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Applied Mathematics
An Art and a Science

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WHAT IS APPLIED MATHEMATICS?

IT HAS BEEN SAID that France is divided into 40 million Frenchmen. To raise the question of definition for applied mathematics is to invite perhaps as many answers as there are applied mathematicians or rather people who call themselves by that name. There have been numerously quoted remarks on this subject, some by individuals bearing great names in mathematics. I personally recall a lecture by the late Harry Bateman in which the applied mathematician was characterized as a "mathematician without mathematical conscience." There is no doubt that the statement reflects the existence of a deep and insoluble conflict—the contradiction between the ideal of clarity, precision, and rigor of mathematics as a science and the adventurous and imaginative urge of the mind, whether it is striving for the ultimate in its impatient search for truth and artistic expression or laboring under the fierce compulsions of the pragmatic and competitive world of technology. One could understand the feelings of the artist who undoubtedly benefits from the scientific study of colors but who would be constantly reminded of proceeding with rigorous conformity to the dictates of physics and psychology.

An excellent illustration of the conflicting mental attitudes referred to here is given by Sir Geoffrey Taylor in a recent witty essay.† His advice to those who would seek counsel from a mathematician is to use the approach one would make to a child: "Make your question as simple as possible and don't be disappointed if he finds that he cannot answer the exact question you put to him but can answer a related but rather different one."

Evidently it would be a drastic fallacy to envisage this conflict as opposing the pure and the applied sciences. Indeed it has generally been the rule that many significant discoveries in pure mathematics were made by a process of trial and error which did not exactly reflect a spirit of conformity with the conventional requirements of rigor. This is not to say, of course, that rigor lies outside the realm of aesthetics, as mathematicians well know, but simply that many times, because of human limitations, it cannot be attained except through lengthy and laborious methods.

If mathematicians were to be assembled to agree on a formal definition of the field of applied mathematics, I believe the result would bear a rather embarrassing similarity to the emasculated and noncommittal resolutions that the present generation has resigned itself to accept as the normal product of international political gatherings. As a definition would surely be presumptuous, is it not preferable to recognize that we are dealing here not with an established body of science but rather with a way of life of the scientific mind in general, best understood by a description of its activities and of its peculiar creative tools and processes? It may be truly said that mathematics today is not only the very fabric and soul of the physical sciences but also their universal language, providing the essential symbols through which the key concepts are not merely conveyed, but created, molded, and refined.

If there is a conflict, we should recognize that it is not between people but that it arises in the individual and is reflected with all the gradations and shades of a spectrum—between those whose preoccupation is mainly with the creation of a systematic and rigorous body of science and those whose interest in mathematics is purely pragmatic. Such disharmonies may be sometimes trying, but we should look upon them as a sign of inner vitality. For in this dual aspect and conflict lies the most fertile source of progress, from the early attempts at land surveying in primitive agricultural societies and their influence on the development of abstract geometry to the complexities of present-day technology. The contrast is reflected on many planes in the teaching and practice of applied mathematics, as it is only partly a science and mainly a craft and an art. This is particularly apparent in the teaching, for, as experience shows, it is usually much easier for a student to acquire the formal trappings of a theory than the ability to use it and adapt it to a particular situation.

In this respect, the achievements of von Kármán as an applied mathematician are perhaps better appraised by stressing the imponderable rather than by referring to specific contributions that are amply discussed elsewhere in this issue. He gives innumerable examples of how to tackle difficult and complex problems in widely different fields, providing the student with the opportunity of living through the same creative experience and inspiring him with an indelible vision and an enduring faith in the effectiveness of such methods in pure and applied science.

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† Taylor, G. I., *Rheology for Mathematicians*, Proceedings of the Second International Congress on Rheology; Academic Press, Inc., New York, 1954.

The Interrelation Between Pure and Applied Mathematics

We can best characterize the relationship between pure mathematics and the surrounding sciences as a continuous process of cross-fertilization. From the applied fields questions constantly arise—problems of precise definition, existence, unicity, convergence, and others that are a challenge in pure logic and belong to the classical realm of pure mathematics. In fact, it is sometimes difficult to say whether some questions of this type belong to the pure or applied category. Take, for example, the problem of unicity of solution of the Navier-Stokes equations in fluid dynamics. We may consider this as a problem for a specific equation concerned with a physical problem, but obviously the question immediately arises of the properties of equations of a similar type, devoid of any physical application and of the possibility of generalizing particular theorems to more general classes of objects. One can cite innumerable occurrences of this process. One of the most striking is the recent impressive development of theories in the functional and abstract spaces which found their origin and inspiration in the familiar eigenvalue problems of classical and quantum physics. As another example in a totally different branch of physics we could mention the boundary-layer theory, whose progress is closely associated with the name of von Kármán, and which has led to a new and interesting study of the properties of differential equations when a high-order derivative is multiplied by a vanishingly small coefficient.

In turn, such progress in the domain of pure form and concept suggests new viewpoints and novel ways in the formulation of physical problems. The tensor calculus, a primary conceptual tool in the creation of the theory of Relativity, was itself the result of abstract developments and generalizations arising from the study of geometric surfaces. Here we have a typical example of the stimulation and the molding of physical thinking by purely mathematical concepts.

In this sense, the mathematical sciences may be looked upon as providing not only an indispensable and universal language for our description and understanding of nature but also the spark for the imagination, casting the experimental facts in a new light and suggesting totally new viewpoints and relationships which would otherwise have remained unnoticed.

The Heuristic and Qualitative Functions of Mathematics

This leads us to an even more striking influence of the mathematical viewpoint on physical discovery, one that is often overlooked but is becoming more apparent every day as modern physics reaches deeper into the nature of matter and energy itself. We refer to what might be called the heuristic function of mathematics—i.e., that by which it operates as a catalytic agent for new discoveries of physical laws. There exists a common misconception embodied in the popular belief that science always proceeds in steps, from the collection of unbiased and precise empirical facts to their generalization in the form of systematic scientific laws. The prevalence of this viewpoint, due perhaps to the pervasion of our elementary and undergraduate teaching by the positivism of Auguste Comte, has had an imponderable but pernicious influence on the layman's conception of creative scientific thought and on scientific progress itself. There are numerous instances of scientific discoveries where the pattern is actually reversed, and where mathematical theories determine directly the type and scope of physical measurements and observation. One of the earliest and most striking examples is the discovery of radio waves. The existence of these waves was not suspected and could not be derived mathematically from the laws of electrodynamics as formulated by Faraday and Ampère. It took Maxwell's addition of a term known as the displacement current in the equations to make it possible to derive theoretically the existence of electromagnetic radiation, later confirmed in most striking fashion by Hertz's famous experiment. It must be emphasized that Maxwell's brilliant intuition was essentially a modification of the physical laws motivated solely by considerations of mathematical symmetry, consistency, and aesthetics. Some of the most spectacular advances in physics have been consequences of this type of thinking. Its latest triumph is the experimental confirmation of the existence of the antiproton.

Having considered what we have called the heuristic value of mathematics in the discovery of physical laws, let us turn our attention to another aspect which can be characterized as the qualitative in contrast with the quantitative. It is represented by the approximate laws and the mathematical models whose object is to give a compact and simple mathematical representation of basic phenomena. In this category we have, for example, sources and sinks, the radiation of a Hertzian dipole, etc. They may or may not be realizable physically, but they furnish a precise formulation of an intuitive concept, combining the advantage of simplicity while containing the essence of the physics. Such models are further useful as building blocks in the representation of a more actual and complex situation. We are referring here more particularly to the use of such models in mathematical physics as a powerful method for acquiring insight and understanding. We shall have occasion to come back to this later in connection with the subject of engineering sciences, where the greater complexity of the problems increases the need for such models and requires an unusual skill on the part of the applied mathematician.

One might argue that it should be simple for a mathematician and an applied scientist to join efforts in such attempts, each as a specialist in his own field; but in fact this always proves to be extremely difficult because of the high degree of compromise required between the use of sufficiently simple mathematical techniques and the inclusion of the most significant physical parameters.

Before ending these remarks, I should like to add a slight note on the critical side. It may occur, and it has happened, that physical thinking has been confused by a choice of inappropriate mathematical techniques. Such occurrences, however, are fairly infrequent and usually quickly corrected by subsequent investigators.

The Role of Numerical Analysis and Computers

Finally, let us consider the more obvious applications of mathematics, those that are primarily of a computational nature. This is a field in which, as a rule, the student trained in the techniques of pure mathematics finds his way more easily. I do not wish to enter into a detailed discussion of this vast field which has acquired a tremendous impetus since the advent of electronic calculators. I wish merely to point out two aspects of the numerical approach to scientific problems, which are not always given clear recognition.

One function of numerical analysis is to act as an instrument for empirical discovery. As an example we may cite the law of asymptotic distribution of prime numbers which was first formulated tentatively by Gauss after a study of prime number tables. The power of modern computing machines opens vast new possibilities along this line. Recent numerical work on the evolution of a complex dynamical system with nonlinear coupling terms promises to throw new light on the validity of the axiomatic foundations of statistical mechanics.

Aside from this role in fundamental science, another important function of the numerical work—and one which does not seem to have reached its full development—is to act as a substitute for elaborate physical experimentation. This is of particular importance in areas where the fundamental laws are well known and when it is desired to develop new principles applicable directly to more complex situations. This is, for instance, the case if we want to derive from the elementary laws of particle interaction some understanding of the general behavior of groups of particles such as atoms and crystals. Calculation of a sufficient number of cases may lead to the formulation of general laws much as they would arise from a systematic experimental program. There are of course drastic limitations in this approach. It does, however, possess the advantage over the purely experimental methods of enormous flexibility in the choice of parameters and variables. Furthermore it opens the way to the study of phenomena under conditions unattainable under physical testing, such as those of extremely high pressure and temperature.

As we shall have occasion to point out below, there seems to be a whole field as yet unexploited which lies at the borderline of pure science and engineering. As an example among hundreds, I might mention the calculation of electromagnetic radiation and scatter by various objects and antennas of elementary geometrical shape.

Finally, it cannot be too strongly emphasized that, no matter how powerful the numerical tools of analysis and the computing devices, they may predict the behavior of a particular system, but they never as such furnish an understanding or a grasp of the qualitative effect of each of the factors involved. They can never therefore substitute for the necessity of mathematical models and simplified theories. In fact, such simplified models not only are essential in the efficient planning of computing programs but act as the prime source of invention and imaginative design.

Mathematics and Engineering Education

It is perhaps in engineering that the most exciting challenge arises for broadening the use of mathematical methods. We can distinguish two major roles to be played by mathematics in the engineering field. One can be characterized as conceptual. With the ever-increasing complexity and the expanding domain of technology, it is essential for those engaged in research and development to possess broad knowledge and insight in connection with the problems investigated. It is in the very nature of mathematics to be synthetic and to provide compact formulation for a vast body of learning in widely different and apparently unrelated fields. We have in recent years become more familiar with what has come to be known as analog computers. However, it is not so much in its application to computing that the concept of the analog is most useful. There are innumerable phenomena obeying laws formulated by the same equations. Familiarity with one type of phenomenon immediately leads to understanding of a large class of others falling in the same analog category. One of the well-known elementary examples is the theory of the electric circuit with capacity and inductance. The differential equation for the electrical variable is identical with that of a spring-mass system, thus bringing the invisible electrical phenomenon within the immediate grasp of common experience.

What I wish to emphasize here is a point that seems to have been generally overlooked not only by the layman but also by some who are interested in the promotion of mathematical courses in the engineering schools—namely, the power of mathematics as an educational tool. It is clear that the example of the electric circuit cited above may be multiplied ad infinitum. To cite only a few, such models as the beam on elastic foundation occur repeatedly in a wide variety of problems governed by the same simple differential equation. Phenomena that involve diffusion, heat and electrical conduction, laminar viscous friction, etc., may all be explained rapidly if the student possesses the mastery of a single type of differential equation. Aeronautical engineers are well acquainted with the far-reaching analogy between the laws of electrostatics and magnetism, and the basic principles of subsonic aerodynamics. The more recent developments in the field of compressible flow have shown the usefulness of the Lorentz transformation in airfoil theory, thereby familiarizing the student of aeronautics with one of the fundamental aspects of a remotely different subject—the theory of Relativity.

This interrelation goes even further than is usually apparent because many courses have been taught as specialized fields, and the classroom presentation has usually been encumbered by the addition of unessential features required by traditional approaches in form and concept. Enormous strides could be made in the training of engineering students if a unified treatment of science subjects could be introduced, eliminating much duplication and confusion while reaching simultaneously the dual goal of timesaving and a deeper understanding of the subject.

No doubt not all students possess the ability to grasp the abstract concepts involved or the imagination to master the techniques and a change in the direction indicated would certainly require a rather basic reorientation of our present way of thinking in engineering education, if only in the selection and encouragement of the more gifted student, and the creation of more than one level in the quality and scope of training.

We should realize that there has been a gradual breakdown of the barriers between the various branches of engineering and an enormous expansion of the knowledge in all fields. A more extensive use of the mathematical tool furnishes the only hope of coping with the educational problems involved in these two trends, and in our losing battle against increasing specialization.

From the viewpoint of the teacher, the job of presenting the subject matter in such fundamental fashion is also not an easy one. Many subjects have to be completely rethought. Formulating a problem or a physical theory so that all its essential features are properly represented in the simplest mathematical model is an art. It cannot be taught in systematic fashion but, like all craftmanship, must be acquired from example and mastered through practice. The difficulty lies not generally in the mathematics but in the proper evaluation of the various factors in a physical situation as to their essential or secondary character—a point that we have already stressed above in connection with mathematical models.

Not the least contribution of von Kármán to the advancement of the engineering sciences has been through his teaching. His lectures here and abroad have left their imprint on the thinking and outlook of generations of students. They have learned by direct example how complex phenomena could be reduced to the essential and how the mathematical language can be made to express ideas and formulate scientific problems with simplicity, elegance, and precision, by a skillful balance between sophistication and misleading crudeness. The fact that many of these former students, at present in leading and influential positions, have thus come to realize the power and practicability of mathematical methods is in itself a substantial factor in the progress of our present administration of engineering research and development.

During the years preceding World War II, an attempt was made to crystallize this approach to engineering problems and applied mathematics in the form of a textbook written by von Kármán in collaboration with this writer. As its title *Mathematical Methods in Engineering* infers, the emphasis was not on the teaching of mathematics or engineering but on the art of formulating problems mathematically. As such it constituted perhaps a pioneering attempt. However, in appraising the effectiveness of such writings, one should remember that, especially in the learning of a craft, the written word is never a substitute for the teacher.

Mathematics in Engineering Research and Development

We should also keep in mind that, while the subject of engineering mathematics bears strong resemblance to mathematical physics, there are also some differences in both the scope and the methods. It is clear, of course, that much of the fundamental knowledge in physics is of immediate use in the engineering sciences. However, there have been many areas of physics which have been left undeveloped. This has happened to problems that did not lie directly in the path of the major objective of physics—namely, the quest into the fundamental nature of matter and energy. It is especially true for such problems as those of boundary layer and turbulence, which, in addition to being on the sideline of fundamental physics and exhibiting considerable theoretical difficulty, have been investigated primarily because of their great technological interest. This also explains why applied mechanics and applied mathematics have been closely associated in their historical development.

And this leads us to another difference between engineering mathematics and mathematical physics. Many engineering systems are exceedingly complex. Furthermore the compromise between theoretical simplicity and accuracy generally lies at a different level from that of pure science because of the engineer's greater emphasis on the practicability of a theory and the margins of error usually accepted in technological problems. Therefore the degree of simplification required in the mathematical formulation of an engineering problem is often greater, with the accompanying demand of unusual specialized ability on the part of the analytical engineer.

One of the usual mathematical difficulties that occur in the problems of the engineering sciences in contrast to those in mathematical physics is nonlinearity. Only an extremely small class of nonlinear differential equations has been successfully treated by the mathematician. This is especially true in the case of partial differential equations, and much progress has been achieved in this field by the ingenuity of the individual worker in applied mechanics and the engineer. Among these von Kármán stands in a unique position. He has given an excellent review of such problems and methods, including many original viewpoints, in the article *The Engineer Grapples with Nonlinear Problems.** It is also in connection with such problems that the art of linearization has its greatest value and makes the largest demand on the ingenuity and flair of the applied mathematician. The remarkable success of the linear airfoil theory, particularly in the nonstationary case, is an outstanding example of the power and fertility of this approach. Along the same line, one of the most ingenious discoveries is the so-called Kutta-Joukowski principle, which approximates the effect of viscosity on the flow around an airfoil by a potential flow with circulation, transforming an insuperable mathematical problem into a simple one and leading to many new and fruitful physical viewpoints.

The importance of the conceptual and theoretical approach should also be emphasized in connection with experimental programs in engineering research and development. A huge amount of experimental data has been accumulated in the past at enormous cost in time and energy. A large portion of this data, now buried in files and forgotten, has contributed little to technological advances because of the absence of coordinating theoretical concepts.

Having discussed the conceptual role played by mathematics in engineering, let us now consider the more evident one that can be characterized as that of a super slide rule. While this viewpoint is familiar and might be said to constitute the popular notion, I believe there are also some current misconceptions. It is true that electronic computers are fast becoming instruments of enormous power and flexibility. This, however, requires of the engineer an even higher standard in scientific training. A keen appreciation of the capabilities of the machines in terms of other available tools is necessary, as well as a competent evaluation of the reliability and accuracy of the input data. As experience shows, it is often possible to solve complex problems by skillful simplified analysis, statistical methods, or synthetic mathematical theorems without recourse to the expensive brute force procedure of the large computers. This type of approach has the added advantage of bringing out certain general principles that afford a better understanding of the problem and may contribute unexpected and useful results.

Everything points to an ever-increasing extension of theoretical and numerical methods versus trial-and-error testing in design and development. There are several reasons for this trend: first the availability of the new computing tools themselves, then the increasing cost and complexity of prototypes, and finally the constant advances in all fields of macroscopic physics and a better knowledge of the laws and principles applicable in the engineering sciences. This last factor has often been underestimated in discussions among engineers about the relative merits of the so-called theoretical and "practical" approaches.

* Fifteenth Josiah Willard Gibbs Lecture of the American Mathematical Society, 1940, Bulletin, Amer. Math. Soc., Vol. 46, No. 8, 1940.

In addition to the factors cited above, it may truly be said that a new dimension has been added to the analysis of engineering systems by the application of the methods of operations research and programing through which difficult design problems may be treated analytically for optimization, under conditions involving a high degree of compromise between various technological and even human factors.

A well-conceived development program should reduce the most expensive phase—the actual physical testing—to the minimum required by our inevitable ignorance of certain factors in the physical laws, the hazards of materials quality, and operational uncertainties. As a consequence, testing becomes more and more in the nature of a spot check and verification or a means of answering definite and specific questions.

The Future

We stand at the threshold of a new era, and the future presents unlimited horizons. We have referred only casually to the applications of mathematics to systems analysis, operations research, and programing and have made no mention of information theory, the theory of games, economics, or the biological and social sciences. In some of them, the role of mathematics is still in its early stage but holds tremendous promise.

It must be expected that the rapid progress and expansion of electronic calculators will considerably enhance the practical value of mathematical theories. With wider application of new inventions, such as the semiconductor elements, miniaturization, and the introduction of automatic coding, we may visualize the development of small-size analog and digital computers of enormous speed and flexibility, requiring no specialized skill for their operation. The widespread use of such powerful calculating tools will certainly promote new interests and advances in pure mathematics. Many problems that were only of academic interest in the past now become key questions related to the applications of the new computers.

In the educational field we should recognize three distinct areas—the teaching of mathematics to science and engineering students, mathematics as a cultural subject, and the teaching of pure mathematics and its techniques as a distinct science and specialization. We have already discussed the advantages of a more unified outlook in the training of science and engineering students. There is no doubt that considerable difficulties shall be encountered in raising the present standards, but should we not pay more attention to the selection of students particularly gifted to benefit from such a change? We must keep in mind that an important purpose of advanced teaching, as opposed to mass education, is to stimulate existing talent and help it discover itself.

In the field of research and development, progress will depend not only on the technical personnel but also to a great extent on those at the policy-making and administrative levels. Here again improved educa-

(*Continued on page 489*)

Applied Mathematics

(*Continued from page 410*)

tional programs can ensure that such positions are occupied by leaders who possess a thorough grasp of the nature and potentialities of applied mathematics and are capable of making an efficient choice and use of facilities and personnel. In this connection it may be well to emphasize that in many cases it is not so much a question of organizing and blueprinting as it is one of creating a suitable climate.

While the pragmatic viewpoint is important, certainly the cultural aspects of mathematics should not be overlooked. The fundamental discoveries in pure mathematics must be looked upon as major contributions to our culture. We are all familiar with the famous paradox of Zeno illustrated by the story of Achilles and the tortoise. The object since antiquity of endless argumentation by philosophers, this paradox finds its clarification in the theory of infinite series. That it is essential for those in the applied fields to be acquainted with the more academic questions and the discipline of rigor of pure mathematics is not only true because this knowledge and experience may suddenly acquire unsuspected practical value, but it is also of importance simply for the same purpose that motivates the study of the liberal arts—the achievement of an intimate communion with the great historical currents of thought and the practice of a formative intellectual discipline.

Finally, if we wish to carry this viewpoint to its ultimate conclusion, it is to be hoped that the teaching of mathematics as a purely cultural subject, stripped of its technicalities and the narrow viewpoint of the specialist, may become a requirement in our general curriculum. The impact of the mathematical sciences on our society is becoming more evident every year, influencing the language and concepts of our culture, and expanding rapidly into fields that until today were looked upon as the exclusive domain of those trained in the traditional methods of economics, law, and business administration. If the means of accomplishment remain a question for debate, we should never lose sight of the fact that, in many ways, the final purpose is humanistic, and that intellectualism should never be promoted for its own sake or at the expense of the spiritual, the artistic, and the creative. Thus can we hope to meet the challenge of the future and fulfill its bold promise.