NOT FOR SALE

REPRINT FROM

DEFORMATION AND FLOW OF SOLIDS

COLLOQUIUM MADRID SEPTEMBER 26–30, 1955

EDITOR

R. GRAMMEL

SPRINGER-VERLAG / BERLIN · GÖTTINGEN · HEIDELBERG 1956
(PRINTED IN GERMANY)

Variational and Lagrangian Methods in Viscoelasticity

By **M. A. Biot**, New York (N.Y.)

1. Introduction. The time history of a thermodynamic system perturbed from equilibrium under the assumption of linearity obeys certain differential equations. Starting from ONSAGER's reciprocity relations we have shown [*1*] how they may be derived from generalized concepts of free energy and dissipation function. This provides a most fruitful link between physical chemistry, thermodynamics, and mechanics, and leads to a very general formulation of relations between stress and strain in linear viscoelasticity in operational form. The outline of this development is given in section 2. The matrix relating stress and strain is formally identical with the matrix of twenty-one distinct coefficients in the theory of Elasticity. In linear viscoelasticity the elements of this matrix are functions of the differential time operator. The form of this operator is also derived from the theory. Variational formulation of deformation and stress field problems are outlined in section 3 and lead to generalizations of LAGRANGE's equations with operational coefficients. We also introduce a general principle expressing the formal correspondence between problems in viscoelasticity and Elasticity. By the latter it is possible to carry over almost all solutions of the theory of Elasticity into that of a corresponding problem of viscoelasticity. Thus we uncover in one stroke a vast area of solved problems for viscoelastic media. This approach provides a compact and synthetic formulation of linear viscoelasticity. As an example we derive in section 4 some general properties of a medium with a uniform relaxation spectrum. An outline is also given of a new approach to the dynamics of plates or shells for isotropic or anisotropic media. It includes the classical theories for elastic materials as first order approximation. The method is also, of course, applicable to improving the theory of plates and shells in the purely elastic case when the effect of increasing thickness is taken into account.

The section 5 deals with large deformations and it is shown that the same methods are also applicable to this case. The approach to the nonlinear problem is different from the traditional one followed by the mathematician in the elastic case. Thus it is possible to separate the nonlinear effects of purely geometric origin from those arising from the physical relations between stress and strain. This leads to a treatment of plates and shells with large deformations which parallels the one outlined above for the linear case.

2. Viscoelastic stress-strain relations derived from thermodynamics. A thermodynamic derivation of the stress-strain relations in linear viscoelasticity for the most general case of anisotropy has been established by the writer. It is based on Onsager's reciprocity relations. We have shown [*1*, *2*] that a thermodynamic system in the vicinity of equilibrium is in its linear ranges of behaviour entirely defined by two quadratic invariants. A generalized free energy

$$V = \frac{1}{2} \sum^{ij} a_{ij} q_i q_j \tag{2.1}$$

and a generalized dissipation function

$$D = \frac{1}{2} \sum^{ij} b_{ij} \dot{q}_i \dot{q}_j . \tag{2.2}$$

Both are positive definite forms and the q's are incremental state variables defining the deviations of the thermodynamic system from equilibrium. The function is a generalization of Helmholtz's free energy concept to include the case of nonuniform temperature. It is defined in references [*1*] and [*2*] as

$$V = T S, \tag{2.3}$$

where S is the entropy of a total isolated system, by the adjunction of a large heat reservoir at the equilibrium temperature T. The dissipation function is defined as

$$D = \frac{1}{2} T \dot{S}, \tag{2.4}$$

where $\dot{S}$ is the rate of entropy production, in the total isolated system — expressed in terms of rate variables $\dot{q}$.

When a system is under the action of perturbing generalized forces Q similar concepts may be introduced by adding large energy reservoirs to the isolated systems, the total energy of the perturbing reservoirs being $\sum Q q$.

The total entropy S' of this new system is then given by

$$T S' = V - \sum Q q . \tag{2.5}$$

This constitutes a generalization of the GIBBS free energy concept to nonuniform temperature. Application of the ONSAGER reciprocity relation to the total system with the "forces" $\partial S'/\partial q$ and conjugate "fluxes" $\dot{q}$ leads to the differential equation

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i. \tag{2.6}$$

These are the equations of a spring dashpot system or in the electric analogy a resistance capacity network. An interesting case is that of a system with a great number of unobserved coordinates with conjuguate forces equal to zero. This is the analogue of a large R. C. network with a small number of outlets or pair terminals. The voltages of the terminals and the total quantities of electricity flowing at these terminals are related by an impedance matrix. Consider now an elementary cube of viscoelastic material oriented along the coordinate axes. If we attribute its visco-elastic properties to a large number of unobserved internal state variables associated with chemical, electrical, thermal effects, etc., we may assimilate this element to an impedance for which the observed input forces are the nine stress components σ_{ij} and the associated coordinates the nine strain components, e_{ij}. They are, therefore, related by

$$\sigma_{\mu\nu} = \sum^{ij} P^{ij}_{\mu\nu}\, e_{ij}\,, \tag{2.7}$$

where $P^{ij}_{\mu\nu}$ is an operator analogous to an impedance matrix. The strain sensor is defined in terms of the displacement vector u_i as

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.8}$$

We have shown that if we consider the system represented by equations (2.6) with a generalized free energy and dissipation function and assume a large number of coordinates the operator is

$$P^{ij}_{\mu\nu} = \int_0^\infty \frac{p}{p+r}\, D^{ij}_{\mu\nu}(r)\,\gamma(r)\,dr + D^{ij}_{\mu\nu} + p\,D'^{\,ij}_{\mu\nu} \tag{2.9}$$

with

$$p = \frac{d}{dt}\,.$$

The operator has the following symmetry properties:

$$P^{\mu\nu}_{ij} = P^{\mu\nu}_{ji} = P^{\nu\mu}_{ij} = P^{ij}_{\mu\nu}\,. \tag{2.10}$$

These are the same as in the case of the elastic moduli of the theory of elasticity. There are twenty-one distinct operators which constitute the formal analogue of these moduli. We have also demonstrated [1] that expression (2.9) is a general formula valid whether the coefficient matrices of V or D are singular or not. The variable r represents a distribution of internal relaxation constants with a density or relaxation spectrum $\gamma(r)$. The integral expression may be considered as containing formally the case of a discreet spectrum when it is replaced by a finite or infinite summation. The reader will recall the significance of these operators. Take for instance a relation such as

$$\sigma = \left(\frac{p}{p+r} + \alpha + \beta p\right) \varepsilon , \tag{2.11}$$

where ε is a strain and σ a stress. If the deformation ε jumps from zero to unity at $t = 0$ and remains constant thereafter, it is represented by

$$\varepsilon = 1(t) . \tag{2.12}$$

The first term in (2.11) is

$$\sigma = \frac{p}{p+r} 1(t) = e^{-rt} 1(t) . \tag{2.13}$$

This represents an exponential relaxation of stress. The term α represents an elasticity while the last one represents NEWTONian viscosity

$$\sigma = \beta p \varepsilon = \beta \dot{\varepsilon} . \tag{2.14}$$

If ε varies in an arbitrary way the first term is the integral transform

$$\sigma = \frac{p}{p+r} \varepsilon(t) = e^{-rt} \int_0^t e^{r\tau} d\varepsilon(\tau) . \tag{2.15}$$

Attention is also called to the general significance of the term viscoelasticity in the present theory. It is to be taken in a very general sense and includes for instance the thermoelastic effect in which the energy dissipation of a perfectly elastic body arises through temperature variations associated with the volume changes and the resulting exchange of heat through conduction. The theory of thermoelastic dissipation has already been developed by ZENER [3] but from a less general viewpoint.

A thermodynamic approach to relaxation phenomena was also given by STAVERMAN [4] and MEIXNER [5].

3. Variational principles and Lagrangian methods. It is possible to formulate the fundamental laws of dissipative phenomena by means of variational principles. One such principle refers to minimum rate of entropy production for the stationary or nonstationary state. We have established [2] that the rate of dissipation is a minimum for a given power input of the disequilibrium forces. Although this statement was proved only for linear thermodynamics, there are indications that it is a particular case of a much more general principle. In the case of viscoelasticity where the particular stress-strain relations has been expressed operationally we have also shown [2] that it is possible to formulate a different variational principle which establishes a powerful tool for the calculations of stress fields or deformations. We define an operational invariant

$$I = \frac{1}{2} \sum^{\mu\nu} \sum^{ij} P^{ij}_{\mu\nu} e_{\mu\nu} e_{ij}, \tag{3.1}$$

which is the formal equivalent of the elastic potential energy, per unit volume and we introduce the volume integral

$$J = \iiint_\tau I \, d\tau. \tag{3,2}$$

The variational principle is then stated as

$$\delta J = \iiint_\tau \bar{G} \, \delta \bar{u} \, d\tau + \iint_S \bar{F} \, \delta \bar{u} \, dS. \tag{3.3}$$

This is an identity valid for all variations of the displacement field $\bar{u}$. The integrals on the right-hand side are respectively the virtual work of the body force $\bar{G}$ and the surface boundary force $\bar{F}$. Proof of the variational equations may be established by evaluating the variation of J integrating by parts and showing that this leads to the equations of equilibrium for the stress field. It may also be derived as in reference [2] as a particular case of a variational principle of interconnected thermodynamic systems. The systems in this case are the elements of the continuum considered as infinitesimal cells.

The variational equation (3.3) may be extended readily to include dynamics by using D'ALEMBERT's principle. The inertia force is then included in the body force. The amounts to replacing $\bar{G}$ by $\bar{G} - \varrho \frac{\partial^2 \bar{u}}{\partial t^2}$, ϱ being the mass density. The variational equation becomes

$$\delta J + \iiint_\tau \varrho \frac{\partial^2 u}{\partial t^2} \delta \bar{u} \, d\tau = \iint_\tau \bar{G} \, \delta \bar{u} \, d\tau + \iint_S \bar{F} \, \delta \bar{u} \, dS. \tag{3.4}$$

With the operator $p = \frac{d}{dt}$ we introduce a kinetic energy invariant

$$T = \frac{1}{2} p^2 \iiint_\tau \varrho \, u^2 \, d\tau. \tag{3.5}$$

The variational principle is then written

$$\delta J + \delta T = \iiint_\tau \bar{G} \, \delta \bar{u} \, d\tau + \iint_S F \, \delta \bar{u} \, dS. \tag{3.6}$$

An interesting application is obtained by the use of generalized coordinates. If the field is expressed in terms of n discreet coordinates q_i we write

$$\bar{u} = \sum^{i} \bar{u}_i q_i, \tag{3.7}$$

where $\bar{u}_i$ are field distributions represented by fixed function of x, y, z then

$$\left.\begin{aligned} \delta J &= \sum^{i} \frac{\partial J}{\partial q_i}\,\delta q_i, \\ \delta T &= \sum^{i} \frac{\partial T}{\partial q_i}\,\delta q_i. \end{aligned}\right\} \tag{3.8}$$

Defining a generalized force by

$$Q_i = \iiint_{\tau} \bar{G}\,\bar{u}_i\,d\tau + \iint_{S} \bar{F}\,\bar{u}_i\,dS \tag{3.9}$$

the variational equation (3.6) leads to the n equations for q_i

$$\frac{\partial J}{\partial q_i} + \frac{\partial T}{\partial q_i} = Q_i. \tag{3.10}$$

Note that this is an operational expression since J and T contain the operator p. Hence these equations are integro-differential equations in Lagrangeian form.

If we define the kinetic energy in the usual way, i.e.

$$T = \frac{1}{2}\iiint_{\tau} \varrho\,\dot{u}^2\,d\tau \tag{3.11}$$

we may replace $\frac{\partial T}{\partial q_i}$ above by $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)$ and the equations assume the more familiar form

$$\frac{\partial J}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = Q_i. \tag{3.12}$$

If the operators in J correspond to pure elasticity and Newtonian viscosity then

$$P^{ij}_{\mu\nu} = D^{ij}_{\mu\nu} + p\,D'^{\,ij}_{\mu\nu} \tag{3.13}$$

hence

$$J = \frac{1}{2}\sum^{ij} a_{ij}\,q_i\,q_j + \frac{1}{2}\,p\sum^{ij} b_{ij}\,q_i\,q_j \tag{3.14}$$

putting

$$\left.\begin{aligned} V &= \frac{1}{2}\sum^{ij} a_{ij}\,q_i\,q_j, \\ D &= \frac{1}{2}\sum^{ij} b_{ij}\,\dot{q}_i\,\dot{q}_j \end{aligned}\right\} \tag{3.15}$$

equations (3.12) become

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = Q_i. \tag{3.16}$$

This is the usual form of Lagrange's equations for an elastic system with Newtonian damping. An important principle may also be formulated relating to the formal correspondance between a large class of equations of the theory of elasticity and viscoelasticity. Because of the identical properties of the operators $P^{ij}_{\mu\nu}$ and the elastic moduli we may generally extend the formulas of the classical theory of elasticity to viscoelasticity provided the elastic moduli are replaced by the corresponding operators. This leads immediately to a large class of solutions of problems.

We shall refer to this as the *correspondence rule.*

In particular the various cases of geometric symmetry are the same. For instance a system with cubic symmetry is characterized by three operators and an isotropic medium by two.

The variational formulation above indicates that the correspondence rule also applies to the approximate solutions of Elasticity derived by energy methods.

It should be noticed that the above principles do not depend on the particular form (2.9) of the operators but only on their symmetry property (2.10). Hence they are valid also in the case when there are internal dynamic degrees of freedom, i.e., microscopic kinetic energy. The latter appears in the particular operational form of J.

4. Some specific applications. We shall now consider some specific applications of the principles formulated above. We shall first derive a general theorem for a viscoelastic medium with a homegeneous relaxation spectrum. This is a case for which the operators are of the form

$$P^{ij}_{\mu\nu} = C^{ij}_{\mu\nu}\, P\,, \tag{4.1}$$

where $C^{ij}_{\mu\nu}$ are constants and P is an invariant operator

$$P = \int_0^\infty \frac{p}{p+r}\,\gamma(r)\,d\,r + \alpha + p\beta\,. \tag{4.2}$$

In this case

$$J = P\,J' \tag{4.3}$$

with

$$J' = \frac{1}{2}\iiint_\tau \sum^{\mu\nu}\sum^{ij} C^{ij}_{\mu\nu}\, e_{\mu\nu}\, e_{ij}\, d\,\tau\,. \tag{4.4}$$

This invariant is the potential energy of an elastic medium of moduli $C^{ij}_{\mu\nu}$. We consider the deformation for the case of negligible inertia effect. The variational equation (3.3) may be written after multyplying by the inverse operator P^{-1}

$$\delta J = \iiint_\tau \bar{G}'\,\delta\bar{u}\,d\,\tau + \iint_S \bar{F}'\,\delta\bar{u}\,d\,S\,. \tag{4.5}$$

The forces $\bar{G}'$ and $\bar{F}'$ are body forces and boundary forces obtained by applying to their actual values the operator P^{-1}

$$\left.\begin{aligned} \bar{G}' &= P^{-1}\,\bar{G}\,,\\ \bar{F}' &= P^{-1}\,\bar{F}\,. \end{aligned}\right\} \tag{4.6}$$

We conclude from (4.5) that the *deformations are the same as for an elastic medium under the action of the transformed forces $\bar{G}'$, $\bar{F}'$*. The stresses are given by

$$\sigma_{\mu\nu} = P\sum^{ij} C^{ij}_{\mu\nu} e_{ij}, \tag{4.7}$$

where e_{ij} are the elastic strains due to the transformed forces $\bar{G}'$, $\bar{F}'$. Because of linearity they may also be written

$$e_{ij} = P^{-1} e'_{ij} \tag{4.8}$$

where e'_{ij} is the elastic strain due to the original forces $\bar{G}$, $\bar{F}$. Hence

$$\sigma_{\mu\nu} = \sum^{ij} C^{ij}_{\mu\nu} e'_{ij}. \tag{4.9}$$

The stress is therefore the same as for the elastic case under the original forces.

This applies in particular to an incompressible isotropic medium by considering the invariant made up of the product of the stress deviator by the strain. A single operator is then factorized and the above reasoning may be repeated leading to a known theorem by ALFREY [*6*].

The theory of deformation of viscoelastic media furnish a fertile field for the application of LAGRANGEian and variational methods. The use of generalized coordinates constitutes a powerful method of approach to the dynamics of plates and shells. It is also of great flexibility and permits the gradual introduction of thickness corrections with a degree of accuracy adjusted to the practical requirements.

This general method was introduced by the writer in references [*2*, *7*]. It applies also of course to purely elastic plates and shells since this a particular case of viscoelasticity.

The method is best illustrated by an example. It is valid for the most general case of anisotropy of the material and applied without difficulty. However, for simplicity we shall assume an isotropic material. In this case we have seen [*1*] that the stress-strain law is

with

$$\left.\begin{aligned} \sigma_{\mu\nu} &= 2Q\, e_{\mu\nu} + \delta_{\mu\nu} R e, \\ e &= e_{xx} + e_{yy} + e_{zz}, \\ \delta_{\mu\nu} &= \begin{cases} 1 & \mu = \nu, \\ 0 & \mu \neq \nu. \end{cases} \end{aligned}\right\} \tag{4.10}$$

The operators are the two invariations

$$\left.\begin{aligned} Q &= \int_0^\infty \frac{p}{p+r} Q(r)\gamma(r)\,dr + Q + p\,Q', \\ R &= \int_0^\infty \frac{p}{p+r} R(r)\gamma(r)\,dr + R + p\,R'. \end{aligned}\right\} \tag{4.11}$$

We consider a plate of uniform thickness h, the x, y plane coinciding with the plane of symmetry, and the z axis being directed across the thickness. The boundaries of the plate are at $z = \pm h/2$. We propose to find the deformation of the plate by expaunding the displacement field u, v, w, into a TAYLOR series in z. We put

$$\left.\begin{aligned} u &= \sum^{n} u_n z^n, \\ v &= \sum^{n} v_n z^n, \\ w &= \sum^{n} w_n z^n. \end{aligned}\right\} \tag{4,12}$$

The coefficients $u_n(x, y)$, $v_n(x, y)$, $w_n(x, y)$ are unknown functions of x, y and implicitly also of the time operator p. In order to find the equations for these unknown functions we apply the variational equation (3.6). In this case

$$\left.\begin{aligned} I &= \frac{1}{2}(2Q + R)(e_{xx}^2 + e_{zz}^2) + R e_{zz} e_{xx} + 2Q e_{zx}^2 . \\ J &= \iiint_{\tau} I \, d\tau, \\ T &= \frac{1}{2} p^2 \varrho \iiint_{\tau} (u^2 + v^2 + w^2) \, d\tau . \end{aligned}\right\} \tag{4.13}$$

Since we aim principally to illustrate the method let us further simplify the problem by assuming a cylindrical deformation parallel with the x, z plane. Hence $v = 0$ and $u_n v_n$ are functions only of x. Finally we expand u, w, to the third order in the thickness and put

$$\left.\begin{aligned} u &= u_1 z + u_3 z^3, \\ w &= w_0 + w_2 z^2 . \end{aligned}\right\} \tag{4.14}$$

If we introduce the condition that the shear stress is zero at $z = \pm h/2$ we find

$$u_3 = -\frac{1}{3}\frac{d w_2}{d x} - \frac{4}{3h^2}\left(\frac{d w_0}{d x} + u_1\right). \tag{4.15}$$

This leaves only three unknown functions u, w_0, w_2 of x. The variational principle is applied by first integrating along z

$$\left.\begin{aligned} \delta \int_0^l d x \int_{-h/2}^{+h/2} I \, d z + \frac{1}{2} p^2 \varrho \, \delta \int_0^l d x \int_{-h/2}^{+h/2} (u^2 + w^2) \, d z \\ = f \int_0^l \delta w_0 \, d x + \frac{f h^2}{12} \int_0^l \delta w_2 \, d x . \end{aligned}\right\} \tag{4.16}$$

In these expressions quantities of order higher than z^3 are neglected. The right-hand side represents the virtual work of applied load f uniformly distributed along the thickness. The integrals along z are readily evaluated. We are then left with single integrals with respect to x and a variational problem yielding three EULER differential equations obtained by cancelling the variation due to δu_1, δw_0, δw_2. These three equations are

$$\left.\begin{aligned} &-(2Q+R)\frac{h^3}{12}\frac{d^2 u_1}{dx^2} - \frac{Rh^3}{6}\frac{dw_2}{dx} + \frac{Qh}{3}\left(\frac{dw_0}{dx} + u_1\right) + p^2\frac{h^3}{12}\varrho u_1 = 0,\\ &4(2Q+R)w_2 + 2R\frac{du_1}{dx} + p^2\varrho w_2 = \frac{f}{h},\\ &-\frac{Qh}{3}\frac{d}{dx}\left(\frac{dw_0}{dx} + u_1\right) + p^2\varrho h w_0 + p^2\varrho\frac{h^3}{12}w_0 = f. \end{aligned}\right\} \quad (4.17)$$

Elimating u_1 w_2 we find

$$\left.\begin{aligned} &\frac{d^4 w_0}{dx^4} - \frac{p^2\varrho}{Q}\left[3\beta + \frac{4Q+R}{4(Q+R)}\right]\frac{d^2 w_0}{dx^2} = \frac{\beta\gamma}{B_1}(f - p^2\varrho h w_0) - \frac{3\beta}{Qh}\frac{d^2 f}{dx^2}\\ &\text{with}\\ &\beta = 1 - \frac{p^2\varrho h^2}{48(2Q+R)}, \quad \gamma = 1 + \frac{p^2\varrho h^2}{4Q}, \quad B_1 = \frac{4Q(Q+R)}{2Q+R}\frac{h^3}{12}. \end{aligned}\right\} \quad (4.18)$$

In conformity with the correspondence rule, for the purely elastic case, Q and R become LAMÉ constants. Putting $p = 0$ i.e., for the static case at zero frequency we obtain the classical equation of the elastic beam with the addition of a shear deflection term.

A more direct method of deriving some simplified plate equations is to apply the correspondence rule to the classical equation of flexural deformation of plates which may be written

$$\nabla^4 w_0 = \frac{12}{h^3}\frac{2\mu+\lambda}{4\mu(\lambda+\mu)}f, \quad (4.19)$$

where λ and μ are LAMÉ constants. Replacing these constants by the corresponding operators R and Q we obtain the equations for the viscoelastic plate

$$\nabla^4 w_0 = \frac{12}{h^3}\frac{2Q+R}{4Q(R+Q)}f. \quad (4.20)$$

We see that the deflection is proportional to that of an elastic plate under a load derived from a transformation of the actual load f to which the time operator on the right-hand side has been applied. For instance, if the operator is expanded in partial fractions

$$\frac{2Q+R}{4Q(R+Q)} = \sum^{n}\frac{A_n}{p+\alpha_n} + A_0, \quad (4.21)$$

the transformed load is

$$\frac{2Q+R}{4Q(R+Q)}f = \sum^{n} A_n e^{-\alpha_n t}\int_0^t e^{\alpha_n \tau} f(\tau)\,d\tau + A_0 f(t). \quad (4.22)$$

Because of some general theorems derived in reference [1] the roots α are real. Equation (4.20) is for the nondynamical case. An inertia term $p^2 \varrho h w_0$ could be added on the left-hand side.

5. Nonlinear problems associated with large deflections. In dealing with nonlinear problems of deformation of solids it is important to distinguish between the nonlinearity arising from geometric properties of the deformation field and that due to the nonlinearity of the stress-strain relations. The former is essentially a mathematical problem while the latter is closely related to the physical nature of the material. An approach to a nonlinear theory of Elasticity which emphasizes this separation was developed by the writer in a series of publications some years ago [8, 9, 10, 11]. This constitutes a departure from the traditional approach to finite strain by the mathematician.

It was found that the equilibrium equations for the stress field are

$$\frac{\partial}{\partial x^\nu}[(1+e)\sigma_{\nu i}] + \frac{\partial}{\partial x^\nu}[\sigma_{i\nu}\omega_{i\mu} - \sigma_{i\mu}e_{\mu\nu}] + X_i\varrho = 0\,. \tag{5.1}$$

In these equations the stress components σ_{ij} are referred to axes which rotate locally with the material. The rotation tensor is

$$\omega_{i\mu} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^\mu} - \frac{\partial u_\mu}{\partial x^i}\right). \tag{5.2}$$

The stress is a function of the strain components referred to the same rotated axes namely

$$\varepsilon_{\mu\nu} = e_{\mu\nu} + \frac{1}{2}(\omega_{i\mu}e_{i\nu} + \omega_{i\nu}e_{i\mu}) + \frac{1}{2}\omega_{i\mu}\omega_{i\nu}\,. \tag{5.3}$$

The body force is X_i per unit mass and summation signs are omitted. The gradients $\frac{\partial \sigma_{\nu i}}{\partial x^\nu}$ in equations (5.1) contain the second order terms of physical origin due to the nonlinearity of the stress-strain relations. The others contain the second order terms of geometric origin which arise from the product of the stress by the strain and rotation components. When introducing the stress-strain relations in the latter terms, it is only necessary to use the first order effects.

The above equations were derived for an elastic body but are applicable to solids in general. For instance, we may consider a viscoelastic solid and assume that the linear stress-strain relations in operational form

$$\sigma_{\mu\nu} = P^{ij}_{\mu\nu}e_{ij} \tag{5.4}$$

are valid throughout the range of strain involved. In that case the three equations for the displacement field are obtained by replacing in the three equilibrium equations (5.1) the value of $\sigma_{\mu\nu}$ by its operational expression (5.4). Since we are now dealing with nonlinear equations care must be exercised to locate the operators in the proper place so that they operate only on the quantities to which they were originally attached. Since the equations contain time operators we thus obtain three nonlinear integro-differential equations for the displacement field $\bar{u}$ of the solid.

The present approach also leads immediately to the theory of incremental stress and deformation for a body under initial stress. By linearizing the equilibrium equations (5.1) in the vicinity of an initial state of stress $S_{\mu\nu}$ we obtain linear equilibrium equations for the incremental stresses. These equations are

$$\frac{\partial}{\partial x^\nu} \sigma_{\nu i} + \frac{\partial}{\partial x^\nu} (S_{\nu i} e) + \frac{\partial}{\partial x^\nu} (S_{i\nu} \omega_{i\mu} - S_{i\mu} e_{\mu\nu}) + \varrho \Delta X_i = 0 . \quad (5.5)$$

The incremental body force is ΔX_i. These equations, if we formulate a relation between incremental stress and deformation, lead to the solution of stability problems of elastic or plastic prestressed fields. In particular, it leads to solutions of incremental stability problems of viscoelastic prestressed fields if we assume incremental stress-strain relations of the type (5.4). The nature of the incremental stress-strain relation appropriate for various materials is essentially a physical problem which still remains to be investigated.

Of special interest here is the possible introduction of variational and Lagrangeian methods in the formulation of problems of large deformation of elastic and anelastic solids. To this effect we follow a procedure which we have introduced in the elastic theory. We define a variational invariant

$$\delta J = \iiint_\tau \tau_{\mu\nu} \delta \varepsilon_{\mu\nu} d\tau , \quad (5.6)$$

where

$$\tau_{\mu\nu} = (1 + \varepsilon) \sigma_{\mu\nu} - \frac{1}{2} (\sigma_{\alpha\mu} \varepsilon_{\alpha\nu} + \sigma_{\alpha\nu} \varepsilon_{\alpha u}) \quad (5.7)$$

and $\varepsilon_{\mu\nu}$ is given by (5.3). In these expressions the $\sigma_{\mu\nu}$ components may be expressed in terms of $e_{\mu\nu}$ by means of operators. In the case of a viscoelastic material it is, therefore, an operational invariant of the same type as (3.2). The variational identity in the present case is

$$\delta J = \iiint_\tau \bar{G} \delta \bar{u} d\tau + \iint_S \bar{F} \delta \bar{u} dS , \quad (5.8)$$

which must be valid for all variations of the displacement field.

This variational principle is derived from [*9*] and [*10*] where we have shown that it is equivalent to the equilibrium equations (5.1) of the stress field. Making use are before of d'Alembert's principle we may introduce the inertia effect and write

$$\delta J + \delta T = \iiint_\tau \bar{G} \delta \bar{u} d\tau + \iint_S \bar{F} \delta \bar{u} dS . \quad (5.9)$$

This equation opens the way to a systematic treatment of nonlinear problems by methods of generalized coordinates entirely analogous to the linear case. Since the equations are now nonlinear we must take care that the operators remain attached to those quantities upon which they initially operate. We could treat problems of vibrations of plates and shells by expanding the displacement field in a power series of the transverse thickness coordinate, and obtain simplified equations with any order of approximation desired as exemplified above. We could for instance generalize the Kármán-Foeppl equations for the finite deflection of elastic plates to include higher order effects of the thickness and auy viscoelastic stress-strain law whether isotropic or anisotropic. Finally, it should be pointed out that a modified correspondence rule may be applied in the nonlinear case. Consider for instance the Kármán-Foeppl equations of finite deformation of an elastic plate. If we follow the derivation of these equations and replace the Lamé constants λ and μ by the operators R and Q we obtain

$$\left.\begin{aligned} \frac{4Q(R+Q)}{2Q+R}\,\frac{h^3}{12}\,\nabla^4 w_0 &= f + \frac{\partial^2 F}{\partial y^2}\,\frac{\partial^2 w_0}{\partial x^2} - 2\,\frac{\partial^2 F}{\partial x\,\partial q}\,\frac{\partial^2 w_0}{\partial x\,\partial q} + \frac{\partial^2 F}{\partial x^2}\,\frac{\partial^2 w_0}{\partial y^2}\,, \\ \frac{Q+R}{Q(2Q+3R)\,h}\,\nabla^4 F &= \left(\frac{\partial^2 w_0}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w_0}{\partial x^2}\,\frac{\partial^2 w_0}{\partial y^2}\,. \end{aligned}\right\} \quad (5.10)$$

These are two nonlinear integro-differential equations for the finite deflection of an isotropic viscoelastic plate.

References

[1] Biot, M. A.: J. appl. Phys. **25**, 1385 (1954).
[2] Biot, M. A.: Phys. Rev. **97**, 1463 (1955).
[3] Zener, C.: Phys. Rev. **53**, 90 (1938).
[4] Staverman, A. J.: Proceedings of the Second International Congress on Rheology 1954, 134.
[5] Meixner, J.: Z. Naturforsch. 9a, 654 (1954).
[6] Alfrey, T.: Mechanical Behaviour of High Polymers. New York 1948.
[7] Biot, M. A.: Phys. Rev. **98**, 1869 (1955).
[8] Biot, M. A.: Proceedings Fifth International Congress of Applied Mechanics. Cambridge, U.S.A. 1938, 117.
[9] Biot, M. A.: Phil. Mag. **27**, 468 (1939).
[10] Biot, M. A.: Z. angew. Math. Mech. **20**, 89 (1940).
[11] Biot, M. A.: J. appl. Phys. **11**, 522 (1940).