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Theory of Elasticity and Consolidation for a Porous Anisotropic Solid

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The author's previous theory of elasticity and consolidation for isotropic materials [J. Appl. Phys. 12, 155–164 (1941)] is extended to the general case of anisotropy. The method of derivation is also different and more direct. The particular cases of transverse isotropy and complete isotropy are discussed.

1. INTRODUCTION

THE theory of consolidation deals with the settlement under loading of a porous deformable solid containing a viscous fluid. In a previous publication[1] a consolidation theory was developed for isotropic materials. The purpose of the present paper is to extend the theory to the most general case of anisotropy. The method by which the theory is derived is also more general and direct. The same physical assumption is introduced, that the skeleton is purely elastic and contains a compressible viscous fluid. The theory may therefore also be considered as a generalization of the theory of elasticity to porous materials. It is applicable to the prediction of the time history of stress and strain in a porous solid in which fluid seepage occurs. The general equations derived in Sec. 2 are applied to the case of transverse isotropy in Sec. 3. This is a case of particular interest in the application of the theory to soils and natural rock formations, since transverse isotropic is the type of symmetry usually acquired by rock under the influence of gravity. For an isotropic material the equations reduce to a simple form given in Sec. 4. They are shown to coincide with the equations derived in reference 1. Application of the theory to specific cases was made previously,[2–4] and it was shown that the operational calculus offers a very powerful tool for the solution of consolidation problems in which a load is applied to the material at a given instant and the time history of the settlement is to be calculated. These methods are directly applicable to the more general nonisotropic case. More general solutions of the equations have been developed and will be presented in a forthcoming publication.

2. GENERAL EQUATIONS FOR THE ANISOTROPIC CASE

Let us consider an elastic skeleton with a statistical distribution of interconnected pores. This porosity is usually denoted by

$$f=V_p/V_b, \tag{2.1}$$

where V_p is the volume of the pores contained in a sample of bulk volume V_b. It is understood that the term "porosity" refers as is customary to the effective porosity, namely, that encompassing only the intercommunicating void spaces as opposed to those pores which are sealed off. In the following, the word "pore" will refer to the effective pores while the sealed pores will be considered as part of the solid. It will be noted that a property of the porosity f is that it represents also a ratio of areas

$$f=S_p/S_b, \tag{2.2}$$

i.e., the fraction S_p occupied by the pores in any cross-sectional area S_b of the bulk material. It must be assumed, of course, that the pores are randomly distributed in location but not necessarily in direction. That this relation holds may be ascertained by integrating S_p/S_b over a length of unity in a direction normal to the cross section S_b. The value of this integral then represents the fraction f of the volume occupied by the pores. It is seen that the ratio S_p/S_b is also independent of the direction of the cross section.

The stress tensor in the porous material is

$$\begin{Bmatrix} \sigma_{xx}+\sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}+\sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}+\sigma \end{Bmatrix}, \tag{2.3}$$

with the symmetry property $\sigma_{ij}=\sigma_{ji}$.

The partial components of this tensor do not have the conventional significance. If we consider a cube of unit size of the bulk material, σ represents the total normal tension force applied to the fluid part of the faces of the cube. Denoting by p the hydrostatic pressure of the fluid in the pores we may write

$$\sigma=-fp. \tag{2.4}$$

The remaining components σ_{xx}, σ_{xy}, etc., of the tensor are the forces applied to that portion of the cube faces occupied by the solid.

We shall now call our attention to this system of fluid and solid as a general elastic system with conservation properties. The solid skeleton is considered to have compressibility and shearing rigidity, and the fluid may be compressible. The deformation of a unit cube is assumed to be completely reversible. By deformation is meant here that determined by both strain tensors in the solid and the fluid which will now be defined. The average displacement components of the solid is designated by u_x, u_y, u_z, and that of the fluid by U_x, U_y, U_z.

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[1] M. A. Biot, J. Appl. Phys. 12, 155–164 (1941).
[2] M. A. Biot, J. Appl. Phys. 12, 426–430 (1941).
[3] M. A. Biot and F. M. Clingan, J. Appl. Phys. 12, 578–581 (1941).
[4] M. A. Biot and F. M. Clingan, J. Appl. Phys. 13, 35–40 (1942).

The strain components for the solid and the fluid, respectively, are

$$e_{xx}=\frac{\partial u_x}{\partial x}\quad \tfrac{1}{2}e_{xy}=\tfrac{1}{2}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)\quad \text{etc.}\qquad (2.5)$$

$$\epsilon_{xx}=\frac{\partial U_x}{\partial x}\quad \tfrac{1}{2}\epsilon_{xy}=\tfrac{1}{2}\left(\frac{\partial U_y}{\partial x}+\frac{\partial U_x}{\partial y}\right)\quad \text{etc.}$$

By a generalization of the procedure followed in the classical theory of elasticity (5) we may write for the elastic potential energy V the expression

$$2V=\sigma_{xx}e_{xx}+\sigma_{yy}e_{yy}+\sigma_{zz}e_{zz}+\sigma_{yz}e_{yz}+\sigma_{zx}e_{zx}+\sigma_{xy}e_{xy}+\sigma\epsilon,\qquad (2.6)$$

with

$$\epsilon=\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}.$$

If we assume that the seven stress components are linear functions of the seven strain components the expression $2V$ is a homogeneous quadratic function of the strain. This function is a positive definite form with twenty-eight distinct coefficients. The stress components are given by the partial derivatives of V as follows:

$$\partial V/\partial e_{xx}=\sigma_{xx}\quad \partial V/\partial e_{xy}=\sigma_{xy},\quad \text{etc.,}\qquad (2.7)$$
$$\partial V/\partial\epsilon=\sigma.$$

This is written

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\\ \sigma\end{pmatrix}=\begin{pmatrix}c_{11}&c_{12}&c_{13}&c_{14}&c_{15}&c_{16}&c_{17}\\ &c_{22}&c_{23}&c_{24}&c_{25}&c_{26}&c_{27}\\ &&c_{33}&c_{34}&c_{35}&c_{36}&c_{37}\\ &&&c_{44}&c_{45}&c_{46}&c_{47}\\ &&&&c_{55}&c_{56}&c_{57}\\ &&&&&c_{66}&c_{67}\\ &&&&&&c_{77}\end{pmatrix}\begin{pmatrix}e_{xx}\\ e_{yy}\\ e_{zz}\\ e_{yz}\\ e_{zx}\\ e_{xy}\\ \epsilon\end{pmatrix}.\qquad (2.8)$$

Because the matrix of coefficients is that of a quadratic form we have the symmetry property

$$c_{ij}=c_{ji}.\qquad (2.9)$$

The total stress field (2.3) of the bulk material satisfies the equilibrium equations

$$\frac{\partial}{\partial x}(\sigma_{xx}+\sigma)+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+\rho X=0;$$
$$\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial}{\partial y}(\sigma_{yy}+\sigma)+\frac{\partial\sigma_{yz}}{\partial z}+\rho Y=0;\qquad (2.10)$$
$$\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial}{\partial z}(\sigma_{zz}+\sigma)+\rho Z=0,$$

where ρ is the mass density of the bulk material and X, Y, Z, the body force per unit mass. Substituting in (2.10) the stress components as functions of the strains from (2.8) we obtain three equations for the six unknown displacement $u_x\cdots U_x\cdots$. Three further equations between these unknowns are obtained by introducing the law governing the flow of a fluid in a porous material.

We introduce here a generalized form of Darcy's law for a nonisotropic material

$$\begin{bmatrix}-\partial p/\partial x+\rho_f X\\ -\partial p/\partial y+\rho_f Y\\ -\partial p/\partial z+\rho_f Z\end{bmatrix}=\begin{bmatrix}k_{xx}&k_{xy}&k_{xz}\\ k_{yx}&k_{yy}&k_{yz}\\ k_{zx}&k_{zy}&k_{zz}\end{bmatrix}\begin{bmatrix}\dot U_x-\dot u_x\\ \dot U_y-\dot u_y\\ \dot U_z-\dot u_z\end{bmatrix},\qquad (2.11)$$

where ρ_f is the mass density of the fluid. The matrix k_{ij} constitutes a generalization of Darcy's constant if we include in it the viscosity coefficient. The average velocities of the fluid and solid are denoted by $\dot U_x\cdots\dot u_x\cdots$.

The symmetry of the coefficients

$$k_{ij}=k_{ji}\qquad (2.12)$$

results from the existence of a dissipation function such that the rate of dissipation of the energy in the porous material at rest is expressed by the positive definite quadratic form

$$2D=\sum^{ij}k_{ij}\dot U_i\dot U_j.\qquad (2.13)$$

If we multiply Eq. (2.11) by f and take (2.4) into account we obtain

$$\begin{pmatrix}\partial\sigma/\partial x+\rho_1 X\\ \partial\sigma/\partial y+\rho_1 Y\\ \partial\sigma/\partial z+\rho_1 Z\end{pmatrix}=\begin{pmatrix}b_{xx}&b_{xy}&b_{xz}\\ b_{yx}&b_{yy}&b_{yz}\\ b_{zx}&b_{zy}&b_{zz}\end{pmatrix}\begin{pmatrix}\dot U_x-\dot u_x\\ \dot U_y-\dot u_y\\ \dot U_z-\dot u_z\end{pmatrix},\qquad (2.14)$$

with $\rho_1=\rho_f f=$ the mass of fluid per unit volume of bulk material. The three equations obtained by combining (2.10) and (2.8) in addition to the three Eqs. (2.14) determine the six unknown displacement components for the fluid and the solid.

3. THE CASE OF TRANSVERSE ISOTROPY

The above equations are valid for the most general case of a symmetry. In practice, however, materials will be either isotropic or exhibit a high degree of symmetry which greatly simplifies the equations. Let us consider first the case of a material which is axially symmetric about the z axis. This type of symmetry is referred to by Love[5] as transverse isotropy (page 160). The expression for the strain energy in this case is

$$2V=(A+2N)(e_{xx}+e_{yy})^2+Ce_{zz}^2+2F(e_{yy}+e_{xx})e_{zz}+L(e_{yz}^2+e_{zx}^2)+N(e_{xy}^2-4e_{xx}e_{yy})+2M(e_{xx}+e_{yy})\epsilon+2Qe_{zz}\epsilon+R\epsilon^2.\qquad (3.1)$$

This expression is invariant under a rotation around the z axis. It is written in such a way as to bring out expressions such as $e_{xy}^2-4e_{xx}e_{yy}$ and $e_{xx}+e_{yy}$ which are invariant under a rotation about the z axis. The coefficient $A+2N$ is written this way for reasons of conformity.

[5] A. E. H. Love, *A Treatise On the Mathematical Theory of Elasticity* (Dover Publications, New York, 1944).

Since A does not appear in any other term, the quantity $A+2N$ is an independent coefficient which could have been written as P [see (4.5)]. The stress-strain relations derived from (2.7) and (3.1) are

$$\begin{aligned} \sigma_{xx}&=2Ne_{xx}+A(e_{xx}+e_{yy})+Fe_{zz}+M\epsilon;\\ \sigma_{yy}&=2Ne_{yy}+A(e_{xx}+e_{yy})+Fe_{zz}+M\epsilon;\\ \sigma_{zz}&=Ce_{zz}+F(e_{xx}+e_{yy})+Q\epsilon;\\ \sigma_{yz}&=Le_{yz};\\ \sigma_{zx}&=Le_{zx};\\ \sigma_{xy}&=Ne_{xy};\\ \sigma&=M(e_{xx}+e_{yy})+Qe_{zz}+R\epsilon. \end{aligned} \tag{3.2}$$

There are therefore in this case eight elastic coefficients.

The equations of flow contain two coefficients of permeability, one in the z direction, the other in the x, y plane, and may be written

$$\begin{aligned} \partial\sigma/\partial x+\rho_1X&=b_{xx}(\dot{U}_x-\dot{u}_x);\\ \partial\sigma/\partial y+\rho_1Y&=b_{xx}(\dot{U}_y-\dot{u}_y);\\ \partial\sigma/\partial z+\rho_1Z&=b_{zz}(\dot{U}_z-\dot{u}_z). \end{aligned} \tag{3.3}$$

These equations along with the stress-strain relation (3.2) and the equilibrium relations (2.10) yield six equations for the six displacement components in the case of transverse isotropy.

4. THE CASE OF ISOTROPY

In the case of complete isotropy the strain energy function (3.1) becomes

$$\begin{aligned} 2V=&(A+2N)(e_{xx}+e_{yy}+e_{zz})^2\\ &+N(e_{yz}^2+e_{zx}^2+e_{xy}^2-4e_{yy}e_{zz}\\ &-4e_{zz}e_{xx}-4e_{xx}e_{yy})\\ &+2Q(e_{xx}+e_{yy}+e_{zz})\epsilon+R\epsilon^2. \end{aligned} \tag{4.1}$$

We put

$$e=e_{xx}+e_{yy}+e_{zz}. \tag{4.2}$$

The stress-strain relations derived from (2.7) are

$$\begin{aligned} \sigma_{xx}&=2Ne_{xx}+Ae+Q\epsilon;\\ \sigma_{yy}&=2Ne_{yy}+Ae+Q\epsilon;\\ \sigma_{zz}&=2Ne_{zz}+Ae+Q\epsilon;\\ \sigma_{yz}&=Ne_{yz};\\ \sigma_{zx}&=Ne_{zx};\\ \sigma_{xy}&=Ne_{xy};\\ \sigma&=Qe+R\epsilon. \end{aligned} \tag{4.3}$$

There are in this case four elastic constants, and this checks with the result obtained in reference 1. The equations of flow contain a single coefficient b. They are written

$$\begin{aligned} \partial\sigma/\partial x+\rho_1X&=b(\dot{U}_x-\dot{u}_x);\\ \partial\sigma/\partial y+\rho_1Y&=b(\dot{U}_y-\dot{u}_y);\\ \partial\sigma/\partial z+\rho_1Z&=b(\dot{U}_z-\dot{u}_z). \end{aligned} \tag{4.4}$$

We shall assume that there is no body force and put $X=Y=Z=0$. Substitution of expression (4.3) into the equilibrium Eq. (2.10) for the stresses and the flow Eq. (4.4) yield the six equations

$$\begin{aligned} &N\nabla^2\bar{u}+(P-N+Q)\,\mathrm{grad}e+(Q+R)\,\mathrm{grad}\epsilon=0\\ &\mathrm{grad}(Qe+R\epsilon)=b(\partial/\partial t)(\bar{U}-\bar{u}). \end{aligned} \tag{4.5}$$

We have put $P=A+2N$.

Taking the divergence of the second equation we may also write

$$\begin{aligned} &N\nabla^2\bar{u}+(P-N+Q)\,\mathrm{grad}e+(Q+R)\,\mathrm{grad}\epsilon=0\\ &Q\nabla^2e+R\nabla^2\epsilon=b(\partial/\partial t)(\epsilon-e). \end{aligned} \tag{4.6}$$

In the previous theory (1) we had obtained these equations by a different method and in a different form. To show their equivalence we write the stress-strain relations by eliminating ϵ from Eqs. (4.3)

$$\begin{aligned} \sigma_{xx}&=2Ne_{xx}+\left(A-\frac{Q^2}{R}\right)e+\frac{Q}{R}\sigma;\\ \sigma_{yy}&=2Ne_{yy}+\left(A-\frac{Q^2}{R}\right)e+\frac{Q}{R}\sigma;\\ \sigma_{zz}&=2Ne_{zz}+\left(A-\frac{Q^2}{R}\right)e+\frac{Q}{R}\sigma;\\ \sigma_{yz}&=Ne_{yz};\\ \sigma_{zx}&=Ne_{zx};\\ \sigma_{xy}&=Ne_{xy}. \end{aligned} \tag{4.7}$$

Substituting these in the equilibrium relation (2.10) we find

$$\begin{aligned} &N\nabla^2\bar{u}+[P-N-Q^2/R]\,\mathrm{grad}e\\ &\qquad+(Q+R)/R\,\mathrm{grad}\sigma=0. \end{aligned} \tag{4.8}$$

We also derive from (4.4)

$$\nabla^2\sigma=b\frac{\partial}{\partial t}(\epsilon-e)=\frac{b}{R}\frac{\partial\sigma}{\partial t}-b\frac{Q+R}{R}\frac{\partial e}{\partial t}. \tag{4.9}$$

Equations (4.8) and (4.9) are in the form obtained in reference 1. We note that the significance of σ in that reference is equivalent to $-\sigma/f$ in our present notation.

Consider now the case of an incompressible material. This corresponds to the condition

$$e(1-f)+f\epsilon=0. \tag{4.10}$$

Since this must be satisfied for all values of σ we derive from the last relation (4.3) that both R and Q are infinite with the condition

$$Q/R=(1-f)/f. \tag{4.11}$$

Since $A-Q^2/R=S$ must remain finite the stress strain law becomes

$$\begin{aligned}
\sigma_{xx} &= 2Ne_{xx}+Se+\frac{1-f}{f}\sigma; \\
\sigma_{yy} &= 2Ne_{yy}+Se+\frac{1-f}{f}\sigma; \\
\sigma_{zz} &= 2Ne_{zz}+Se+\frac{1-f}{f}\sigma; \\
\sigma_{yz} &= Ne_{yz}; \\
\sigma_{zx} &= Ne_{zx}; \\
\sigma_{xy} &= Ne_{xy}.
\end{aligned} \tag{4.12}$$

Substituting these expressions in the equilibrium relations (2.10) we derive

$$N\nabla^2\bar{u}+(N+S)\,\mathrm{grad}e+(1/f)\,\mathrm{grad}\sigma=0 \tag{4.13}$$

and from (4.9)

$$\nabla^2\sigma=-\frac{b}{f}\frac{\partial e}{\partial t}. \tag{4.14}$$

Taking the divergence of (4.13)

$$(2N+S)\nabla^2 e+\frac{1}{f}\nabla^2\sigma=0. \tag{4.15}$$

Hence (4.14) may be written

$$f^2(2N+S)\nabla^2 e=b\frac{\partial e}{\partial t}. \tag{4.16}$$

This is the equation of heat conduction. Equations (4.13) and (4.16) coincide with those obtained in reference 1 for the incompressible case.