Reprinted from THE PHYSICAL REVIEW, Vol. 98, No. 6, 1869–1870, June 15, 1955
Printed in U. S. A.

Deformation of Viscoelastic Plates Derived from Thermodynamics

M. A. BIOT
Shell Development Company, New York, New York
(Received April 13, 1955)

IN a previous publication,[1] the problem of bending of a viscoelastic plate of uniform thickness h was used as an example of application of variational principles in irreversible thermodynamics. We considered a two-dimensional deformation parallel with the x, y plane and corresponding displacement components u, w, along the x and z directions. The coordinate z is directed along the thickness. We assumed that the displacement could be represented by

$$u=zu_1(x), \quad w=w_0(x). \tag{1}$$

It should be pointed out that this representation is a particular case of the more general one where the displacements are expanded in a Taylor series of the coordinate z, namely,

$$u=\sum_{n=0}^{\infty} u_n z^n, \quad v=\sum_{n=0}^{\infty} v_n z^n, \quad w=\sum_{n=0}^{\infty} w_n z^n. \tag{2}$$

The u_n, v_n, w_n are functions of x and y. Application of the variational method with such expressions leads to a very general theory of viscoelastic plates represented by partial differential equations in x and y with time operator coefficients. This, of course, applies to plates of constant or variable thickness. The question then arises as to the number of terms required in the expression (2) in order to represent the deformation with adequate accuracy. This can be answered, of course, only by considering each individual problem and will depend on the type of material and deformation as well as on the accuracy requirements.

The approximation (1) represents an expansion to the first order and was introduced only for the purpose of presenting a simplified example of the variational method. The approximation is only valid for materials which elongate with a small lateral contraction and therefore does not apply in the case of incompressibility. A better approximation is given by expanding u and w to the second order in z. Because of the particular nature of the problem, we may assume that u is an odd function and w an even function of z. Hence expressions (2), expanded to the second order, are

$$u=u_1 z, \quad w=w_0+w_2 z^2. \tag{3}$$

The invariant in this case is

$$I=\tfrac{1}{2}(2Q+R)(e_{xx}^2+e_{zz}^2)+Re_{xx}e_{zz}+2Qe_{zx}^2, \tag{4}$$

with the time operators P and Q. The terms e_{xx}^2, e_{zz}^2 and $e_{xx}e_{zz}$ are all of the second order z^2, while e_{zx}^2 contains a term independent of z and terms of the order z^2 and z^4. To be consistent, of course, we should retain at least all terms of the order z^2. However, we shall drop both z^2 and z^4 terms in e_{zx}^2. There is some justification for this from a physical standpoint in the present case. Furthermore, it leads to agreement with the classical theory of bending of plates in the elastic case. Applying, then, the variational method as indicated in reference 1, we find

$$\begin{gathered} -(2Q+R)\frac{h^3}{12}\frac{d^2u_1}{dx^2}-\frac{Rh^3}{6}\frac{dw_2}{dx}+Qh\left(\frac{dw_0}{dx}+u_1\right)=0, \\ Qh\frac{d}{dx}\left(\frac{dw_0}{dx}+u_1\right)=-f, \\ 2(2Q+R)w_2+R\frac{du_1}{dx}=0. \end{gathered} \tag{5}$$

In this derivation, we assume that the load f applied to the plate is distributed uniformly throughout the thickness. The last equation is equivalent to the condition that the stress component σ_{zz} vanishes. Eliminating all variables except w_0 in (5), we find

$$\frac{d^4w_0}{dx^4}=\frac{f}{B_1}-\frac{1}{Qh}\frac{d^2f}{dx^2}. \tag{6}$$

This equation is of the same form as found previously[1] except for the operator B, which is

$$B_1=\tfrac{1}{3}h^3Q(Q+R)/(2Q+R). \tag{7}$$

This operator coincides with the coefficient in the theory of bending of plates for the elastic case.

[1] M. A. Biot, Phys. Rev. 97, 1463 (1955).