Thermoelasticity and Irreversible Thermodynamics

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A unified treatment is presented of thermoelasticity by application and further developments of the methods of irreversible thermodynamics. The concept of generalized free energy introduced in a previous publication plays the role of a "thermoelastic potential" and is used along with a new definition of the dissipation function in terms of the time derivative of an entropy displacement. The general laws of thermoelasticity are formulated in a variational form along with a minimum entropy production principle. This leads to equations of the Lagrangian type, and the concept of thermal force is introduced by means of a virtual work definition. Heat conduction problems can then be formulated by the methods of matrix algebra and mechanics. This also leads to the very general property that the entropy density obeys a diffusion-type law. General solutions of the equations of thermoelasticity are also given using the Papkovitch-Boussinesq potentials. Examples are presented and it is shown how the generalized coordinate method may be used to calculate the thermoelastic internal damping of elastic bodies.

1. INTRODUCTION

SOME new methods and concepts in linear irreversible thermodynamics have been developed in previous publications[1,2] by the writer. In particular, variational principles were introduced in conjunction with a generalization of the definition of free energy to include systems with nonuniform temperature. Our purpose is to develop more extensively this line of approach for problems of thermoelasticity and heat conduction.

The expression for the generalized free energy is developed for thermoelasticity and is referred to as a thermoelastic potential. A new expression for the dissipation function is defined in terms of a vector field representing the rate of entropy flow.

The term thermoelasticity encompasses a large category of phenomena. It includes the general theory of heat conduction, thermal stresses, and strains set up by thermal flow in elastic bodies and the reverse effect of temperature distribution caused by the elastic deformation itself leading to thermoelastic dissipation. It is well known that the latter is an important cause of internal damping in elastic bodies.

The use of generalized coordinates and the variational method leads to concepts of generalized forces of the Lagrangian type which are applicable to both the mechanical and the purely thermal problems. This

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[1] M. A. Biot, Phys. Rev. **97**, 1463–1469 (1955).
[2] M. A. Biot, J. Appl. Phys. **25**, 1385–1391 (1954).

unified treatment is a result of some very basic physical laws obeyed by all irreversible phenomena.[1] As a flexible and compact formulation of a wide class of problems it brings out hidden relationships with other phenomena and situates the particular field of thermoelasticity in a broader frame.

After two introductory sections to establish the basic equations in the usual way, Sec. 4 points out the identity of these equations with those of a porous elastic body containing a compressible viscous fluid. The analogy provides a useful intuitive model for thermoelasticity and heat conduction. Section 5 established general solutions of these equations. A rather significant result is obtained, namely, that the distribution of entropy satisfies a diffusion equation. Instead of deriving the thermoelasticity equations directly from thermodynamics we prefer to follow the inverse process of showing that they lead to a variational formulation which coincides with a general principle of irreversible thermodynamics. This is done in Secs. 6 and 7. It requires the use of an entropy flow field which plays the role of a coordinate conjugate to the temperature. The concept of thermoelastic potential is introduced and the dissipation function is discussed in terms of the rate of entropy production. It is also shown how the general principle of minimum entropy production applies to thermoelasticity and heat conduction. Generalized coordinates and admittance matrices are discussed in Sec. 8. This leads to the treatment of practical thermal problems by matrix algebra in analogy with stress and vibration analysis methods. The concept of generalized thermal force is defined by a virtual displacement method as in mechanics. The virtual displacement in this case is the variation of entropy flow. The matrix equation also indicates that the propagation of entropy as a diffusion process is quite general and applies to the most general case of anisotropy. Extension to anisotropic media and to dynamics is briefly treated in Secs. 9 and 10. As an illustration, a classical problem in pure heat conduction is solved by the present methods in Sec. 11. An interesting aspect of this solution is shown by its formulation in terms of a continuous relaxation spectrum in analogy with the treatment of viscoelastic media[2] and the appearance of a steady-state variable which is a linear function of time. The last section deals with thermoelastic damping. It illustrates on the simple example of a cantilever rod how vibration problems with thermoelastic damping may be treated by the generalized coordinate method. Such an approach can be used, for instance, to evaluate the thermoelastic damping of crystals in piezoelectric oscillators.

Thermoelastic damping and its experimental verification were the object of extensive work by Zener.[3,4] The present treatment brings these phenomena into the general frame of irreversible thermodynamics and its variational formulation.

2. THE ENTROPY OF AN ELASTIC ELEMENT

The thermodynamic variables defining the state of an element are its absolute temperature $T+\theta$ and the strain components e_{ij}. We are dealing here with the linear problem where θ is an increment of temperature above a reference absolute temperature T for the state of zero stress and strain. The strain components are defined in terms of the displacement field $u_x u_y u_z$ of the medium, by,

$$e_{xx} = \frac{\partial u_x}{\partial x} \quad e_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}. \quad (2.1)$$

These equations of state are the relations between the stress, the strain, and the temperature. We write

$$
\begin{aligned}
\sigma_{xx} &= 2\mu e_{xx} + \lambda e - \beta\theta \\
\sigma_{yy} &= 2\mu e_{yy} + \lambda e - \beta\theta \\
\sigma_{zz} &= 2\mu e_{zz} + \lambda e - \beta\theta \\
\sigma_{xy} &= \mu e_{xy} \\
\sigma_{yz} &= \mu e_{yz} \\
\sigma_{zx} &= \mu e_{zx}.
\end{aligned}
\quad (2.2)
$$

In these expressions λ and μ are the Lamé constants, for isothermal deformation,

$$e = e_{xx} + e_{yy} + e_{zz}$$

$$\beta = \left(\lambda + \frac{2\mu}{3}\right)\alpha_t \quad (2.3)$$

and α_t is the coefficient of thermal expansion. Consider an element of unit size. Denote by U its internal energy and h the heat absorbed by it. Conservation of energy requires

$$dh = dU - \sum^{\mu\nu} \sigma_{\mu\nu} de_{\mu\nu}. \quad (2.4)$$

This is the first law of thermodynamics. The entropy is

$$ds = \frac{dh}{T_1} = \frac{dU}{T_1} - \frac{1}{T_1}\sum^{\mu\nu} \sigma_{\mu\nu} de_{\mu\nu} \quad (2.5)$$

with $T_1 = T + \theta$. The summation $\sum$ is extended to all distinct pairs of μ, ν. This may also be written

$$ds = \frac{1}{T_1}\frac{\partial U}{\partial T_1}dT_1 + \frac{1}{T_1}\sum^{\mu\nu}\left[\frac{\partial U}{\partial e_{\mu\nu}} - \sigma_{\mu\nu}\right]de_{\mu\nu}. \quad (2.6)$$

The second law of thermodynamics requires that ds be an exact differential in T_1 and $e_{\mu\nu}$. This implies

$$\frac{\partial \sigma_{\mu\nu}}{\partial e_{ij}} = \frac{\partial \sigma_{ij}}{\partial e_{\mu\nu}} \quad (2.7)$$

[3] C. Zener, Phys. Rev. **53**, 90–99 (1938).
[4] C. Zener, *Elasticity and Anelasticity of Metals* (The University of Chicago Press, Chicago, 1948).

which is verified by the equations of state (2.2). It also implies that

$$\frac{\partial U}{\partial e_{\mu\nu}} = \sigma_{\mu\nu} - T_1 \frac{\partial \sigma_{\mu\nu}}{\partial T_1}. \qquad (2.8)$$

From (2.2) we find for $\mu = \nu$

$$\frac{\partial U}{\partial e_{\mu\nu}} - \sigma_{\mu\nu} = T_1\beta \qquad (2.9)$$

and for $\mu \neq \nu$

$$\frac{\partial U}{\partial e_{\mu\nu}} - \sigma_{\mu\nu} = 0. \qquad (2.10)$$

Also because of (2.4) we have

$$\frac{dh}{dT_1} = \frac{\partial U}{\partial T_1} = c \qquad (2.11)$$

where c is the specific heat of the unit volume for $e_{\mu\nu} = 0$, i.e., in the absence of deformation. Since we consider a linear theory, c is assumed independent of the temperature in the vicinity of the equilibrium temperature. Introducing (2.9), (2.10), and (2.11) into the expression for the entropy gives

$$ds = c\frac{dT_1}{T_1} + \beta de. \qquad (2.12)$$

Integrating, we find,

$$s = c \log\left(1 + \frac{\theta}{T}\right) + \beta e \qquad (2.13)$$

with a constant of integration such that $s = 0$ for $T_1 = T_1 e = 0$. For small changes of temperature this is limited to the linear term in θ

$$s = (c\theta/T) + \beta e. \qquad (2.14)$$

We note that Ts represents the amount of heat h absorbed by the element for small changes of volume and temperature.

$$h = c\theta + T\beta e. \qquad (2.15)$$

We obtain the well-known result that under adiabatic transformation positive dilatation produces a cooling. In this case

$$c\theta = -T\beta e. \qquad (2.16)$$

Substitution in the stress-strain relations (2.2) shows that for adiabatic deformations the Lamé constant λ is replaced by $\lambda + \beta^2(T/c)$ and μ remains unchanged.

3. THE EQUATIONS OF THE TEMPERATURE AND DEFORMATION FIELDS

We shall now derive differential equations for the displacement field

$$\mathbf{u} = (u_x, u_y, u_z) .$$

of the solid and the distribution of temperature θ. The equations of equilibrium of a stress field with a body force X, Y, Z, per unit mass in a continuum of mass density ρ are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho X = 0$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho Y = 0 \qquad (3.1)$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho Z = 0.$$

Substituting expressions (2.2) for the stress components and assuming the body force to be zero for simplicity, we find

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu)\, \text{grad}\, e - \beta\, \text{grad}\, \theta = 0. \qquad (3.2)$$

These are three equations for $\mathbf{u}$ and θ. The fourth equation is provided by the law of heat conduction. The heat absorbed by an element is h, per unit volume. Hence with a coefficient k of heat conduction we find

$$\frac{\partial h}{\partial t} = k\, \text{div}\, \text{grad}\, \theta = k\nabla^2 \theta. \qquad (3.3)$$

Introducing the expression (2.15) for h yields

$$k\nabla^2 \theta = c\frac{\partial \theta}{\partial t} + T\beta \frac{\partial e}{\partial t}. \qquad (3.4)$$

This equation, together with the three equations (3.2), determine the time history of the deformation and thermal fields. As expected because of the cooling and heating associated with a change of volume, the two fields are coupled through the coefficient β. As may be seen from the derivation this coupling is a consequence of the laws of thermodynamics.

Equations (3.2) and (3.4) may also be expressed in terms of the local entropy density s instead of the temperature θ by using relation (2.14).

4. THE ANALOGY WITH THE THEORY OF ELASTICITY OF POROUS MATERIALS

In earlier papers[5,6] we have established the equations for the deformation and fluid seepage of an elastic porous solid whose pores are filled with a compressible viscous fluid. We will show that these equations are the same as the thermoelasticity Eqs. (3.2) and (3.4) with the temperature playing the same role as the fluid pressure. Equations relating the stress field to the

[5] M. A. Biot, J. Appl. Phys. 12, 155–164 (1941).
[6] M. A. Biot, J. Appl. Phys. 26, 182–185 (1955).

strain may be written

$$\sigma_{xx}+\sigma=2Ne_{xx}+Se-\alpha p$$
$$\sigma_{yy}+\sigma=2Ne_{yy}+Se-\alpha p$$
$$\sigma_{zz}+\sigma=2Ne_{zz}+Se-\alpha p \qquad (4.1)$$
$$\sigma_{yz}=Ne_{yz}$$
$$\sigma_{zx}=Ne_{zx}$$
$$\sigma_{xy}=Ne_{xy}$$

with $\alpha=(Q+R)f/R$. These relations are derived from Eqs. (4.7) of reference 6 after replacing σ by

$$\sigma=-fp \qquad (4.2)$$

where f is the porosity and p the fluid pressure. The total stress field in the bulk material is $\sigma_{xx}+\sigma$, etc., and σ_{xy}, etc. Equations (4.1) correspond to relations (2.2) with the correspondence

$$\theta \leftrightarrow p$$
$$\beta \leftrightarrow \alpha. \qquad (4.3)$$

Equilibrium equations for the stress field are similar to (3.1) except that $\sigma_{xx}\sigma_{yy}\sigma_{zz}$ are replaced by $\sigma_{xx}+\sigma$, etc. Substituting expressions (4.1) into the equilibrium equations yields,

$$N\nabla^2\mathbf{u}+(N+S)\,\mathrm{grad}\,e-\alpha\,\mathrm{grad}p=0. \qquad (4.4)$$

This corresponds to Eq. (3.2). Finally, we consider the relation between the fluid pressures, the solid strain and the fluid content. This was written in reference 6 as

$$\sigma=Qe+R\epsilon \qquad (4.5)$$

where $\epsilon=\mathrm{div}\mathbf{U}$ and $\mathbf{U}$ is the absolute displacement vector of the fluid. In order to bring out the analogy with thermoelasticity we introduce the increment of fluid content of a porous element which is $\zeta=-f(\epsilon-e)$. We may write (4.5) as

$$\zeta=\frac{p}{M}+\alpha e$$

with

$$1/M=f^2/R. \qquad (4.6)$$

This is the analog of Eq. (2.15) giving the amount of heat absorbed by an element. The entropy acquired is h/T. We have the following correspondence

$$\zeta \leftrightarrow h/T$$
$$1/M \leftrightarrow c/T. \qquad (4.7)$$

Darcy's law expressing proportionality between the relative fluid flow and the pressure gradient may be written

$$k_1\,\mathrm{div}\,\mathrm{grad}p=k_1\nabla^2 p=\partial\zeta/\partial t \qquad (4.8)$$

with $k_1=f^2/b$ a permeability coefficient. This is written

$$k_1\nabla^2 p=\frac{1}{M}\frac{\partial p}{\partial t}+\alpha\frac{\partial e}{\partial t} \qquad (4.9)$$

which corresponds to Eq. (3.4). We notice that k_1, corresponds to k/T where k is the thermal conductivity.

5. GENERAL SOLUTIONS OF THE EQUATIONS OF THERMOELASTICITY

By a similar procedure[7] for the case of a porous medium general solutions may be established for the equations of thermoelasticity. Let us transform Eqs. (3.2) and (3.4) by introducing the entropy density instead of the temperature. We find

$$\mu\nabla^2\mathbf{u}+\left(\lambda+\mu+\frac{\beta^2 T}{c}\right)\mathrm{grad}e-\frac{\beta T}{c}\mathrm{grad}s=0$$

$$\frac{k}{c}\frac{2\mu+\lambda}{\left(2\mu+\lambda+\beta^2\dfrac{T}{c}\right)}\nabla^2 s=\frac{\partial s}{\partial t}. \qquad (5.1)$$

The second equation shows the remarkable property that it is the entropy which satisfies the diffusion equation. This has a deeper physical significance since it corresponds to the propagation of disorder, a process which in the present case follows a diffusion type law. In the analogous case of a porous material the fluid content ζ represents the entropy and is also found to satisfy a diffusion equation. A solution for the displacement field $\mathbf{u}$ is found by introducing the Papkovitch-Boussinesq potentials,[8,9]

$$\mathbf{u}=-\mathrm{grad}(\psi_0+\mathbf{r}\cdot\boldsymbol{\psi})+B\boldsymbol{\psi} \qquad (5.2)$$

with

$$\nabla^2\boldsymbol{\psi}=0$$

$$B=\frac{2(\lambda+2\mu+\beta^2 T/c)}{(\lambda+\mu+\beta^2 T/c)} \qquad (5.3)$$

and a vector $\mathbf{r}=(x,y,z)$. Substituting in the first three equations (5.1) yields,

$$\mathrm{grad}\left[\left(\lambda+2\mu+\frac{\beta^2 T}{c}\right)\nabla^2\psi_0+\frac{\beta T}{c}s\right]=0. \qquad (5.4)$$

We may drop the grad operator since this amounts to adding a constant to the right-hand side of the equation, hence also to adding to ψ_0 a quadratic function of the coordinates. However, this is also equivalent to adding a linear function of the coordinates to $\boldsymbol{\psi}$. Hence, we write

$$\left(\lambda+2\mu+\beta^2\frac{T}{c}\right)\nabla^2\psi_0+\frac{\beta T}{c}s=0. \qquad (5.5)$$

[7] M. A. Biot, J. Appl. Mech. (to be published).
[8] J. Boussinesq, *Application des Potentiels à l'Etude de l'Equilibre et des Mouvements des Solides Elastiques* (Gauthier-Villars, Paris, 1885).
[9] P. E. Papkovitch, Compt. rend. **195**, 513–515, 754–756 (1932).

Because of the last Eq. (5.1) ψ_0 must satisfy

$$\nabla^2\left(K_0\nabla^2-\frac{\partial}{\partial t}\right)\psi_0=0 \tag{5.6}$$

with

$$K_0=\frac{k}{c}\frac{2\mu+\lambda}{(2\mu+\lambda+\beta^2T/c)}.$$

This equation is satisfied by a sum of the solutions of the two equations

$$\nabla^2\psi_1=0 \tag{5.7}$$

$$K_0\nabla^2\psi_2=\frac{\partial\psi_2}{\partial t}$$

namely,

$$\psi_0=\psi_1+\psi_2. \tag{5.8}$$

It should be further pointed out that all other solutions such as those of reference 7 are immediately applicable to thermoelasticity. The fluid displacement plays the role of S.

6. VARIATIONAL FORMULATION

We shall now derive variational principles corresponding to basic equations of thermoelasticity derived in Sec. 3. This is done by introducing two invariants. The first one is an integral over the volume τ

$$V=\int\int\int_{\tau}\left(W+\frac{1}{2}\frac{c}{T}\theta^2\right)d\tau \tag{6.1}$$

where W is the isothermal mechanical energy

$$2W=(\lambda+2\mu)e^2+\mu(e_{yz}^2+e_{zz}^2+e_{xy}^2 \\ -4e_{yy}e_{zz}-4e_{zz}e_{xx}-4e_{xx}e_{yy}). \tag{6.2}$$

The quantity V will be referred to as the *thermoelastic potential*. In order to formulate the variational principle, the integrand $W+\frac{1}{2}c\theta^2$ must be expressed in terms of two vector fields. One is the displacement field

$$\mathbf{u}=(u_x,u_y,u_z) \tag{6.3}$$

of the solid. The other is defined in terms of a vector S which represents the amount of heat which has flown in a given direction divided by the absolute temperature T. We call it the entropy flow or entropy displacement. Conservation of energy requires

$$-\mathrm{div}\,\mathbf{S}=s. \tag{6.4}$$

where s is the entropy density. From (2.14) we may write

$$\theta=\frac{T}{c}(s-\beta e). \tag{6.5}$$

The temperature in expression (6.1) is then written in terms of the two vectors fields S and u as

$$\theta=-\frac{T}{c}\,\mathrm{div}(\mathbf{S}+\beta\mathbf{u}). \tag{6.6}$$

The second invariant is defined as

$$D=\frac{1}{2}\int\int\int_{\tau}\frac{T}{k}\left(\frac{\partial\mathbf{S}}{\partial t}\right)^2d\tau. \tag{6.7}$$

We will show in the next section that V and D represent generalizations of the concept of free energy and dissipation function as introduced by the writer in reference 1. The vector $\partial\mathbf{S}/\partial t$ represents the entropy rate of flow field. With these invariants a variational principle can be derived directly from thermodynamics. At this time, however, we prefer to derive it independently by demonstrating directly its equivalence to Eqs. (3.2) and (3.4) of thermoelasticity.

It is convenient to modify somewhat the expression for D by writing instead, the operational expression

$$D=\frac{1}{2}p\int\int\int\frac{T}{k}(\mathbf{S})^2d\tau \tag{6.8}$$

with $p=d/dt$. The introduction of such an operational expression is motivated by an advantage of simplicity in deriving a variational principle. When calculating the variation, the operator p is treated as a constant and it is only in the final differential equations that it is replaced by an actual differential. The justification for this procedure lies in the fact that it leads to the correct differential equations for the field. The same remark applies to the procedure in Secs. 9 and 10. It is seen that it avoids the usual manipulation required in deriving Lagrange's equations. We have introduced operational invariants of a more general nature in reference 1. The variational principle is written as

$$\delta V+\delta D=\int\int_A(\mathbf{F}\cdot\delta\mathbf{u}+\theta\mathbf{n}\cdot\delta\mathbf{S})dA. \tag{6.9}$$

The variation applies to all six components at the two vector fields w and S. The integral on the right-hand side is extended to the boundary A of the volume τ. The boundary force per unit area is $\mathbf{F}$ and $\mathbf{n}$ is the unit normal to the boundary directed toward the interior. By applying the six independent variations to the six vector components, $\delta\mathbf{u},\delta\mathbf{S}$ we may verify that the total variation of $V+D$ is

$$\delta V+\delta D=\int\int_A(\mathbf{F}\cdot\delta\mathbf{u}+\theta\mathbf{n}\cdot\delta\mathbf{S})dA$$

$$-\int\int\int_{\tau}[\mu\nabla^2\mathbf{u}+(\lambda+\mu)\,\mathrm{grad}\,e-\beta\,\mathrm{grad}\,\theta]\cdot\delta\mathbf{u}d\tau$$

$$+\int\int\int_{\tau}\left(\mathrm{grad}\,\theta+p\frac{T}{k}\mathbf{S}\right)\cdot\delta\mathbf{S}d\tau. \tag{6.10}$$

Substituting in the variational equations (6.9) the surface integrals cancel out and we are left with the condition that the volume integrals vanish identically.

This implies the two vector equations

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu)\, \mathrm{grad}\, e - \beta\, \mathrm{grad}\, \theta = 0$$

$$\mathrm{grad}\, \theta + p\frac{T}{k}\mathbf{S} = 0. \qquad (6.11)$$

Applying the divergence operator to the second equation and substituting

$$\mathrm{div}\,\mathbf{S} = -s = \frac{c\theta}{T} - \beta e \qquad (6.12)$$

we find

$$\nabla^2 \theta = p\left(\frac{c}{k}\theta + \frac{T\beta}{k}e\right). \qquad (6.13)$$

Equations (6.11) and (6.13) are identical with the basic thermoelasticity equations (3.2) and (3.4). We have thus established that the variational equation (6.9) is equivalent to these equations.

The above derivation implies that the material is homogeneous. It is easily verified that the variational principle applies also for nonhomogeneous materials, i.e., when λ, μ, c, β, and k are functions of the coordinates. This fact is particularly useful in solving problems of heat conduction and thermoelasticity in composite materials.

It is interesting to note that the surface integral may be interpreted as a generalized virtual work where the temperature vector $\mathbf{n}\theta$ represents a force and $\delta\mathbf{S}$ a virtual entropy displacement. The analogy with the theory of porous materials is complete if θ represents the fluid pressure and $\delta\mathbf{S}$ the virtual volume flow of the fluid.

7. THE GENERALIZED FREE ENERGY AND THE RATE OF ENTROPY PRODUCTION

We should bear in mind that the variational principle (6.9) is a particular case of a more general one in irreversible thermodynamics as formulated in reference 1. The invariant V in the general theory is a generalized free energy while D is a generalized dissipation function defined in terms of entropy production.

In order to show this, let us calculate the internal energy per unit volume. Relations (2.9) and (2.10) may be written by introducing W and taking into account the stress-strain relations (2.2). We find

$$\frac{\partial U}{\partial e_{\mu\nu}} = \frac{\partial W}{\partial e_{\mu\nu}} + \beta T \delta_{\mu\nu} \qquad (7.1)$$

with

$$\delta_{\mu\nu} = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu. \end{cases}$$

Also taking into account (2.11)

$$\frac{\partial U}{\partial \theta} = c. \qquad (7.2)$$

Hence

$$U = W + \beta T e + c\theta. \qquad (7.3)$$

The generalized free energy was defined as[1]

$$V = \iiint_{\tau} (U - sT)\, d\tau. \qquad (7.4)$$

The entropy s expanded to the second order is

$$s = c \log\left(1 + \frac{\theta}{T}\right) + \beta e = \frac{c\theta}{T} - \frac{1}{2}c\frac{\theta^2}{T^2} + \beta e. \qquad (7.5)$$

We derive

$$V = \iiint_{\tau}\left(W + \frac{1}{2}\frac{c\theta^2}{T}\right) d\tau. \qquad (7.6)$$

This expression coincides with expression (6.1) for the thermoelastic potential. The term generalized free energy was introduced[1] to emphasize the fact that in contrast to the classical definition it is defined here for the case where the temperature distribution departs from uniformity by an amount locally defined by θ.

It is important to recognize that the concept of generalized free energy extends beyond the linear range of small departures from equilibrium and is quite general. As in reference 1 we adjoin a reservoir at the constant temperature T and consider the partial system of entropy s plus this reservoir as a total isolated system of entropy s'. We may write

$$Ts' = Ts - h \qquad (7.7)$$

where h is the heat acquired by the partial system from the reservoir. Conservation of energy requires that

$$U - \mathcal{E} = h. \qquad (7.8)$$

U is the internal energy of the partial system and $\mathcal{E}$ the external work performed on the system. Hence, we may write

$$-Ts' = v - \mathcal{E} \qquad (7.9)$$

with

$$v = U - Ts \qquad (7.10)$$

the generalized free energy.

Since s' tends toward a maximum for equilibrium v must tend toward a minimum if $\mathcal{E} = 0$. Moreover for a reversible transformation $s' = 0$ and

$$v = \mathcal{E}. \qquad (7.11)$$

The change of v is a measure of the work available. We may split the free energy into two terms

$$v = v_T + v_c \qquad (7.12)$$

where v_T is the classical isothermal value and v_c the value due to changes of temperature while maintaining

all other variables constant. This may be written

$$v_c = U_c - Ts_c = \int_0^\theta c d\theta - T \int_0^\theta \frac{c d\theta}{T+\theta}$$

$$v_c = \int_0^\theta \frac{c\theta}{T+\theta} d\theta. \tag{7.13}$$

The specific heat for constant values of all variables except the temperature is denoted by c.

It is interesting to note that this new concept introduces the Carnot efficiency $\theta/T+\theta$ in an integrated form.

If we expand v_c in a series, the first term is

$$v_c = \tfrac{1}{2}c\theta^2. \tag{7.14}$$

This checks with the integrand in (7.6). The v_T term is the isothermal free energy which in the case of thermoelasticity coincides with the mechanical energy.

Attention is called to the close relationship between Gibbs[10] concept of available energy or "availability" as defined by Keenan[11] and the present definitions. This availability may be defined as

$$\phi = U - Ts + PV_R$$

where PV_R is the work done by the system on a large reservoir at pressure P.

We also introduced an invariant D which plays the role of a dissipation function. It was defined in terms of a vector $\partial S/\partial t$, which is analogous to a velocity field and represents the rate of entropy flow or entropy flux. It may be shown to measure the rate of entropy production.

A flow of heat $T\dot{S}$ per unit time and per unit area from a temperature $T+\theta$ to T corresponds to a rate of entropy production.[12]

$$R = T\left[\frac{\dot{S}}{T} - \frac{\dot{S}}{T+\theta}\right] \cong \frac{\dot{S}\theta}{T}. \tag{7.15}$$

The rate of entropy production due to a temperature gradient is, therefore,

$$R = \frac{\partial S/\partial t}{T} \cdot \text{grad}\,\theta \tag{7.16}$$

per unit volume. Since

$$T(\partial S/\partial t) = k\,\text{grad}\,\theta, \tag{7.17}$$

where k is the thermal conductivity, the rate of entropy production is

$$R = \dot{S}^2/k. \tag{7.18}$$

[10] *The Collected Works of J. Willard Gibbs* (Longmans Green and Company, New York, 1931), Vol. 1.
[11] J. H. Keenan, Brit. J. Appl. Phys. 2, 183–192 (1951).
[12] S. R. De Groot, *Thermodynamics of Irreversible Processes* (Interscience Publishers, Inc., New York, 1952).

The invariant D is

$$D = \frac{1}{2}\int\int\int TR d\tau = \frac{1}{2}\int\int\int \frac{T}{k}\dot{S}^2 d\tau \tag{7.19}$$

which represents a quantity proportional to the total rate of entropy production in the volume τ. A principle of minimum entropy production may also be stated in accordance with the general theory of reference 1. However, it is easier to express it in terms of generalized coordinates as done in the next section.

8. GENERALIZED COORDINATES AND THERMOELASTIC ADMITTANCE

The variational principle is a powerful tool for the formulation of thermoelastic problems in any system of coordinates including generalized coordinates. If we represent the solid displacement and the entropy displacement fields as

$$\mathbf{u} = \sum \mathbf{u}_i(x,y,z)q_i$$
$$\mathbf{S} = \sum \mathbf{S}_i(x,y,z)q_i \tag{8.1}$$

the q_i constitute a set of n generalized coordinates. The thermoelastic potential and the dissipation function are then positive definite quadratic forms

$$V = \tfrac{1}{2}\sum^{ij} a_{ij}q_i q_j$$
$$D = \tfrac{1}{2}p\sum^{ij} b_{ij}q_i q_j. \tag{8.2}$$

Applying the variational principle with arbitrary variations δq_n leads to n first-order differential equations

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial q_i} = Q_i. \tag{8.3}$$

Q_i plays the role of a generalized Lagrangian force defined by

$$Q_i = \int\int_A (\mathbf{F}\cdot\mathbf{u}_i + \theta\mathbf{n}\cdot\mathbf{S}_i)dA. \tag{8.4}$$

It is similar to the usual definition of the force by the method of virtual work. In analogy with porous materials the term $\theta\mathbf{n}\cdot\mathbf{S}_i$ of the surface integral represents the virtual work of a pressure θ on a total fluid volume inflow $\mathbf{S}_i\cdot\mathbf{n}$. It introduces the concept of *generalized thermal force*. The set of linear differential equations (8.3) may be written

$$\sum^j a_{ij}q_j + p\sum^j b_{ij}q_j = Q_i \tag{8.5}$$

with symmetric matrices $a_{ij}=a_{ji}$ and $b_{ij}=b_{ji}$. We have shown in reference 2 that their operational solution is

$$q_i = \sum_{j=1}^n A_{ij}Q_j. \tag{8.6}$$

The symmetric admittance matrix $A_{ij} = A_{ji}$ is

$$A_{ij} = \sum^s \frac{C_{ij}^{(s)}}{p + \lambda_s} + C_{ij}, \qquad (8.7)$$

where $-\lambda_s$ are real and negative or zero and are the roots of the determinant

$$\det |a_{ij} + p b_{ij}| = 0. \qquad (8.8)$$

An impedance matrix may also be defined if we apply forces only to k coordinates and leave the others free. In this case, we have shown that

$$Q_i = \sum_{j=1}^{k} Z_{ij} q_j \quad i = 1 \cdots k, \qquad (8.9)$$

where the symmetric impedance matrix $Z_{ij} = Z_{ji}$ is

$$Z_{ij} = \sum^s \frac{p}{p + r_s} D_{ij}^{(s)} + D_{ij} + p D_{ij}'. \qquad (8.10)$$

In this case $-r_s$ are the roots of the determinant of the subsystem obtained by eliminating in Eqs. (8.5) lines and columns from 1 to k.

The characteristic roots λ_s corresponds to normal modes of the type

$$q_i = \phi_i^{(s)} e^{-\lambda_s t} \qquad (8.11)$$

which may be called relaxation modes and are solutions of Eqs. (8.4) in the absence of applied generalized forces. They are orthogonal, i.e.,

$$\sum^{ij} a_{ij} \phi_i^{(s)} \phi_j^{(r)} = \sum^{ij} b_{ij} \phi_i^{(s)} \phi_j^{(r)} = 0. \qquad (8.12)$$

These modes may be used as generalized coordinates in which case the matrix $a_{ij} + p b_{ij}$ is diagonalized. It should be noted that Eqs. (8.5) assume a characteristic form if we consider separate generalized coordinates for **u** and **S**. We put

$$\mathbf{u} = \sum^i \mathbf{u}_i q_i \quad \mathbf{S} = \sum^s \mathbf{S}_s q_s. \qquad (8.13)$$

With partitioned matrices, Eqs. (8.4) become

$$\left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{12}' & M_{22} \end{array} \right] \left[\begin{array}{c} q_i \\ q_s \end{array} \right] + \left[\begin{array}{c|c} O & O \\ \hline O & N \end{array} \right] \left[\begin{array}{c} \dot{q}_i \\ \dot{q}_s \end{array} \right] = \left[\begin{array}{c} Q_i \\ Q_s \end{array} \right] \quad (8.14)$$

where the matrices M and N are constants and M_{12}' is the transposed of M_{12}. The case $Q_s = 0$ corresponds to the application of mechanical forces and $Q_i = 0$ to the case of pure thermal forces. We may eliminate q_i by matrix multiplication. We find

$$[M_{22} - M_{12}' M_{11}^{-1} M_{12}] q_s + N \dot{q}_s$$
$$= Q_s - M_{12}' M_{11}^{-1} Q_i; \quad (8.15)$$

this may be solved in normal coordinates or pure

thermoelastic relaxation modes, with thermal variables q_s. It corresponds to the diffusion equation (5.5). It is of considerable interest to point out that according to this result, and the results in Sec. 9, the propagation of entropy as a diffusion process, which was derived in Sec. 5 for isotropic materials is also valid for the general case of anisotropy.

With reference to extremum principles we may also apply a general theorem established in reference 1. Equations (8.3) are equivalent to stating that the rate of entropy production or the dissipation D is a minimum for all possible fields of entropy flow rate satisfying the condition that the power input of the disequilibrium forces is constant. In the present case this condition reads

$$\sum^i \left(Q_i - \frac{\partial V}{\partial q_i} \right) \dot{q}_i = 0. \qquad (8.16)$$

The disequilibrium force $Q_i - (\partial V / \partial q_i)$ depends on the instantaneous configuration of the mechanical and thermal forces.

9. ANISOTROPIC MEDIA

In the most general anisotropic media, the equation of state relating stress deformation and temperature are written

$$\sigma_{\mu\nu} = \sum^{ij} C_{\mu\nu}^{ij} e_{ij} - \beta_{\mu\nu} \theta \qquad (9.1)$$

where $\beta_{\mu\nu}$ is a second-order six component tensor corresponding to thermal dilatation and $C_{\mu\nu}^{ij}$ are the twenty-one components of the isothermal elastic modulus tensor. The summation with respect to i, j is extended to all distinct combination pairs of i, j. This is because of our definition of e_{ij}. The entropy is still given by Eq. (2.6) and Eqs. (2.7), (2.8), and (2.10) are still valid. We derive

$$C_{\mu\nu}^{ij} = C_{ij}^{\mu\nu}. \qquad (9.2)$$

Also Eq. (2.8) leads to

$$\frac{\partial U}{\partial e_{\mu\nu}} = \sigma_{\mu\nu} + T_1 \beta_{\mu\nu}. \qquad (9.3)$$

Hence

$$ds = \frac{1}{T_1} \frac{\partial U}{\partial T_1} dT_1 + \sum^{\mu\nu} \beta_{\mu\nu} de_{\mu\nu} \qquad (9.4)$$

or

$$ds = c \frac{dT_1}{T_1} + \sum^{\mu\nu} \beta_{\mu\nu} de_{\mu\nu}. \qquad (9.5)$$

Integrating this exact differential we find the entropy

$$s = c \log \left(1 + \frac{\theta}{T} \right) + \sum^{\mu\nu} \beta_{\mu\nu} e_{\mu\nu}. \qquad (9.6)$$

We have chosen the constant of integration such that $s = 0$ for $e_{\mu\nu} = \theta = 0$. The expression sT is the heat

absorbed h per unit volume. Retaining only the linear term in θ we have

$$h = c\theta + T \overset{\mu\nu}{\sum} \beta_{\mu\nu}e_{\mu\nu}. \qquad (9.7)$$

This generalizes Eq. (2.14). The field equations for the displacement vector u_i are first the three equations obtained by substituting the stresses into the equilibrium equations (3.1). The fourth equation is obtained from the law of heat conduction.

$$\frac{T\partial S_i}{\partial t} = -\overset{j}{\sum} k_{ij}\frac{\partial\theta}{\partial x_j}, \qquad (9.8)$$

where TS_i is the total heat flow vector, and $k_{ij} = k_{ji}$ is the thermal conductivity tensor. The symmetry conforms with Onsager's reciprocity relations.[12] We have

$$T\,\mathrm{div}\mathbf{S} = -h = -c\theta - T \overset{\mu\nu}{\sum} \beta_{\mu\nu}e_{\mu\nu}. \qquad (9.9)$$

Combining with (9.11)

$$\overset{ij}{\sum}\frac{\partial}{\partial x_i}\left[k_{ij}\frac{\partial\theta}{\partial x_j} \right] = -c\frac{\partial\theta}{\partial t} - T\sum \beta_{\mu\nu}\frac{\partial e_{\mu\nu}}{\partial t}. \qquad (9.10)$$

This generalizes Eq. (3.4) to the anisotropic case. We note that the summation μ, ν is extended to all six pairs of indices, while the summation i, j is extended to all nine permutations of indices.

It is of interest to express these equations in the general variational form of irreversible thermodynamics. Proceeding as in the isotropic case, we have

$$\partial U/\partial\theta = c. \qquad (9.11)$$

From (9.1) and (9.3) we derive

$$\frac{\partial U}{\partial e_{\mu\nu}} = \overset{ij}{\sum} C_{\mu\nu}{}^{ij}e_{ij} + T\beta_{\mu\nu}. \qquad (9.12)$$

Hence,

$$U = W + T \overset{\mu\nu}{\sum} \beta_{\mu\nu}e_{\mu\nu} + c\theta, \qquad (9.13)$$

where W is the mechanical isothermal energy

$$W = \tfrac{1}{2} \overset{\mu\nu}{\sum}\, \overset{ij}{\sum} C_{\mu\nu}{}^{ij}e_{ij}e_{\mu\nu}. \qquad (9.14)$$

The summation is for distinct combinations i, j, and μ, ν_1. The thermoelastic potential is

$$V = \iiint_\tau (U - sT)d\tau \qquad (9.15)$$

$$= \iiint_\tau \left(W + \frac{1}{2}\frac{c}{T}\theta^2 \right)d\tau. \qquad (9.16)$$

This result could also have been derived from the general expression (7.12) since W represents the isothermal free energy v_T.

We must also express the rate of entropy production as a generalized dissipation function. Following the same procedure as for the isotropic case the rate of entropy production per unit volume is written

$$R = -\frac{1}{T}\overset{i}{\sum}\frac{\partial S_i}{\partial t}\frac{\partial\theta}{\partial x_i}. \qquad (9.17)$$

Expressing the heat conduction law by means of a resistivity matrix λ_{ij} inverse of the thermal conductivity k_{ij} we write

$$-T\overset{j}{\sum}\lambda_{ij}\frac{\partial S_i}{\partial t} = \frac{\partial\theta}{\partial x_i}. \qquad (9.18)$$

Hence

$$R = \overset{ij}{\sum}\lambda_{ij}\frac{\partial S_i}{\partial t}\frac{\partial S_j}{\partial t}. \qquad (9.19)$$

The generalized dissipation function is

$$D = \frac{T}{2}\iiint_\tau R\,d\tau = \frac{T}{2}\iiint_\tau \overset{ij}{\sum}\lambda_{ij}\frac{\partial S_i}{\partial t}\frac{\partial S_j}{\partial t}d\tau. \qquad (9.20)$$

As above we write instead of D the operational invariant

$$D = \tfrac{1}{2}pT\iiint_\tau \overset{ij}{\sum}\lambda_{ij}S_iS_j\,d\tau. \qquad (9.21)$$

In order to apply the variational method, we must express θ in terms of $\mathbf{u}$ and $\mathbf{S}$. This is obtained from (9.19)

$$V = \iiint_\tau\left[W + \frac{1}{2}\frac{T}{C}(\mathrm{div}\mathbf{S} + \overset{\mu\nu}{\sum}\beta_{\mu\nu}e_{\mu\nu})^2 \right]d\tau. \qquad (9.22)$$

The variational principle is again

$$\delta V + \delta D = \iint_A (\mathbf{F}\cdot\delta\mathbf{u} + \theta\mathbf{n}\cdot\delta\mathbf{S})dA, \qquad (9.23)$$

which must be verified for all possible variations of the vectors $\mathbf{u}$ and $\mathbf{S}$. This principle may be considered as a consequence of a more general one in irreversible thermodynamics.[1] It may also be justified by verifying that it leads to the correct differential equations of thermoelasticity as established in the foregoing.

It may also be shown as in Sec. 4 that the complete equivalence between the thermoelastic problem and the theory of elasticity for porous materials extends to the anisotropic case by comparing the above equations with those of reference 6. This is a consequence of a deeper thermodynamic identity between the two problems. The variational formulation leads to Eqs. (8.3) with generalized coordinates and to the existence of relaxation modes and a thermoelastic admittance (8.7).

10. DYNAMICAL EQUATIONS

We will show now how the principles developed in the previous sections may be extended to include the

dynamics of thermoelastic media. This is important among other things, in the evaluation of damping and resonant frequencies of piezoelectric oscillators.

We shall consider the most general case of anisotropic elastic body. If we introduce an inertia term, the equilibrium equations (3.11) are replaced by

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z}+\rho\left(X-\frac{\partial^2 u_x}{\partial t^2}\right)=0$$

$$\frac{\partial \sigma_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{yz}}{\partial z}+\rho\left(Y-\frac{\partial^2 u_y}{\partial t^2}\right)=0 \qquad (10.1)$$

$$\frac{\partial \sigma_{zx}}{\partial x}+\frac{\partial \sigma_{zy}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}+\rho\left(Z-\frac{\partial^2 u_z}{\partial t^2}\right)=0.$$

In order to obtain the field equations for this case, we may proceed entirely as above, in Secs. 3 and 9. The only difference is in the appearance of an additional inertia term in the equation. If the body force is not neglected, it will, of course, also appear in the equations.

It can be verified directly that these field equations are equivalent to the variational formulation

$$\delta V+\delta D=\int\int_A(\mathbf{F}\cdot\delta\mathbf{u}+\theta\mathbf{n}\cdot\delta\mathbf{S})dA$$

$$+\int\int\int_\tau\rho\mathbf{B}\cdot\delta\mathbf{u}d\tau-\int\int\int\rho\frac{\partial^2\mathbf{u}}{\partial t^2}\cdot\delta\mathbf{u}d\tau. \quad (10.2)$$

The volume integrals on the right side represent the virtual work of the body force $\mathbf{B}$ and of the inertia force. The generalized free energy V and dissipation function D are given by (9.16) and (9.21). The variational equation (10.2) may also be considered as a consequence of general variational principles in irreversible thermodynamics combined with d'Alembert's principle. We may write it in a more compact form by introducing an invariant corresponding to the kinetic energy. With

$$\mathcal{T}=\tfrac{1}{2}p^2\int\int\int_\tau\rho\mathbf{u}^2 d\tau \qquad (10.3)$$

and

$$p^2=d^2/dt^2,$$

we may write

$$\delta\mathcal{T}=\int\int\int_\tau\rho p^2\mathbf{u}\cdot\delta\mathbf{u}d\tau=\int\int\int_\tau\rho\frac{\partial^2\mathbf{u}}{\partial t^2}\cdot\delta\mathbf{u}d\tau. \quad (10.4)$$

Hence the variational equation (10.2) becomes,

$$\delta V+\delta D+\delta\mathcal{T}$$

$$=\int\int_A(\mathbf{F}\cdot\delta\mathbf{u}+\theta\mathbf{n}\cdot\delta\mathbf{S})dA+\int\int_\tau\rho\mathbf{B}\delta\mathbf{u}d\tau. \quad (10.5)$$

As in Sec. 7, the variational method may be used to introduce equations of the Lagrangian type for gen-

eralized coordinates. These generalized coordinates q_i are associated with normalized configurations of the fields $\mathbf{u}_i$, $\mathbf{S}_i$ and defined by relations (8.1). The generalized force Q_i conjugate to q_i is defined by the virtual work of the surface forces, the surface temperatures, and the body forces, i.e.,

$$Q_i=\int\int_A(\mathbf{F}\cdot\mathbf{u}_i+\theta\mathbf{n}\cdot\mathbf{S}_i)dA+\int\int\int_\tau\rho\mathbf{B}\cdot\mathbf{u}_i d\tau. \quad (10.6)$$

The dynamical equations are then

$$\frac{\partial V}{\partial q_i}+\frac{\partial D}{\partial q_i}+\frac{\partial\mathcal{T}}{\partial q_i}=Q_i. \qquad (10.7)$$

With V and D expressed as quadratic forms (8.2) and

$$\mathcal{T}=\tfrac{1}{2}p^2\sum^{ij}m_{ij}q_iq_j, \qquad (10.8)$$

we derive the system of equations

$$\sum^j a_{ij}q_j+p\sum^j b_{ij}q_j+p^2\sum^j m_{ij}q_j=Q_i. \qquad (10.9)$$

Remembering that p is a time differential operator, this is also written

$$\sum^j a_{ij}q_j+\sum^j b_{ij}\dot{q}_j+\sum^j m_{ij}\ddot{q}_j=Q_i. \qquad (10.10)$$

These equations are the same as the differential equations of a damped dynamical system under external forces Q_i. Attention is called to the fact that in the thermoelastic problem the generalized forces represent both the applied mechanical forces and applied temperatures at the boundary. The reader will notice that the Lagrangian equations (10.10) may be written in the usual way

$$\frac{\partial V}{\partial q_i}+\frac{\partial D}{\partial\dot{q}_i}+\frac{d}{dt}\frac{\partial\mathcal{T}}{\partial\dot{q}_i}=Q_i \qquad (10.11)$$

where, D and $\mathcal{T}$ are the usual quadratic invariants of $\dot{q}_i$ representing the dissipation function and the kinetic energy

$$D=\tfrac{1}{2}\sum^{ij}b_{ij}\dot{q}_i\dot{q}_j$$

$$\mathcal{T}=\tfrac{1}{2}\sum^{ij}m_{ij}\dot{q}_i\dot{q}_j. \qquad (10.12)$$

As regards wave propagation addition of the inertia terms yields field equations for the dilatation e and the entropy density s. We find

$$\left(\lambda+2\mu+\frac{\beta^2 T}{c}\right)\nabla^2 e-\frac{\beta}{c}T\nabla^2 s=\rho\frac{\partial^2 e}{\partial t^2}$$

$$-\frac{\beta T}{c}\nabla^2 e+\frac{T}{c}\nabla^2 s=\frac{T}{k}\frac{\partial s}{\partial t}. \quad (10.13)$$

The dilatational wave propagates with a diffuse front, characterized by a finite group velocity and an infinite signal velocity. Rotational waves are unaffected by the thermal phenomenon and propagate with no attenuation or dispersion at the velocity $(\mu/\rho)^{\frac{1}{2}}$. These waves behave as those in a porous medium containing a massless fluid. Waves in a porous medium will be discussed in a forthcoming publication.

11. APPLICATION TO HEAT CONDUCTION

Elasticity problems have been treated frequently by variational methods. The new feature of the present approach is the extension of such methods to heat conduction problems whether or not the phenomenon is coupled with an elasticity problem. As an illustration, we shall treat by these methods the well-known problem of a plate suddenly heated to a given constant temperature on one side. The heated side brought to the temperature θ_0 at the time $t=0$ is located at $y=0$. The other side at $y=h$ is maintained at the temperature $\theta=0$.

We choose for the entropy displacement vector a cosine series distribution whose coefficients are the generalized coordinates q_n,

$$S = \sum_{n=0}^{\infty} q_n \cos\pi n\frac{y}{h}. \tag{11.1}$$

The temperature is

$$\theta = -\frac{T}{c}\,\mathrm{div}\,\mathbf{S} = -\frac{T}{c}\frac{dS}{dy}$$

$$\theta = -\frac{T}{c}\frac{\pi}{h}\sum_{n=1}^{\infty} nq_n \sin\pi n\frac{y}{h}. \tag{11.2}$$

Since we neglect the elastic effect, the thermoelastic potential depends only on the temperature

$$V = \frac{1}{2}\int_0^h \frac{c\theta^2}{T}dy = \frac{1}{2}\frac{T}{c}\int_0^h \left(\frac{dS}{dy}\right)^2 dy$$

$$V = \frac{\pi^2}{4h}\frac{T}{c}\sum_{n=1}^{\infty} n^2 q_n^2. \tag{11.3}$$

The generalized dissipation function is

$$D = \frac{1}{2}p\frac{T}{k}\int_0^h S^2 dy$$

$$D = \frac{1}{2}p\frac{Th}{k}\left[q_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} q_n^2\right]. \tag{11.4}$$

We notice that V and D contain no cross-coupling terms, the problem is therefore diagonalized from the start. We must now evaluate the generalized force Q_n associated with the coordinate q_n. This is a purely

thermal force due to the application of a temperature

$$\theta_0(t) = \theta_0 z(t) \tag{11.5}$$

at $y=0$. The function $z(t)$ is the unit step function, zero for negative time and unity for positive time. The virtual work of the temperature on each coordinate q_n at $y=0$ is

$$Q_n \delta q_n = \theta_0(t)\delta S_n = \theta_0(t)\delta q_n. \tag{11.6}$$

Hence

$$Q_n = \theta_0 z(t). \tag{11.7}$$

The basic equations

$$\frac{\partial V}{\partial q_n} + \frac{\partial D}{\partial q_n} = Q_n \tag{11.8}$$

become

$$pq_0 = \frac{k}{Th}\theta_0 z(t)$$

$$\left(p + \frac{\pi^2 n^2 k}{h^2 c}\right)q_n = \frac{2k}{Th}\theta_0 z(t). \tag{11.9}$$

These are independent differential equations of the first order. Solving them operationally

$$q_0 = \frac{k\theta_0}{Th}\frac{z(t)}{p} = \frac{k\theta_0}{Th}t$$

$$q_n = \frac{2k\theta_0}{Th}\frac{z(t)}{p + \frac{\pi^2 n^2 k}{h^2 c}} = \frac{2\theta_0 hc}{\pi^2 n^2 T}\left[1 - \exp\left(-\frac{\pi^2 n^2 k}{h^2 c}t\right)\right]. \tag{11.10}$$

The temperature distribution is

$$\theta = \frac{2\theta_0}{\pi}\left[\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi\frac{y}{h}\right.$$

$$\left. -\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{\pi^2 n^2 k}{h^2 c}t\right)\sin n\pi\frac{y}{h}\right]. \tag{11.11}$$

For $t=\infty$ a steady state is reached

$$\theta = \frac{2\theta_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi\frac{y}{h} = \theta_0\left(1 - \frac{y}{h}\right). \tag{11.12}$$

This is the linear distribution of temperature associated with a total heat flow

$$TS = Tq_0 = \frac{k\theta_0}{h}t. \tag{11.13}$$

All coordinates tend to zero equilibrium value except for q_0 which represents a steady state. As shown in reference 1 this corresponds to a general property of thermodynamic systems for which a root λ_s of the characteristic determinant vanishes. The displacement S takes the general form given by the thermal ad-

mittance (8.7)

$$S = \frac{k\theta_0}{Th}\frac{1}{p} + \sum_{n=1}^{\infty}\frac{2k\theta_0}{Th}\frac{1}{p+\lambda_n}\cos\pi n\frac{y}{h} \quad (11.14)$$

with

$$\lambda_n = \frac{\pi^2 n^2 k}{h^2 c}.$$

It is interesting to consider the case $h \to \infty$, i.e., an infinitely thick plate. The first term disappears relative to the series.

$$S = \lim_{h\to\infty}\sum_{n=1}^{\infty}\frac{2k\theta_0}{Th}\frac{1}{p+\lambda_n}\cos\pi n\frac{y}{h}. \quad (11.15)$$

Now

$$\Delta\sqrt{\lambda} = (\lambda_{n+1})^{\frac{1}{2}} - (\lambda_n)^{\frac{1}{2}} = \frac{\pi}{h}\left(\frac{k}{c}\right)^{\frac{1}{2}}. \quad (11.16)$$

Going over to differentials

$$\frac{1}{h} = \frac{1}{\pi}\left(\frac{c}{k}\right)^{\frac{1}{2}}\frac{d\lambda}{2\sqrt{\lambda}}. \quad (11.17)$$

Hence the series (11.15) becomes

$$S = \frac{\theta_0(kc)^{\frac{1}{2}}}{\pi T}\int_0^{\infty}\frac{\cos\left(\frac{c\lambda}{k}\right)^{\frac{1}{2}}y}{p+\lambda}\frac{d\lambda}{\sqrt{\lambda}}. \quad (11.18)$$

This is a case of continuous relaxation spectrum with a density distribution $1/\sqrt{\lambda}$ for the relaxation constants λ. Such continuous distributions are in conformity with the operational solutions of general relaxation phenomena developed in reference 2. The more usual form of operational solution is found by performing the integration in (11.18). We find,

$$S = \frac{\theta_0}{T}\left(\frac{kc}{p}\right)^{\frac{1}{2}}\exp\left[-y\left(\frac{cp}{k}\right)^{\frac{1}{2}}\right]. \quad (11.19)$$

The temperature is

$$\theta = -\frac{T}{c}\frac{dS}{dy} = \theta_0\exp\left[-y\left(\frac{cp}{k}\right)^{\frac{1}{2}}\right]. \quad (11.20)$$

This expression could be obtained directly by finding the operational solution of the partial differential equation of heat conduction

$$k\partial^2\theta/\partial y^2 = c\partial\theta/\partial t. \quad (11.21)$$

This shows how the method of spectral density and relaxation modes represent a very general class of solutions of partial differential equations by means of analytic functions of the operator p.

12. THERMOELASTIC DAMPING

Another important class of phenomena included in the present theory is the damping of thermoelastic

origin. As an illustration we shall consider the problem of a vibrating cantilever rod. Let it be of rectangular shape of height h, width a, and length l. The x-direction is along the axis of the rod with its origin at the point of damping. Following the elementary beam theory it is assumed that all cross sections remain plane and perpendicular to the deflected axis.

We shall apply the generalized coordinate procedure of Sec. 10. In order to evaluate the thermoelastic potential V we must calculate the isothermal mechanical energy. An approximate expression obtained from the elementary beam flexure theory is

$$\iiint_\tau W d\tau = \frac{1}{2}\frac{Eah^3}{12}\int_0^l\left(\frac{d^2w}{dx^2}\right)^2 dx \quad (12.1)$$

when E is Young's modulus and w is the deflection at x. We assume a deflection shape

$$w = q_1\left(1 - \cos\frac{\pi}{2}\frac{x}{l}\right) \quad (12.2)$$

with q_1 the tip deflection as generalized coordinate. We find

$$\iiint_\tau W d\tau = \frac{\pi^4}{64}\frac{Eah^3}{12l^3}q_1^2. \quad (12.3)$$

The second term in V involves the temperature θ. It is related to the local entropy and the dilatation e by (6.5). Hence,

$$\frac{1}{2}\frac{c}{T}\iiint_\tau\theta^2 d\tau = \frac{1}{2}\frac{T}{c}\iiint_\tau(s-\beta e)^2 d\tau. \quad (12.4)$$

The local entropy is assumed to vary linearly across the height z and follow a cosine law along x. This corresponds to a similar distribution of θ and e. We put

$$s = \frac{2q_2}{h}z\cos\frac{\pi}{2}\frac{x}{l} \quad (12.5)$$

where q_2 is a generalized coordinate for the entropy. Also, from the beam theory, the dilatation is

$$e = -z(1-2\nu)\frac{d^2w}{dx^2}$$

$$e = -q_1\frac{\pi^2}{4l^2}(1-2\nu)z\cos\frac{\pi}{2}\frac{x}{l}; \quad (12.6)$$

ν is the isothermal Poisson's ratio. This is, of course, an approximation since ν will vary depending upon the frequency. The error, however, will be small. We may write

$$\frac{1}{2}\frac{c}{T}\iiint_\tau\theta^2 d\tau$$

$$= \frac{Tah^3}{24c}\left[\frac{2q_2}{h} + \frac{\pi^2}{4l^2}\beta(1-2\nu)q_1\right]^2\int_0^l\cos^2\frac{\pi x}{2l}dx. \quad (12.7)$$

From the above results

$$V = \int\int\int_\tau \left(W + \frac{1}{2}\frac{c}{T}\theta^2\right)d\tau$$

$$= \tfrac{1}{2}a_{11}q_1^2 + a_{12}q_1q_2 + \tfrac{1}{2}a_{22}q_2^2 \quad (12.8)$$

with

$$a_{11} = \frac{\pi^4}{32}\frac{ah^3}{12l^3}\left[E + \beta^2(1-2\nu)^2\frac{T}{c}\right]$$

$$a_{12} = \frac{\pi^2}{48}\beta(1-2\nu)\frac{Tah^2}{lc} \quad (12.9)$$

$$a_{22} = \frac{1}{6}\frac{Tahl}{c}.$$

The kinetic energy invariant is

$$\mathcal{T} = \tfrac{1}{2}p^2\rho ah\int_0^l w^2 dx. \quad (12.10)$$

Substituting (12.2) we find

$$\mathcal{T} = \tfrac{1}{2}p^2 m_{11}q_1^2 \quad (12.11)$$

with

$$m_{11} = \frac{3\pi - 8}{2\pi}\rho ahl.$$

Finally, we must evaluate D. We assume that the entropy flow S is directed along the z direction, i.e., across the height of the beam. From (6.4) we have

$$\frac{dS}{dz} = s = \frac{2q_2}{h}z\cos\frac{\pi}{2}\frac{x}{l}. \quad (12.12)$$

Assuming there is no heat flow across the boundaries, this corresponds to an entropy displacement.

$$S = \frac{q_2}{h}\left(\frac{h^2}{4} - z^2\right)\cos\frac{\pi x}{2l}. \quad (12.13)$$

Hence, by integrating

$$D = \tfrac{1}{2}p\frac{T}{k}\int\int\int_\tau S^2 d\tau = \tfrac{1}{2}pb_{22}q_2^2 \quad (12.14)$$

with

$$b_{22} = Tah^3l/60k$$

Lagrangian equations are

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial q_i} + \frac{\partial \mathcal{T}}{\partial q_i} = 0. \quad (12.15)$$

Substituting the above expressions, with operators replaced by time derivatives, yields two differential equations for q_1, q_2,

$$m_{11}\ddot{q}_1 + a_{11}q_1 + a_{12}q_2 = 0$$
$$b_{22}\dot{q}_2 + a_{12}q_1 + a_{22}q_2 = 0. \quad (12.16)$$

Eliminating q_2 we may write in mixed operational notation,

$$m_{11}\ddot{q}_1 + \left[a_{11} - \frac{a_{12}^2}{a_{22} + pb_{22}}\right]q_1 = 0. \quad (12.17)$$

For harmonic oscillations $p = i\omega$ and the bracketed coefficient may be considered as a complex spring constant, function of the frequency.

$$a_{11} - \frac{a_{12}^2}{a_{22} + pb_{22}} = a_{11} - \frac{a_{12}^2 a_{22}}{a_{22}^2 + \omega^2 b_{22}^2} + i\frac{a_{12}^2 b_{22}\omega}{a_{22}^2 + \omega^2 b_{22}^2}. \quad (12.18)$$

The real part represents the rigidity modulus which varies from the isothermal to the adiabatic value when the frequency goes from zero to infinity.

We note here that we have assumed Poisson's ratio to be constant in order to simplify the derivation, so that the adiabatic rigidity is only approximate. A more exact value could easily be obtained by introducing the lateral contraction as a third coordinate.

The imaginary term gives the equivalent damping force, which vanishes at zero and infinite frequency and is maximum for

$$\omega = \frac{a_{22}}{b_{22}} = 10\frac{k}{ch^2}. \quad (12.19)$$

This corresponds to a period of the order of the relaxation time of the linear temperature distribution along the transverse coordinate.

It should be noted that for frequencies large relative to the frequency of maximum damping the approximation (12.12) is not valid as regards the distribution of entropy along z. In order to obtain the correct asymptotic values of the damping at high frequency we should introduce additional generalized coordinates for s, e.g., in the form of a Fourier series or a distribution which takes into account the appearance of a "skin effect." The principle of the method, however, remains the same.

13. CONCLUSIONS AND RESULTS

The reader will undoubtedly realize that many developments and concepts have been merely sketched. A clearer understanding will perhaps be gained through a more precise formulation of some of the conclusions and results as listed hereafter.

1. The concept of thermoelastic potential is introduced. This function represents by a single invariant the elastic and thermostatic properties of the medium.

2. The state of the thermoelastic medium is described by means of two vector fields, the solid displacement **u** and an entropy flow field **S**. The depend-

ence of the thermoelastic potential, i.e., of the thermo-static state, on **S** appears only through the divergence of the field **S**. The curl of the entropy flow is therefore an ignorable (or cyclic) coordinate in the same sense as this term is used in the description of dynamical systems.

3. The irreversible properties of the medium are represented by a dissipation function defined in terms of the entropy flow field as a quadratic invariant of the time rate of flow.

4. From the thermoelastic field equations a variational principle is derived by means of the thermoelastic potential and a dissipation function. This permits the thermoelasticity equations including inertia effects to be written in any coordinate system.

5. It is shown that the present variational formulation of thermoelasticity constitutes a particular case of a more general approach previously developed by the writer for linear irreversible thermodynamics.[1] The thermoelastic potential is a particular case of the generalized free energy as defined in reference 1 for systems with nonuniform temperatures. The dissipation function is shown to be proportional to the rate of entropy production. It is also indicated how a general principle of minimum entropy production applies to thermoelasticity.

6. The concept of thermal force is introduced as a generalized force by a principle of virtual work involving the product of the temperature and a virtual entropy displacement. The thermal and mechanical force are thus similarly defined leading to a unified treatment of heat conduction and mechanics.

7. From the variational principle, once the thermoelastic potential and the dissipation function have been expressed in terms of generalized coordinates and the generalized forces evaluated by the method of virtual work, Lagrangian equations may be derived which have the same form as in dynamical dissipative systems.

8. By the use of operational invariants for the dissipation function and kinetic energy the derivation of the Lagrangian equations from the variational principle is considerably simplified.

9. It is shown that the thermoelastic potential is the sum of two terms. The first which is the usual mechanical potential energy of the theory of elasticity is also the isothermal free energy as defined in classical thermostatics. The second term is the volume integral of an expression $c\theta^2/2T$ involving the local excess temperature θ. It is also shown how this fact is a particular case of more general principles as developed in reference 1.

10. If inertia is neglected the entropy density satisfies a diffusion type equation. This is derived explicitly for isotropic medium and it is shown how this must be a general property for other cases.

11. It is shown that the thermoelastic theory is isomorphic to the isothermal theory of elasticity of a porous material containing a viscous fluid (consolidation theory). Results of one theory may be transferred to the other by a simple change of notation. This analogy includes dynamics and wave propagation if the fluid mass is zero. The signal velocity of dilatational waves is infinite and the physical propagation is represented by the group velocity.

12. General solutions of the thermoelasticity equations for an isotropic homogeneous medium are established by analogy with the results obtained in the theory of elasticity of a porous solid. Further solutions developed in reference 7 may be transferred to thermoelasticity by a simple change of notation.

13. The response of the thermoelastic system to mechanical or thermal forces is expressed by means of a thermoelastic admittance matrix. The general form of this matrix has been derived. In particular it leads to expressions analogous to those familiar in viscoelasticity where the stress strain relations are formulated in terms of a relaxation spectrum.

14. The present formulation of thermoelasticity leads to new numerical methods. In particular they are applicable to pure thermal conduction problems. This is illustrated on the simple example of heating of a plate by the use of normal coordinates.

15. Thermoelastic damping effects are also readily evaluated by these new methods and this is shown by treating the problem of thermoelastic behavior of a vibrating plate.