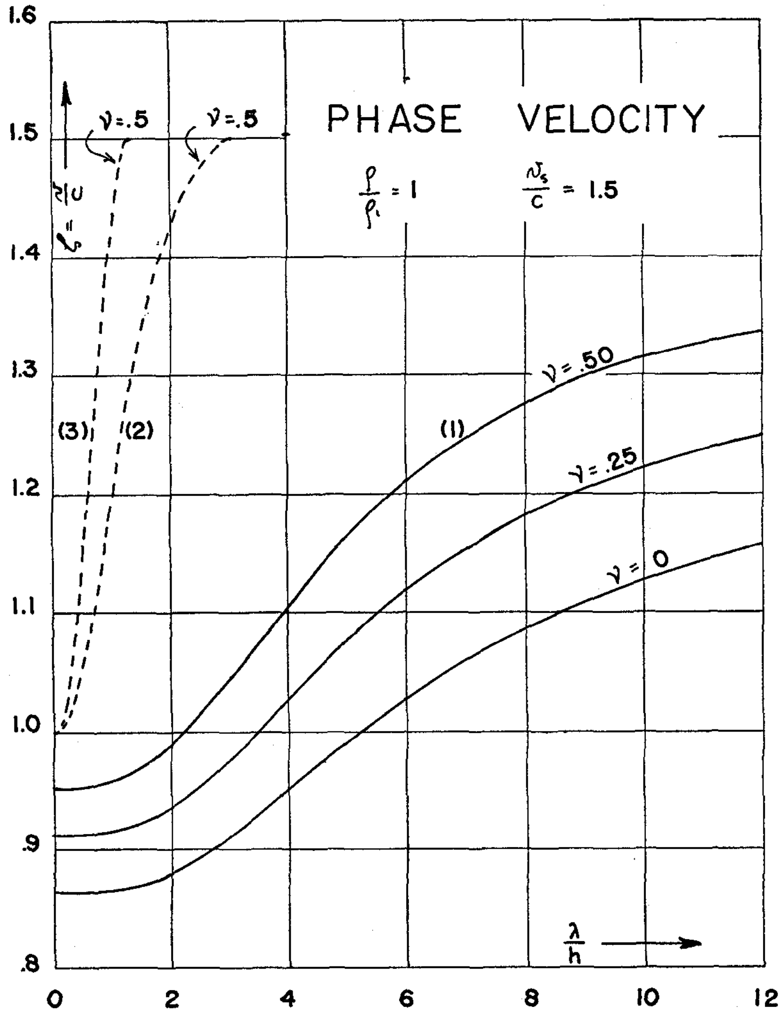

Fig. 3.

represents pure Stoneley waves. The amplitudes for such waves are concentrated near the water-solid interface and decrease exponentially with the distance from the interface in both water and solid. The rate of decrease of the amplitude with the distance from the interface is larger the smaller the wave length.

These waves are practically not influenced by the free surface of the water. As the wave length increases, the phase velocity increases, and for $\lambda/h = 2.21$ it reaches the speed of sound in water ($\zeta = 1$). At this point the pressure distribution in the water is linear starting from zero at the free surface and reaching a maximum at the bottom (fig. 4). This may be verified by putting $(\alpha^2/c^2) - l^2 = 0$ in expressions (1.5) and (1.6). We find

$$p = \rho\alpha^2 y e^{i(lx-\alpha t)} . \tag{3.4}$$

For further increase in the wave length the phase velocity continues to increase. For large wave lengths the influence of the water body tends to disappear and the velocity tends toward an asymptotic value

$$\zeta = \frac{v}{c} = 1.432 .$$

This may also be written

$$v = 0.955\, v_s ,$$

which is the velocity of a pure Rayleigh wave in an incompressible solid ($\nu = \frac{1}{2}$).

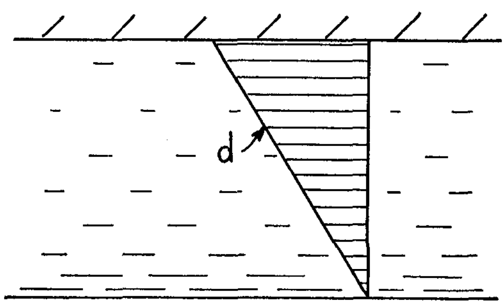

Fig. 4.

It will be noted that this lowest branch of the dispersion curve goes right through the speed of sound in the fluid without any singular behavior at that point. There are *no cut-off properties* associated with this point. This is a consequence of the mathematical fact that the function

$$\frac{\tan \sqrt{1 - z^2}}{\sqrt{1 - z^2}}$$

which appears in equation (3.2) is regular at the point $z = 1$.

As mentioned above, there are also an infinite number of other dispersion branches which correspond to modes of higher order. Two of these branches are shown as dotted lines in figure 3. Phase velocities for these branches are always higher than the speed of sound in water. These branches exhibit a cut-off for phase velocities equal to the rotational wave velocity v_s in the bottom; in this case, for $\zeta = 1.5$.

Other values of the parameters have also been considered, and the lowest branch dispersion curves have been plotted. Figure 3 shows the effect of the Poisson ratio ν on the lowest branch of the phase-velocity dispersion. This branch is plotted for

$$\frac{\rho}{\rho_1} = 1 \qquad \frac{v_s}{c} = 1.5$$

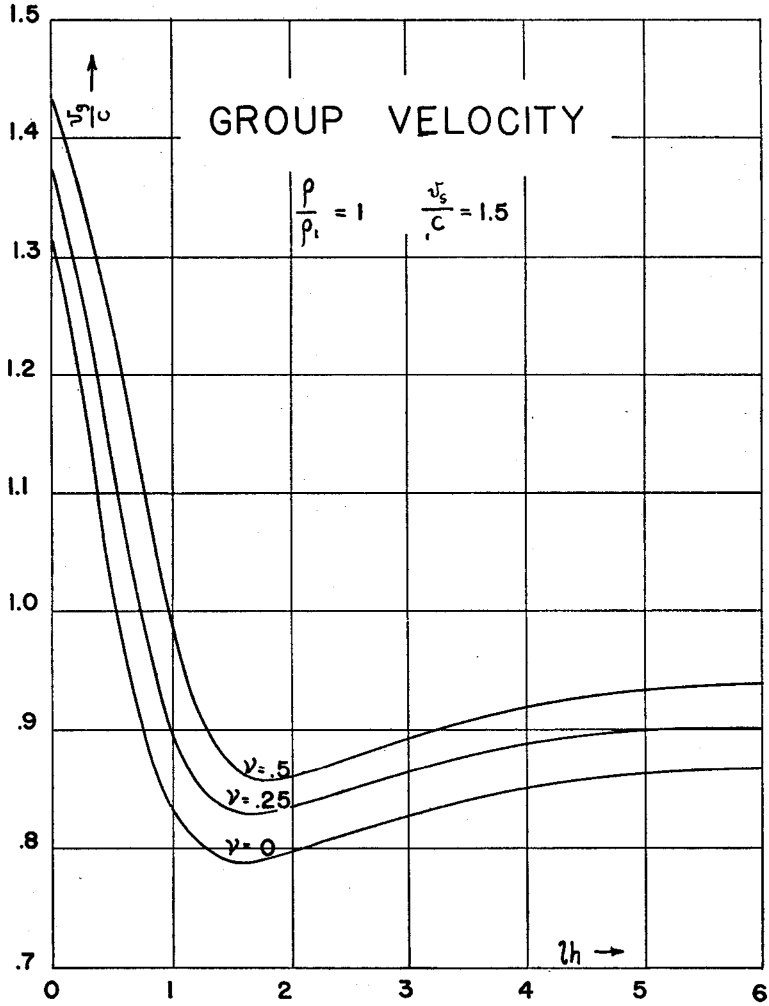

Fig. 5.

and three values of the Poisson ratio:

$$\nu = 0,\ 0.25,\ 0.50\ .$$

The corresponding group velocities for the lowest branch and the same values of the parameters have been plotted in figure 5. The group velocity v_g is given by the derivative

$$v_g = \frac{d\alpha}{dl}\ . \tag{3.5}$$

This may be written

$$\frac{v_g}{c} = \frac{d(\zeta lh)}{d(lh)}\ , \tag{3.6}$$

where the right-hand side represents the derivative of ζlh considered as a function of $lh = 2\pi(h/\lambda)$. The value of v_g/c is plotted in figure 5 as a function of lh.

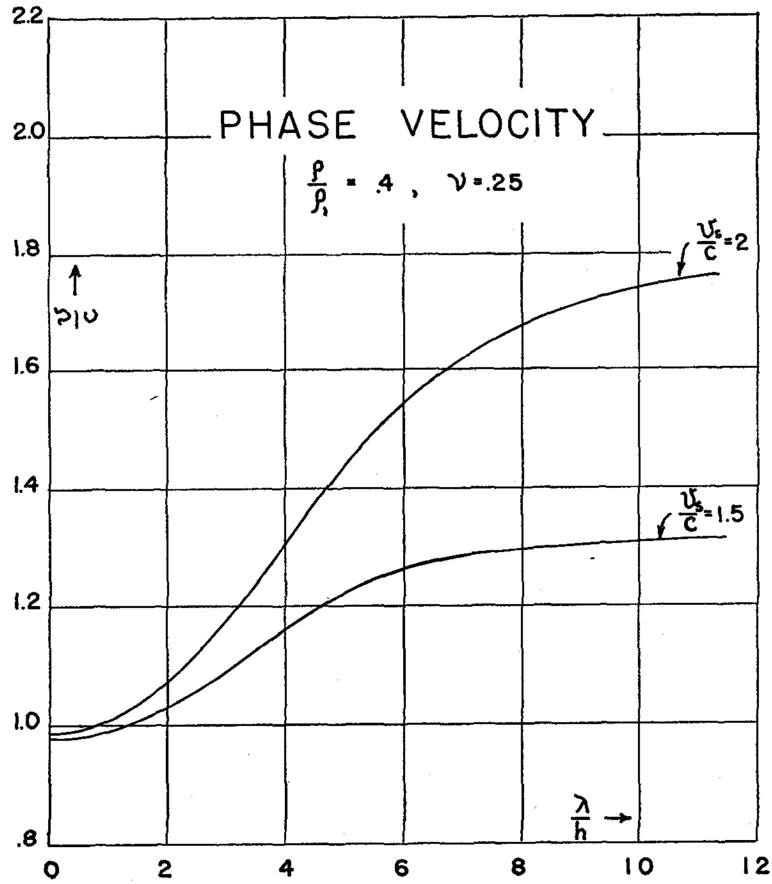

Fig. 6.

The lowest branch of the phase velocity is also plotted, in figure 6, for

$$\frac{\rho}{\rho_1} = 0.4 \qquad \nu = 0.25$$

and two values of v_s/c

$$\frac{v_s}{c} = 1.5 \quad \text{and} \quad 2\,.$$

This affords an estimate of the influence of the parameter v_s/c and, by comparison with figure 3, an estimate of the influence of ρ/ρ_1.

In figure 7 are plotted the group velocity curves corresponding to the phase velocities of figure 6.

4. Waves at the interface of a massless solid and an incompressible fluid. It should be pointed out that the propagation of waves at the interface of two media is not essentially related to the existence of body waves in each medium. This is illustrated by the following idealized case of two semi-infinite media, one being an

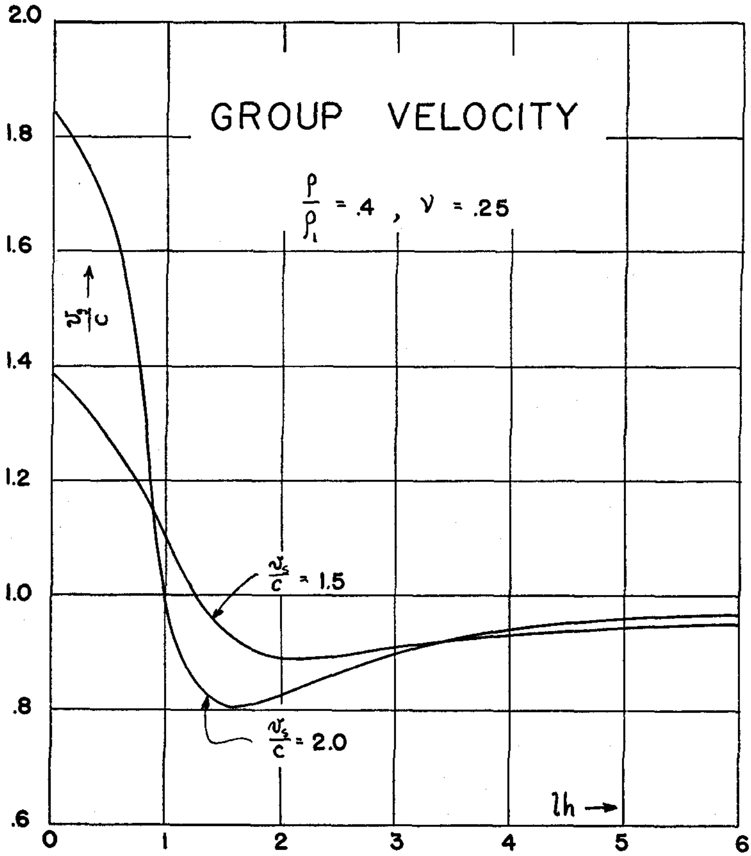

Fig. 7.

incompressible liquid and the other a massless elastic solid. It will be shown that surface waves still propagate at the interface.

This result could be derived by introducing the limiting values for $c = \rho_1 = 0$ and $hl = \infty$ in the final formula (3.3). However, in order to give a better understanding of the physical nature of the phenomenon, we shall repeat the steps followed in the preceding analysis and apply them to the limiting case.

Let us first consider the massless solid. A sinusoidally distributed load on the surface produces a sinusoidal deflection proportional to the load and independent of the frequency. This deflection can be derived from the theory of elasticity, or, what amounts to the same thing, by using the wave theory and then introducing the limiting value zero for the density. This can be done by considering expression (2.13), obtained previously.

We note that

$$\zeta_2^2 = \frac{1 - 2\nu}{2(1 - \nu)} \zeta_1^2 \tag{4.1}$$

$$\rho_1 v_s^2 = G \cdot \tag{4.2}$$

The case $\rho_1 = 0$ corresponds to $\zeta_1 \to 0$ and $\zeta_2 \to 0$. Expanding the right-hand side of (2.13), taking (4.1) and (4.2) into account, yields

$$\frac{\sigma_y}{V} = -\frac{Gl}{1-\nu}. \tag{4.3}$$

Hence, the surface acquires a deflection V proportional to the load σ_y and independent of the frequency in analogy with a distribution layer of massless springs.

We now direct our attention to the fluid. Since it is incompressible, the displacement potential φ as defined in section 1 now satisfies Laplace's equation

$$\nabla^2\varphi = 0. \tag{4.4}$$

This equation has a solution

$$\varphi = Ae^{ly}\cos(lx - \alpha t) \cdot \tag{4.5}$$

We take $y = 0$ to correspond to the interface and assume the fluid to be semi-infinite. Hence (4.5) is the expression required, since it vanishes for $y = -\infty$, that is, at an infinite distance from the interface. The pressure in the fluid is given as before by

$$p = -\rho\frac{\partial^2\varphi}{\partial t^2} = A\alpha^2\rho e^{ly}\cos(lx - \alpha t). \tag{4.6}$$

The vertical displacement is

$$V' = \frac{\partial\varphi}{\partial y} = Ale^{ly}\cos(lx - \alpha t). \tag{4.7}$$

The ratio of pressure to vertical displacement at the interface $y = 0$ is

$$\frac{p}{V'} = \frac{\alpha^2\rho}{l}. \tag{4.8}$$

At this interface we also have the boundary condition of continuity between fluid and solid, $p = -\sigma_y$, $V = V'$. Hence

$$\frac{p}{V'} = -\frac{\sigma_y}{V}. \tag{4.9}$$

Introducing (4.3) and (4.8), this condition becomes

$$\frac{Gl}{1-\nu} = \frac{\alpha^2\rho}{l}. \tag{4.10}$$

This is the condition under which waves propagate at the interface with the phase velocity

$$v = \frac{\alpha}{l} = \sqrt{\frac{1}{1-\nu}\frac{G}{\rho}}. \tag{4.11}$$

We see that the expression for the phase velocity of these waves depends on the ratio of the elastic constant G of the solid and the density ρ of the fluid. They are waves truly characteristic of the interface and do not depend on the existence of body waves in either medium. One can imagine as a physical model for such waves an interface of mercury and cork material.

It is also interesting to note that, as the wave propagates, the energy at one particular location travels alternately across the interface, being in the form of kinetic energy in the fluid at one instant and of potential energy in the solid during the next.

5. Coupling between Stoneley waves and SOFAR waves. A short-period arrival traveling with approximately the speed of sound in water has been observed on a large number of seismograms of earthquakes occurring at sea. This arrival has been designated as the T phase. In an analysis of this phenomenon it was suggested by I. Tolstoy and M. Ewing[4] that the T phase could be due to waves traveling either as SOFAR waves or as waves in the complete water-solid system, or both. The SOFAR channel is a layer of minimum sound velocity at a depth of approximately 700 fathoms. In such a layer the dispersion curve for the phase velocity must be such that this velocity increases for increasing wave length with values of this velocity slightly lower than the velocity of sound in the water near the bottom. If we direct our attention to the dispersion curve obtained in figure 6 for the phase velocity of the Stoneley wave at short wave lengths, we see that this curve is close to the type of dispersion curve that can be expected in the SOFAR channel. The results obtained in the present report therefore suggest the possibility that strong coupling could at times arise between the SOFAR waves and the Stoneley waves with the familiar features of coupled waves such as an alternating transfer of energy from one type of wave to the other.

[4] I. Tolstoy and M. Ewing, "The T Phase of Shallow-Focus Earthquakes," *Bull. Seism. Soc. Am.*, 40: 25–51 (1950).