"DYNAMICS OF VISCOELASTIC ANISOTROPIC MEDIA"

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Abstract

New principles are introduced for the formulation of dynamic problems in viscoelasticity for the most general case of anisotropy. One is a variational principle derived from irreversible thermodynamics. The other a correspondence principle between viscoelasticity and elasticity. Applications to the dynamics of viscoelastic plates and rods are developed as examples. The variational principles are applied in conjunction with a very general method of approximation by expanding the deformation in a power series across the thickness. The method is applicable to viscoelastic shells. The concept of partial mode as a generalization of that of vibration mode for the elastic case is also introduced.

1. Introduction

General stress strain laws in anisotropic linear viscoelasticity were derived in operational form from irreversible thermodynamics by the writer[1] [1]. In a subsequent publication [2] it was shown that the stress strain relations are a particular case of a more general formulation of irreversible thermodynamics applicable to a wide variety of phenomena of the linear relaxation or viscous type when mechanical electrical thermal chemical effects, etc. are simultaneously present and may be coupled to each other. Variational principles were also formulated and a brief outline was given of their applicability to viscoelasticity. We present here a more extensive treatment of methods derived from these principles to calculate stresses and deformation in viscoelastic media. The principles are generalized also to include all dynamic phenomena such as wave propagation and vibrations. The methods are applied mostly to examples of

1. Numbers in brackets refer to the bibliography at the end of the paper.

plates and rods of isotropic materials but they apply without difficulty to much more general problems including materials with a high degree of anisotropy.

The general variational principle is established in section 2. In the case of viscoelasticity it may be considered as an extension of the principle of virtual work to an operational form. Another principle which we call the principle of correspondence is a consequence of the formal analogy between the operational tensor and the elastic moduli and is formulated in section 3. A complete correspondence exist between the two so that all static and dynamic solutions of elasticity may be immediately transposed into a corresponding viscoelasticity solution by simply replacing the elastic constants by operators. Because of the variational principle of section 2 the correspondence also extends to all Elasticity solutions obtained by variational methods. In the more restricted domain of static and isotropic problems, the existence of associated solutions in elasticity and viscoelasticity was also noticed by Lee. [5]

Illustrations of the principles are shown in sections 4 and 5 by application to the dynamics of plates and rods. The variational principle is a very flexible and powerful tool for the simplified formulation of complicated problems by the use of partial generalized coordinates. It is shown how this method may be used to establish equations for vibrations and wave propagation in plates and rods to any degree of accuracy by expanding the displacements in a power series along the thickness coordinate. More appropriate series such as combinations of powers and trigonometric functions or other functions may also be used depending on the problem. The method is readily applicable to the dynamics of viscoelastic shells. Equations are also obtained from the principle of correspondence.

In section 6 we discuss the concept of partial mode which constitutes a generalization of the vibration modes of elastic media. A solution for the moving load on a beam or plate is also indicated.

The treatment of plates and rods are intended here more as examples of the methods and is of course far from exhaustive. A more complete discussion should be carried out ultimately regarding the best choice of generalized coordinates to achieve a suitable compromise between simplicity and accuracy.

Extension of the present methods to include non-linear problems of large deflections will be presented in a forthcoming publication. [7] A brief account of the method as applied to plates was given in references [2] and [8].

2. Variational Principles Derived from Irreversible Thermodynamics

We consider a viscoelastic continuum which is linear and may have a most general anisotropy. The small strain tensor in terms of the displacement u_i and the coordinate x_i is

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.1}$$

From the principles of irreversible thermodynamics we have shown [1] that under the assumption of linearity the strain is related to the stress $\sigma_{\mu\nu}$ by

$$\sigma_{\mu\nu} = \sum^{ij} P^{ij}_{\mu\nu}\, e_{ij} \tag{2.2}$$

The tensor $P^{ij}_{\mu\nu}$ is an operational expression

$$P^{ij}_{\mu\nu} = \sum^{s} \frac{p}{p+r_s} D^{ij,s}_{\mu\nu} + D^{ij}_{\mu\nu} + p\, D'^{ij}_{\mu\nu} \tag{2.3}$$

in the time operator

$$p = \frac{d}{dt} \tag{2.4}$$

The D's are constant tensors and the r's are decay constants of "internal degrees of freedom."

The summation sign in Eq 2.3 may be replaced by an integral if there are a great number of internal degrees of freedom—i.e. a relaxation spectrum. In a general way we may write

$$P^{ij}_{\mu\nu} = \int_0^\infty \frac{p}{p+r} D^{ij}_{\mu\nu}(r)\,\gamma(r)\,dr + D^{ij}_{\mu\nu} + p\, D'^{ij}_{\mu\nu} \tag{2.5}$$

where $\gamma(r)$ is a spectral density function which may not necessarily be continuous and may contain discontinuities of the Dirac function type. This property may of course be also formulated by introducing Stieltjes type integrals.

The tensor has the following symmetry properties

$$P^{ij}_{\mu\nu} = P^{ij}_{\nu\mu} = P^{ji}_{\mu\nu} = P^{\mu\nu}_{ij} \tag{2.6}$$

Because of this symmetry, we may introduce an equivalent six by six matrix relating the six stress components to six strain components e_{ii} and $2e_{ij} = \gamma_{ij}$ where $0 \neq j$ and γ_{ij} represents the shear.

In this system there are twenty one components. This operational tensor constituting a six by six symmetric matrix, which generalize the elastic moduli of the classical Elastic Theory.

The above operational relations are a particular case of the general theory of linear irreversible thermodynamics, as formulated in references [1] and [2]. Being given any system of n thermodynamic state variables, it is possible to derive equations for the time history of the systems under given perturbations if we know two invariant functions. One is a function of the time derivatives of the state variable and is proportional to the rate of internal entropy production. It corresponds to a generalized dissipation function. The other is a function of the state variables and represents a generalized free energy. Relations of the type (Eqs 2.2 and 2.3) arise when we consider a system containing hidden coordinates and we wish to find expression for the observed coordinates only. The assumption that viscoelasticity results from a great number of hidden thermodynamic coordinates lead to the above stress strain relations.

In reference [2] we have formulated variational principles applicable to irreversible thermodynamics. One of these principles is a general law of minimum entropy production. It is applicable to viscoelasticity. Another variational principle applies to the case where certain coordinates are hidden and is therefore particularly adaptable to the case of viscoelastic materials. The principle was formulated for static viscoelastic problems in reference [2]. It may be considered as a generalization of the principle of virtual work of Elasticity. We shall now derive a similar principle applicable to viscoelastic problems which include inertia forces.

We formulate d'Alembert's principle as follows:

$$\sum^{\mu\nu} \iiint_{\tau} \sigma_{\mu\nu}\, \delta e^{\mu\nu} d\tau = \sum^{\mu} \iint_{S} F_{\mu}\, \delta u^{\mu} ds - \sum^{\mu} \iiint_{\tau} \rho \ddot{u}_{\mu}\, \delta u^{\mu} d\tau \qquad (2.7)$$

These equations express that the virtual work of the internal forces is equal to the virtual work of the inertia forces and the boundary forces.

The virtual displacement δu^{μ} and associated strain variations $\delta e^{\mu v}$ are applied in the volume τ and at its boundary s. Forces F_{μ} per unit area are applied at the boundary and ρ is the mass density. The method hinges on the symmetry of the tensor $P^{ij}_{\mu v}$. Because of this symmetry we may write

$$\sum^{\mu v} \sigma_{\mu v} \delta e^{\mu v} = \sum^{\mu v} \sum^{ij} P^{ij}_{\mu v} e_{ij} \delta e^{\mu v}$$

$$= \frac{1}{2} \delta \sum^{\mu v} \sum^{ij} P^{ij}_{\mu v} e_{ij} e^{\mu v} \tag{2.8}$$

We introduce the following two operational invariants integrated over the volume

$$J = \frac{1}{2} \iiint_{\tau} \sum^{\mu v} \sum^{ij} P^{ij}_{\mu v} e_{ij} e^{\mu v} d\tau \tag{2.9}$$

and

$$T = \frac{1}{2} p^2 \iiint \sum^{\mu} \rho u^2_{\mu} d\tau \tag{2.10}$$

The latter corresponds formally to the kinetic energy. With these invariants the variational equation is written

$$\delta (J + T) = \sum^{\mu} \iint_{s} F_{\mu} \delta u^{\mu} ds \tag{2.11}$$

This expresses that the variation of J + T is equal to the virtual work of the boundary forces.

The significance of this variational principle and its usefulness will be illustrated by its application below.

Finally, it should be noted that the variational principle (Eq 2.11) or the principle of correspondence formulated hereafter depend only on the symmetry property of the operational matrix, and not on any particular nature of the operator as a function of p.

The variational principle still applies if the operators are derived from a viscoelastic material where the kinetic energy associated with hidden coordinates is taken into account. The operators would then be similar to the impedance matrix of an electric network with inductance resistance and capacity. In this case the roots of the denominator of the operational tensor (Eq 2.5) could be complex conjugate.

3. Principle of Correspondence

We have seen that the stress-strain relations are identical to those in the classical theory of Elasticity, when the elastic moduli are replaced by the operators (Eq 2.3). With respect to a geometric symmetry as already indicated in reference [1] the same relation holds as in Elasticity. Invariance under geometric symmetry operations require that the number of independent operators be the same as the number of elastic constants in the theory of Elasticity. For instance, cubic symmetry requires three independent operators and isotropy two. Finally, we have just seen that a variational principle can be formulated which is the exact counterpart of the principle of virtual work and d'Alembert's principle in the theory of Elasticity. We may therefore formulate the general rule that a large class of equations of the theory of Elasticity whether derived by direct or variational methods may be extended to the most general type of viscoelastic material provided the elastic constants are replaced by the corresponding operators. We call this the principle of correspondence.

As an example let us consider waves in an isotropic viscoelastic material. The stress-strain relations in this case are written:

$$
\begin{aligned}
\sigma_{xx} &= 2Qe_{xx} + Re \\
\sigma_{yy} &= 2Qe_{yy} + Re \\
\sigma_{zz} &= 2Qe_{zz} + Re \\
\sigma_{xy} &= 2Qe_{xy} \\
\sigma_{yz} &= 2Qe_{yz} \\
\sigma_{zx} &= 2Qe_{zx}
\end{aligned}
\tag{3.1}
$$

$$e = e_{xx} + e_{yy} + e_{zz}$$

This introduces the two operators:

$$
\begin{aligned}
Q(p) &= p\int_0^\infty \frac{Q(r)\gamma(r)dr}{p+r} + Q + Q'p \\
R(p) &= p\int_0^\infty \frac{R(r)\gamma(r)dr}{p+r} + R + R'p
\end{aligned}
\tag{3.2}
$$

In the case of a discrete relaxation spectrum the integrals will be replaced by summations. A scalar ϕ and a vector ψ represent the displacement $\bar{u}$ of the continuum in the form

$$\bar{u} = \text{grad } \phi + \text{Curl } \bar{\psi} \tag{3.3}$$

The functions ϕ and $\bar{\psi}$ satisfy the operational equations

$$\begin{aligned} (R + 2Q)\, \nabla^2_{\phi} &= p^2 \rho \phi \\ Q\, \nabla^2_{\bar{\psi}} &= p^2 \rho \bar{\psi} \end{aligned} \tag{3.4}$$

where ρ is the mass density.

Harmonic dilatational waves of circular frequency ω propagating along x are represented by

$$\phi = \exp(i\omega t) \exp\left[\sqrt{\frac{\rho p^2}{R + 2Q}}\; x\right] \tag{3.5}$$

and the rotational wave by

$$\psi = \exp(i\omega t) \exp\left[\sqrt{\frac{\rho p^2}{Q}}\; x\right] \tag{3.6}$$

The operator p is replaced by $i\omega$ in these expressions.

For a material which is elastic under hydrostatic stress, there is only a single operator Q. In this case, as shown in reference [1] we may write R as

$$R = K - \frac{2}{3} Q \tag{3.7}$$

where K is the elastic bulk modulus.

Attention should be called to the proper application of the principle of correspondence to the case where the applied forces are not the product of a function of the coordinates by a function of time. In such a case we should first transform the applied load into a sum of forces applied at fixed points, each being multiplied by its own function of time. A particular case is that of a moving load. A possible representation of such a moving load could be a Fourier series in the coordinate variable with time dependent coefficients.

We have stated that the principle of correspondence applies to a large class of solutions in the theory of Elasticity. In general it applies to all static solutions and to the dynamic solutions in operational form.

4. Flexural Deformations of Viscoelastic Plates

We consider a plate of thickness h. The x,y plane coincides with the plane of symmetry of the plate and the z axis is directed along the thickness. A load f per unit area of the x,y plane and directed along z is applied. It is assumed that this load is distributed uniformly as a body force along the thickness. It is clear that flexural deformation will correspond to antisymmetry with respect to the x,y plane while extensional deformations will be represented by symmetry with respect to the same plane. For simplicity we shall assume that the thickness h is uniform, since the results can be readily extended to the case of non-uniform thickness. In that case the surfaces of the plates are represented by the planes $z = \pm h/z$.

It should be pointed out that the case of loads applied to the surface of the plate on one side can be treated exactly as hereafter. Such a load may be decomposed into an antisymmetry part of two equal loads acting in the same direction on top and bottom and a symmetric part when loads act in opposite directions. The two systems excite separately flexural and extensional deformations.

We shall first consider flexural deformations and use the variational approach. The method is quite general and applies to materials with complete anisotropy. However, in order to avoid undue heaviness we shall treat the isotropic case where the stress-strain law is given by the operational relations (Eqs 3.1, 3.2).

We now introduce some simplifying assumptions regarding the deformation of the plate. We denote the three displacement components by u, v, w, and express them as Taylor series expansion in z

$$
\begin{aligned}
u &= \sum_{o}^{n} u_n z^n \\
v &= \sum_{o}^{n} v_n z^n \\
w &= \sum_{o}^{n} w_n z^n
\end{aligned}
\tag{4.1}
$$

Where u_n v_n w_n are functions of x,y. Application of the variational principle leads to partial differential equations in x and y for these quantities, with coefficients which are functions of the time operator p . The advantage of this method is that we may introduce gradually terms of higher power z^n as we consider plates of increasing thickness. The problem of course is in practice to decide how many terms are required to achieve a desired amount of accuracy. In order to simplify further the formatism we consider a deformation which is two dimensional, i.e., parallel with the xz plane and independent of y. In this case v is zero and $u_n w_n$ are functions only of x. Since we consider flexural type deformations u is an odd function and w an even function of z. We shall use expansions to the third order in z and put

$$\begin{aligned} u &= u_1 z + u_3 z^3 \\ w &= w_0 + w_2 z^2 \end{aligned} \tag{4.2}$$

The strain components associated with these displacements are

$$\begin{aligned} e_{xx} &= z\frac{du_1}{dx} + z^3\frac{du_3}{dx} \\ e_{zz} &= 2w_2 z \\ 2e_{zx} &= \frac{dw_0}{dx} + u_1 + z^2\left(\frac{dw_2}{dx} + 3u_3\right) \end{aligned} \tag{4.3}$$

Considerable simplification results if we introduce additional constraints in the choice of the four unknown functions $u_1\ u_3\ w_0\ w_2$ in such a way that the shear stress σ_{zx} be made to vanish at the surface of the plate. This requires the condition

$$e_{zx} = 0 \qquad \text{for} \qquad z = \pm\frac{h}{2} \tag{4.4}$$

The function u_3 is then determined in terms of the three others

$$u_3 = -\frac{1}{3}\frac{dw_2}{dx} - \frac{4}{3h^2}\left(\frac{dw_0}{dx} + u_1\right) \tag{4.5}$$

The strain component e_{zx} becomes

$$2e_{zx} = \left(\frac{dw_0}{dx} + u_1\right)\left(1 - \frac{4z^2}{h^2}\right) \tag{4.6}$$

The operational invariant J may be written as the volume integral

$$J = \iiint_\tau I_1\, d\tau \tag{4.7}$$

with

$$I_1 = \frac{1}{2}\,\sigma_{xx}e_{xx} + \frac{1}{2}\,\sigma_{zz}e_{zz} + \sigma_{zx}e_{zx} \tag{4.8}$$

In terms of the strain components and the operators this is

$$I_1 = \frac{1}{2}(2Q + R)(e^2_{xx} + e^2_{zz}) + Re_{zz}e_{xx} + 2Qe^2_{zx} \tag{4.9}$$

Neglecting all powers of z higher than z^3 this may be written

$$I_1 = \frac{1}{2}(2Q + R)\left[\left(\frac{du_1}{dx}\right)^2 + 4w_2{}^2\right]z^2 + 2Rw_2\,\frac{du_1}{dx}\,z^2 + \frac{1}{2}\,Q\left(\frac{dw_0}{dx} + u_1\right)^2\left(1 - \frac{8z^2}{h^2}\right) \tag{4.10}$$

We also must form the invariant T corresponding to the dimetic energy

$$T = \frac{1}{2}\int_0^\ell dx \int_{-\frac{h}{z}}^{\frac{h}{z}} I_2\, dz \tag{4.11}$$

Neglecting again all powers of z higher than z^3 we find

$$I_2 = \frac{1}{2}\,p^2\rho\left[u_1{}^2z^2 + w_0{}^2 + 2w_0w_2z^2\right] \tag{4.12}$$

Finally we must evaluate the virtual work of the external forces. Assuming that at the end points the forces accomplish no virtual work we have

$$\delta W = f\int_0^\ell \delta w_0\, dx + \frac{fh^2}{12}\int_0^\ell \delta w_2\, dx \tag{4.13}$$

Applying now the variational principle (Eq 2.11) yields

$$\delta J + \delta T = f\int_0^\ell \delta w_0\, dx + \frac{fh^2}{12}\int_0^\ell \delta w_2\, dx \tag{4.14}$$

The left-hand side is evaluated by integrating first with respect to z. This leaves us with integrations with respect to x. The variational calculus method gives three differential equations in x obtained by equating the coefficients on both sides of the equation. We find

$$- (2Q + R)\frac{h^3}{12}\frac{du_1}{dx^2} - \frac{Rh^3}{6}\frac{dw_2}{dx} + \frac{Qh}{3}\left(\frac{dw_0}{dx} + u\right) + p^2\frac{h^3}{12}\rho u_1 = 0$$

$$4(2Q + R)w_2 + 2R\frac{du_1}{dx} + p^2\rho w_0 = \frac{f}{h} \tag{4.15}$$

$$-\frac{Qh}{3}\frac{d}{dx}\left(\frac{dw_0}{dx} + u_1\right) + p^2\rho h w_0 + p^2\rho\frac{h^3}{12} w_2 = f$$

At the ends $x = 0$ and $x = \ell$ we have the condition

$$(2Q + R)\frac{du_1}{dx} + 2Rw_2 = 0 \tag{4.16}$$

for a pinned end and the additional condition

$$\frac{dw_0}{dx} + u_1 = 0 \tag{4.17}$$

for a free end.

Elimination of u_1 and w_2 in the three equations (4.15) leads to

$$\frac{d^4 w_0}{dx^4} - \frac{p^2\rho}{Q}\left[3\beta + \frac{4Q + R}{4(Q + R)}\right]\frac{d^2w_0}{dx^2} = \frac{\beta\gamma}{B_1}(f - p^2\rho h w_0) - \frac{3\beta(}{Qh}\frac{d^2f}{dx^2} \tag{4.18}$$

with

$$\beta = 1 - \frac{p^2\rho h^2}{48(2Q + R)} \qquad B_1 = 4Q\frac{(Q + R)}{2Q + R}\frac{h^3}{12}$$

$$\gamma = 1 + \frac{p^2\rho h^2}{4Q}$$

In solving such an equation we must remember of course that the coefficients are time operators.

The plate equations in two dimensions with terms up to the third order in z^3 can be developed as above without difficulty. They lead to partial differential equations in the coordinates x,y. A less accurate equation neglecting shear and rotational inertia may be obtained from the principle of correspondence replacing the Lamé constants λ and μ by the operators R and Q is the classical theory of elastic plates. We find

$$B_1\nabla^4 w_0 + p^2\rho h w_0 = f \tag{4.19}$$

The effect of shear and rotational inertia may also be introduced by similarly applying the principle of correspondence to the equation derived by Mindlin [3] for the elastic case.

Equations for flexural deformations of beams and rods may also be obtained by the present variational method or by the principle of correspondence from elastic theories such as derived by E. G. Volterra.

5. Extensional Deformations of Plates and Rods

Extensional deformations are represented by displacements which are symmetric with respect to the plane $z = 0$. For simplicity in fact we limit the expansion of the displacement to the first order in z and write

$$\begin{aligned} u &= u_0 \\ w &= w_1 z \end{aligned} \tag{5.1}$$

The invariant I_1, is

$$I_1 = \frac{1}{2}(2Q + R)\left[\left(\frac{du_0}{dx}\right)^2 + w_1{}^2 \right] + R\frac{du_0}{dx} w_1 + \frac{1}{2}Qz^2\left(\frac{dw_1}{dx}\right)^2 \tag{5.2}$$

The invariant I_2 corresponding to the kinetic energy is

$$I_2 = \frac{1}{2} p^2 \rho \left[u_0{}^2 + w_1{}^2 z^2 \right] \tag{5.3}$$

Integrating I_1, and I_2 first with respect to z then with respect to x and taking the variations we find the two differential equations

$$\begin{aligned} &- (2Q + R)\frac{d^2u_0}{dx^2} - R\frac{dw_1}{dx} + p^2\rho u_0 = 0 \\ &(2Q + R) w_1 + R\frac{du_0}{dx} - \frac{Qh^2}{12}\frac{d^2w_1}{dx^2} + p^2\frac{h^2}{12}\rho w_1 = 0 \end{aligned} \tag{5.4}$$

More accurate equations would be obtained of course by considering a third order approximation of the type

$$\begin{aligned} u &= u_0 + u_2 z^2 \\ w &= w_1 z + w_3 z^3 \end{aligned} \tag{5.5}$$

and to carry out the derivation as for the flexural case. For instance it is possible to introduce the constraint that $\sigma_{zz} = 0$ at $z = \pm h/2$, in which case

$$w_3 = -\frac{R}{3(2Q + R)}\left(\frac{4u_0}{h^2} + u_2\right) - \frac{4}{3h^2} w_1 \tag{5.6}$$

and we are left only with three unknown functions u_0 u_2 w_1 leading to three differential equations for these quantities. Trigonometric expansion along z could also be used in conjunction or not with power series terms.

We may also use the principle of correspondence by taking advantage of known solutions for the elastic case. Extensional waves in an elastic circular rod were investigated by Mindlin and Herrmann [4]. Replacing the Lamé constants by the corresponding operators R and Q we find the two equations

$$\begin{aligned} &-a^2(2Q + R)\frac{d^2u_0}{dx^2} - 2aR\frac{dw_r}{dx} + pa^2\rho u_0 = 0 \\ &8k_1^2(Q + R)w_r + 4ak_1^2R\frac{du_1}{dx} - a^2k^2Q\frac{d^2w_r}{dx^2} + p^2a^2\rho w_r = 0 \end{aligned} \tag{5.7}$$

The longitudinal displacement is u_0 and the radial displacement at the boundary is w_r. The radius of the cross section is a. The coefficient k_1 denotes the ratio of the Rayleigh wave velocity to the shear wave velocity and

$$k_1^2 = 0.422(2 - k^2) \tag{5.8}$$

The analogy of the rod Eq (5.7) and the plate Eq (5.4) is obvious. The above equation contains correction coefficients k and k_1 not derivable from the theory. A more accurate theory for the rod could be obtained by the use of a third or higher order approximation of the type Eq (5.5).

6. Partial Modes

Consider the problem of forced flexural oscillations expressed by Eq (4.17). If the plate is pinned at $x = 0$, $x = \ell$, the end conditions are satisfied by solutions of the type.

$$w_n = \sin n\pi\frac{x}{\ell} \tag{6.1}$$

If there is no external force, $f = 0$, each of these solutions introduced in the equation yields an equation for p,

$$Z_n(p) = \frac{n^4r^4}{\ell^4} + \frac{p^2\rho}{Q}\left[3\beta + \frac{4Q + R}{4(Q + R)}\right]\frac{n^2r^2}{\ell^2} + p^2\frac{\beta\gamma\rho h}{B_1} = 0 \tag{6.2}$$

Each solution may be called a "partial mode." Each partial mode has its own spectrum of real or complex time constants p_{ns} such that the time history of the free mode is

$$w_n(t) = w_n \sum^{s} C_s \, e^{p_{ns}t} \tag{6.3}$$

It is represented by a superposition of damped oscillations or decaying motions all with the same sinusoidal spatial distribution. The constants C_s depend on the past time history of the partial mode at the time it becomes free.

The set of p_{ns} which are the roots of Eq (6.2) have real negative parts. They may be called the partial spectrum associated with the mode n. Because P, Q and B_1 are operators function of p Eq (6.2) may be of a very high or even infinite degree and lead to an approximately continuous partial spectrum. The response to a transient loading is easily found by the operational method. Consider a constant load with sinusoidal distribution and suddenly applied at $t = 0$. The external force is represented by

$$f = f_n \sin n \frac{\pi}{\ell} x \;\; 1\,(t) \tag{6.4}$$

where $1\,(t)$ is the unit step function

$$1\,(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \tag{6.5}$$

The corresponding operational solution for w_n is

$$w_n(t) = w_n f_n \left(\frac{\gamma}{B_1} + \frac{3n^2\pi^2}{Q\,h\,\ell^2} \right) \frac{\beta}{Z_n(p)} \, 1(t) \tag{6.6}$$

It is seen by expanding the operator in partial fraction or using the method of residues that the roots of $Z_n(p)$, i.e. the partial spectrum will appear as time constants in the exponentials representing the motion. If we denote by α_i the roots of the equations,

$$Q(p) + R(p) = 0 \qquad Q(p) = 0 \qquad Z_n(p) = 0, \tag{6.7}$$

the operator in Eq (6.6) may be expanded in partial fraction.

$$\left(\frac{\gamma}{B_1} + \frac{3n^2r^2}{Qh\ell^2} \right) \frac{\beta}{Z_n(p)} = \sum^{i} \frac{A_i}{p + \alpha_i} \tag{6.8}$$

The operational expression Eq (6.6) becomes

$$w_n(t) = w_n f_n \sum^{i} \frac{A_i}{\alpha_i} (1 - e^{-\alpha_i t}) \tag{6.9}$$

The case of an arbitrary load distribution is taken care of by Fourier expansion and any arbitrary time variation by applying Dubamel's integral.

The case of a moving load is handled in the same way by expressing the moving load as a Fourier Series with coefficients as functions of time. An alternate method is to represent it by a sequence of fixed loads with a Dirac type function of time as a factor for each load.

The concept of partial mode is quite general and has a deeper significance. Its introduction does not require that we first establish the differential equation. We could have started directly by considering amplitudes of mode shapes as generalized coordinates and applying the variational method with the operational invariant expressed in these generalized coordinates. The existence of partial modes is a consequence of the property that in this case the invariant separates into a sum of partial invariants each containing only the coordinate of the associated mode. The variables have then been separated from the start.

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