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The Elastic Coefficients of the Theory of Consolidation

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The theory of the deformation of a porous elastic solid containing a compressible fluid has been established by Biot. In this paper, methods of measurement are described for the determination of the elastic coefficients of the theory. The physical interpretation of the coefficients in various alternate forms is also discussed. Any combination of measurements which is sufficient to fix the properties of the system may be used to determine the coefficients. For an isotropic system, in which there are four coefficients, the four measurements of shear modulus, jacketed and unjacketed compressibility, and coefficient of fluid content, together with a measurement of porosity appear to be the most convenient. The porosity is not required if the variables and coefficients are expressed in the proper way. The coefficient of fluid content is a measure of the volume of fluid entering the pores of a solid sample during an unjacketed compressibility test. The stress-strain relations may be expressed in terms of the stresses and strains produced during the various measurements, to give four expressions relating the measured coefficients to the original coefficients of the consolidation theory. The same method is easily extended to cases of anisotropy. The theory is directly applicable to linear systems but also may be applied to incremental variations in nonlinear systems provided the stresses are defined properly.

1 Introduction

The theory of the deformation of a porous elastic solid containing a viscous compressible fluid was established by Biot in several earlier papers (1, 2, 3, 4).[3] In reference (1) the isotropic case is considered, and in reference (2) the theory is generalized to anisotropic materials. General solutions to the elastic equations are established in reference (3). The theory of wave propagation in such systems is examined in reference (4).

Gassman (5, 6) has examined the properties of "open" and "closed" porous elastic systems corresponding to the case of constant pore pressure ($\sigma = 0$) and constant fluid content ($\zeta = 0$), respectively. The elastic properties of a porous solid were considered by Geertsma (7) who described methods for their measurement with jacketed and unjacketed compressibility tests, in the case of a homogeneous and isotropic porous matrix. Hughes and Cooke (8) described measurements of pore volume and porosity in jacketed compression tests, which would apply to a system containing an incompressible fluid completely filling the pores.

In the present paper methods of measurement are described for the determination of the elastic coefficients of references (1, 2, 3, 4). The physical interpretation of the elastic coefficients in various alternate forms is also discussed.

In Section 2 it is shown that the elastic coefficients A, N, Q, and R may be expressed in terms of four directly measurable coefficients μ, κ, δ, γ, and the porosity factor f. It is pointed out that under certain assumptions the coefficient γ does not need to be measured if we know the fluid compressibility c and already have measured δ and f. Other elastic coefficients μ, λ, α, and M are discussed in Section 3. They may be expressed in terms of the measurable coefficients μ, κ, δ, and γ without reference to porosity. Use of the constants μ, λ, α, and M are more convenient in application to consolidation problems, i.e., when the inertia forces are neglected, while A, N, Q, and R have been introduced in connection with wave propagation (4).

In addition to these four measurements it is possible to measure directly the constant α by a fifth and entirely independent measurement. This redundant information results from the fact that it is possible to define five physically different coefficients, two of which must be equal as a result of the assumption that we are dealing with a conservative elastic system. The redundancy thus provides a check on the validity of this assumption.

In Section 4 the case of transverse isotropy is considered using a different method and different measurements, and the case of general anisotropy is discussed in Section 5.

The measurements described are directly applicable to linear systems. They also will apply to incremental variations in nonlinear systems such as those having unconsolidated porous materials, provided the stresses are defined properly. This will be discussed in Section 6.

2 The Isotropic Case

The stress-strain relations for the isotropic case (2) are

$$\left.\begin{aligned} \sigma_{xx} &= 2Ne_{xx} + Ae + Q\epsilon \\ \sigma_{yy} &= 2Ne_{yy} + Ae + Q\epsilon \\ \sigma_{zz} &= 2Ne_{zz} + Ae + Q\epsilon \\ \sigma_{yz} &= Ne_{yz} \\ \sigma_{zx} &= Ne_{zx} \\ \sigma_{xy} &= Ne_{xy} \\ \sigma &= Qe + R\epsilon \end{aligned}\right\} \quad \text{[1]}$$

in which the σ_{ij} are the forces acting on the solid portions of the faces of a unit cube of porous material, and σ is the force acting on the fluid portions.

The average displacement vector of the solid has the components u_x, u_y, u_z, and that of the fluid U_x, U_y, U_z. The solid strain components are then given by

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$$\left.\begin{aligned} e_{xx} = \frac{\partial u_x}{\partial x},\quad e_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \ldots \text{etc.} \\ e = e_{xx} + e_{yy} + e_{zz} \end{aligned}\right\} \ldots [2]$$

The fluid dilatation ϵ is given by

$$\epsilon = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} \ldots [3]$$

It should be pointed out that this expression is not the actual strain in the fluid but simply the divergence of the fluid-displacement field which itself is derived from the average volume flow through the pores.

In Equations [1] attention is called to the fact that the properties of linearity alone would lead to five distinct elastic coefficients. The assumption that there exists a potential energy for the fluid-solid system reduces to four the number of independent elastic coefficients. This is the reason why the constant Q which appears in the last Equation [1] is the same as in the three first equations. An equivalent way of formulating this property is by stating that the matrix of elastic coefficients is symmetric with respect to the main diagonal. This symmetry property appears throughout when dealing with both isotropic and anisotropic media, and is further illustrated by Equation [30] and in the treatment of anisotropic media in Sections 4 and 5. The symmetry property is the reason for the redundancy in the measurement of the elastic coefficients. As will be pointed out, it is possible to derive the quantity $f(Q + R)/R$ from two independent physical measurements.

Four independent measurements, in addition to the porosity f, are required to fix the four elastic coefficients A, N, Q, and R. Satisfactory combinations of measurements may be made in a variety of different ways, but the most convenient appears to be the combination of measurement of shear modulus, jacketed and unjacketed compressibility of the porous solid, and an unjacketed coefficient of fluid content.

The shear modulus μ of the bulk material is equivalent to N, which is therefore obtained directly.

In the jacketed compressibility test, a specimen of the material is enclosed in a thin impermeable jacket and then subjected to an external fluid pressure p'.

To insure constant internal fluid pressure, the inside of the jacket may be made to communicate with the atmosphere through a tube. The conventional jacketed test is usually performed on a dry specimen, and in that case such precaution is of course not necessary. However, the dry specimen may not exhibit the same properties as the saturated one. As an example of this we may cite the case where the elastic properties result from surface forces of a capillary nature at the interface of the fluid and the solid.

The dilatation of the specimen is measured and a coefficient of jacketed compressibility κ is determined by

$$\kappa = -\frac{e}{p'} \ldots [4]$$

In this test the entire pressure of the fluid is transmitted to the solid portions of the surfaces of the specimen.

Therefore

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p' \ldots [5]$$

Furthermore, the pore pressure remains essentially constant

$$\sigma = 0 \ldots [6]$$

In addition the solid strains will be

$$\left.\begin{aligned} e &= -\kappa p' \\ e_{xx} = e_{yy} = e_{zz} &= -\frac{\kappa p'}{3} \end{aligned}\right\} \ldots [7]$$

From Equations [1] we obtain the two relations

$$\left.\begin{aligned} -p' &= -\frac{2}{3} N\kappa p' - A\kappa p' + Q\epsilon \\ 0 &= -Q\kappa p' + R\epsilon \end{aligned}\right\} \ldots [8]$$

Eliminating ϵ and p gives

$$\frac{1}{\kappa} = \frac{2}{3} N + A - \frac{Q^2}{R} \ldots [9]$$

thus indicating that the quantity $A - (Q^2/R)$ is equivalent to the Lamé coefficient λ of the porous material under conditions of constant pore pressure.

In the unjacketed compressibility test, a sample of the material is immersed in a fluid to which is applied a pressure p'. When the fluid pressure has penetrated the pores completely, the dilatation of the sample is then measured and an unjacketed compressibility coefficient δ is determined by

$$\delta = -\frac{e}{p'} \ldots [10]$$

In this case the pressure acts both on the solid portion $(1 - f)$ and the fluid portion f of the surfaces of the specimen giving

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -(1 - f)p'$$

$$\sigma = -fp' \ldots [11]$$

The strains are given by

$$\left.\begin{aligned} e_{xx} = e_{yy} = e_{zz} &= -\frac{\delta p'}{3} \\ e &= -\delta p' \end{aligned}\right\} \ldots [12]$$

And we obtain from Equations [1] the relation

$$1 - \left(\frac{Q + R}{R}\right) f = \left[\frac{2}{3} N + \left(A - \frac{Q^2}{R}\right)\right] \delta \ldots [13]$$

giving with Equation [9]

$$\left(\frac{Q + R}{R}\right) f = 1 - \frac{\delta}{\kappa} \ldots [14]$$

An independent and redundant measurement of this expression is furnished if we consider the quantity $f(e - \epsilon)$, which is denoted by ζ in Section 3, and represents the volume of fluid which enters the pores of a unit volume of bulk material. If, during a jacketed compressibility test the interior of the jacket is connected to the atmosphere by a tube the fluid volume passing through this tube is equal to $f(e - \epsilon)$. Putting $\sigma = 0$ in the last Equation [1] yields

$$\left(\frac{Q + R}{R}\right) f = \frac{f(e - \epsilon)}{e} \ldots [15]$$

Since we also have measured the solid strain e, this again yields the value of the quantity

$$\frac{(Q + R)f}{R}$$

Measurements of changes in pore volume and porosity in a jacketed test have been made by Hughes and Cooke (8), which are equivalent to measurements of $f(e - \epsilon)$.

To fix the properties of the system, one additional measurement is required which must involve the fluid strain. There is apparently no standard test which would provide a satisfactory measurement, and we must therefore define a new elastic coefficient. Again we consider the volume of fluid which enters the pores of a unit volume of porous material, $f(e - \epsilon)$, but in this case with reference to an unjacketed compressibility test. We may define a coefficent γ of fluid content by

$$\gamma = \frac{f(e - \epsilon)}{p'} \quad [16]$$

for an unjacketed compressibility test. This gives for the fluid strain

$$\epsilon = -\frac{\gamma}{f} p' + e \quad [17]$$

or

$$\epsilon = -\frac{\gamma}{f} p' + \delta p' \quad [18]$$

Expressions [11] and [18] substituted into the last of Equations [1] give

$$f = Q\delta + R\delta + R\frac{\gamma}{f} \quad [19]$$

Summarizing the measurements

$$\left.\begin{aligned} N &= \mu \\ \frac{2}{3}N + \left(A - \frac{Q^2}{R}\right) &= \frac{1}{\kappa} \\ \left(\frac{Q+R}{R}\right) f &= \left(1 - \frac{\delta}{\kappa}\right) \\ f &= Q\delta + R\delta + R\frac{\gamma}{f} \end{aligned}\right\} \quad [\ldots 20]$$

These equations are easily solved for N, A, Q, and R to give their values in terms of the measured parameters

$$\left.\begin{aligned} N &= \mu \\ A &= \frac{\frac{\gamma}{\kappa} + f^2 + (1 - 2f)\left(1 - \frac{\delta}{\kappa}\right)}{\gamma + \delta - \frac{\delta^2}{\kappa}} - \frac{2}{3}\mu \\ Q &= \frac{f\left(1 - f - \frac{\delta}{\kappa}\right)}{\gamma + \delta - \frac{\delta^2}{\kappa}} \\ R &= \frac{f^2}{\gamma + \delta - \frac{\delta^2}{\kappa}} \end{aligned}\right\} \quad [21]$$

The coefficient of fluid content γ may be determined by the following experiment: A unit volume of porous material containing fluid is placed within a closed chamber which has been filled with fluid. Fluid is then injected into the chamber under pressure and the volume of injected fluid is measured. Care must be exercised that the pores of the specimen are completely saturated with the same fluid or mixture of fluids as existed in the original system. In particular for a liquid-saturated solid it is important that no air be trapped in the pores near the surface. The volume of fluid injected per unit pressure will be the sum of the solid compressibility δ, the volume of fluid which has entered the pores γ, and a fixed quantity depending upon the elastic properties of the chamber and the fluid. The porous material is then removed from the chamber and its volume replaced by fluid. The volume of fluid injected per unit pressure is again measured and in this case will be the sum of the same fixed quantity as in the previous measurement, and the fluid compressibility c representing the new unit volume occupied by the fluid. Therefore the difference between the volumes injected with and without the porous material in the chamber will be given by

$$\Delta V = \delta + \gamma - c \quad [22]$$

If the fluid compressibility c is then measured independently, the coefficient of fluid content γ may then be determined. There are of course ways of avoiding this additional measurement of the fluid compressibility, which is not directly relevant, but the procedures involved appear to be more elaborate.

For the special case in which the material of the porous matrix is homogeneous and isotropic and the fluid completely saturates the pores, it is possible to determine the coefficient of fluid content γ *directly from the fluid compressibility* c. Considering the unjacketed test, the pore space in this special case will undergo the same strain as the solid matrix. Therefore the porosity f of the material will not undergo any strain and, if the fluid completely saturates the pores, the fluid dilatation will be given by

$$\epsilon = -cp' \quad [23]$$

This gives for the coefficient of fluid content

$$\gamma = f(c - \delta) \quad [24]$$

This relation will be strictly valid only for materials such that the pore volume and the bulk volume remain in constant ratio; i.e., the porosity does not vary when the specimen is subjected to fluid pressure in an unjacketed test. This of course will be true if the material of the porous matrix is homogeneous, isotropic, and elastically linear.

However, this is not a necessary condition, and we may visualize cases where the matrix material is heterogeneous but behaves approximately like a homogeneous solid so that it undergoes approximately the same strain as the pores in an unjacketed test. Therefore Relation [24] may provide a satisfactory approximation for γ in many cases. On the other hand, one also could imagine a porous matrix which is statistically isotropic but does not behave in isotropic fashion when considered from the standpoint of the individual pores.

Further remarks concerning the validity of Relation [24] will be made in Section 6 in connection with its application to incremental stresses.

3 Elastic Coefficients with Alternative Variables

As shown in reference (3), it is possible to introduce as alternative variables the total forces acting on the surfaces of a unit cube

$$\left.\begin{aligned} \tau_{xx} &= \sigma_{xx} + \sigma \\ \tau_{yy} &= \sigma_{yy} + \sigma \\ \tau_{zz} &= \sigma_{zz} + \sigma \\ \tau_{yz} &= \sigma_{yz} \\ \tau_{zx} &= \sigma_{zx} \\ \tau_{xy} &= \sigma_{xy} \end{aligned}\right\} \quad [25]$$

the fluid pressure p, and the fluid content, already mentioned

$$\zeta = f(e - \epsilon) \qquad [26]$$

We use the Lamé coefficients μ and[4]

$$\lambda = A - \frac{Q^2}{R} \qquad [27]$$

and two new coefficients
$$M = \frac{R}{f^2}$$

$$\alpha = \left(\frac{Q + R}{R}\right) f \qquad [28]$$

With these variables and coefficients Equations [1] may be written

$$\left.\begin{aligned}
\tau_{xx} + \alpha p &= 2\mu e_{xx} + \lambda e \\
\tau_{yy} + \alpha p &= 2\mu e_{yy} + \lambda e \\
\tau_{zz} + \alpha p &= 2u e_{zz} + \lambda e \\
\tau_{yz} &= \mu e_{yz} \\
\tau_{zx} &= \mu e_{zx} \\
\tau_{xy} &= \mu e_{xy} \\
\zeta &= \frac{1}{M} p + \alpha e
\end{aligned}\right\} \qquad [29]$$

An alternate form of these equations is obtained by solving the last equation for p and substituting in the first three. We find

$$\left.\begin{aligned}
\tau_{xx} &= 2\mu e_{xx} + (\lambda + \alpha^2 M)e - \alpha M \zeta \\
\tau_{yy} &= 2\mu e_{yy} + (\lambda + \alpha^2 M)e - \alpha M \zeta \\
\tau_{zz} &= 2u e_{zz} + (\lambda + \alpha^2 M)e - \alpha M \zeta \\
\tau_{yz} &= \mu e_{yz} \\
\tau_{zx} &= \mu e_{zx} \\
\tau_{xy} &= \mu e_{xy} \\
p &= -\alpha M e + M \zeta
\end{aligned}\right\} \qquad [30]$$

These equations express the total stress components τ_{ij} and the pore pressure p in terms of strain components e_{ij}, e, and ζ. We note that, because the same coefficient $-\alpha M$ appears in the first three equations and the last, the matrix of coefficients is symmetric. This again results from the existence of an elastic potential energy with p and ζ acting as conjugate variables. The elastic potential per unit volume is expressed as

$$W = \frac{1}{2}(\tau_{xx}e_{xx} + \tau_{yy}e_{yy} + \tau_{zz}e_{zz} + \tau_{yz}e_{yz} + \tau_{zx}e_{zx} + \tau_{xy}e_{xy} + p\zeta) \qquad [31][5]$$

Equations [30] are equivalent to

$$\left.\begin{aligned}
\tau_{ij} &= \frac{\partial W}{\partial e_{ij}} \\
p &= \frac{\partial W}{\partial \zeta}
\end{aligned}\right\} \qquad [32]$$

We note also from Equations [30] that $\lambda + \alpha^2 M$ plays the role of a Lamé constant λ for $\zeta = 0$; i.e., for a "closed system."

As before, μ may be measured directly as the modulus of shear.

In the jacketed compressibility test, $p = 0$, giving for the fluid content

$$\zeta = \alpha e \qquad [33]$$

Since the fluid pressure remains constant, the significance of α may be seen to be the ratio of change in pore volume to dilatation in a jacketed test. If the interior of the jacket is connected with the atmosphere by a tube we may measure the quantity ζ by the amount of fluid which is flowing through this tube. This will furnish the value of α.

If we designate the external pressure on the jacket by p', we have

$$\left.\begin{aligned}
\tau_{xx} = -p', \quad \tau_{xy} &= 0, \quad \ldots \text{etc.} \\
p &= 0 \\
e &= -p'\kappa
\end{aligned}\right\} \qquad [34]$$

giving with Equations [29] the relation

$$\frac{1}{\kappa} = \frac{2}{3}\mu + \lambda \qquad [35]$$

In the unjacketed test

$$\left.\begin{aligned}
\tau_{xx} = -p', \quad \tau_{xy} &= 0, \quad \ldots \text{etc} \\
p &= p' \\
e &= -\delta p'
\end{aligned}\right\} \qquad [36]$$

giving with Equations [29]

$$-(1 - \alpha)p' = \left(\frac{2}{3}\mu + 2\right)e \qquad [37]$$

Hence

$$\delta = (1 - \alpha)\kappa \qquad [38]$$

$$\alpha = \left(1 - \frac{\delta}{\kappa}\right) \qquad [39]$$

which provides a further interpretation of the coefficient α which is physically different from that given by Equation [33]. That they yield equal values for α results from the symmetry of the coefficient matrix in Equations [30] and this in turn is a consequence of the assumption that there exists an elastic potential energy for the fluid-solid system.

An alternative interpretation of the coefficient α which does not depend on the existence of a potential energy and is equivalent to Equation [39] is given by the first three of Equations [29]. In this case α represents the proportion of fluid pressure which will produce the same strains as the total stress.

The coefficient M may be determined from the coefficient of fluid content γ for the unjacketed test. In this case

$$\zeta = \gamma p' = \frac{1}{M} p' - \alpha \delta p' \qquad [40]$$

giving for M
$$M = \frac{1}{\gamma + \delta - \dfrac{\delta^2}{\kappa}} \qquad [41]$$

Another alternate way of writing Equations [29] is by introducing a stress τ' defined as

$$\left.\begin{aligned}
\tau_{xx}' &= \tau_{xx} + p \\
\tau_{yy}' &= \tau_{yy} + p \\
\tau_{zz}' &= \tau_{zz} + p \\
\tau_{yz}' &= \tau_{yz} \\
\tau_{zx}' &= \tau_{zx} \\
\tau_{xy}' &= \tau_{xy}
\end{aligned}\right\} \qquad [42]$$

[4] μ and λ are designated by S and N in references (2) and (3).

[5] Expression [31] is identical with that introduced in reference (1) in connection with the derivation of the property of symmetry of the coefficients. The notations σ and θ were used instead of p and ζ.

The total stress τ is then represented as a superposition of hydrostatic pressure p, the same as in the fluid pores and a residual component τ' which acts only in the solid matrix. With this definition the first three Equations [29] are replaced by

$$\left.\begin{aligned} \tau_{xx}' - (1-\alpha)p &= 2\mu e_{xx} + \lambda e \\ \tau_{yy}' - (1-\alpha)p &= 2\mu e_{yy} + \lambda e \\ \tau_{zz}' - (1-\alpha)p &= 2\mu e_{zz} + \lambda e \end{aligned}\right\} \ldots\ldots [43]$$

As shown in references (1) and (3) the elastic coefficients μ, λ, α, and M may all be determined without reference to the porosity f. Furthermore if the Darcy equation for volume flow is used, all the equations of consolidation theory may be developed without reference to porosity.

It is interesting to examine the limits of the coefficient α. Consider first the last of Equations [1]

$$\sigma = Qe + R\epsilon \ldots\ldots [44]$$

If a positive fluid force σ is applied to the system at the same time that e is held fixed, a positive fluid strain ϵ must result. Therefore R must be positive. Alternatively, if the fluid force is held constant and a positive solid force is applied, e must be positive. In addition there must be a net increase in the porosity, requiring a negative fluid strain ϵ. Therefore, Q also must be positive. Since both Q and R are positive, it may be seen from the relation

$$\alpha = \left(\frac{Q+R}{R}\right) f \ldots\ldots [28]$$

that α cannot be smaller than f. Alternatively, since the unjacketed compressibility δ cannot be less than zero, it follows from the relation

$$\alpha = \left(1 - \frac{\delta}{\kappa}\right) \ldots\ldots [39]$$

that α cannot be greater than unity.

If the unjacketed compressibility δ is very small compared to the jacketed compressibility κ, we may approximate the value of α by putting $\alpha = 1$. This will be true in some cases for a water-saturated gel or clay (1, 2).[6]

4 Transverse Isotropy

For the case of transverse isotropy the stress-strain relations may be written (2)

$$\left.\begin{aligned} \sigma_{xx} &= Pe_{xx} + Ae_{yy} + Fe_{zz} + M\epsilon \\ \sigma_{yy} &= Ae_{xx} + Pe_{yy} + Fe_{zz} + M\epsilon \\ \sigma_{zz} &= Fe_{zz} + Fe_{yy} + Ce_{zz} + Q\epsilon \\ \sigma_{yz} &= Le_{yz} \\ \sigma_{zx} &= Le_{zx} \\ \sigma_{xy} &= Ne_{xy} \\ \sigma &= Me_{xx} + Me_{yy} + Qe_{zz} + R\epsilon \end{aligned}\right\} \ldots\ldots [45]$$

in which $P = A + 2N$ and there are eight independent elastic coefficients. It will therefore require eight independent measurements to fix the values of the coefficients.

Consider the matrix

$$\begin{bmatrix} P & A & F & M \\ A & P & F & M \\ F & F & C & Q \\ M & M & Q & R \end{bmatrix} \ldots\ldots [46]$$

and let its inverse be represented by

$$\begin{bmatrix} a & d & g & h \\ d & a & g & h \\ g & g & b & m \\ h & h & m & n \end{bmatrix} \ldots\ldots [47]$$

Also let

$$\left.\begin{aligned} \frac{1}{N} &= s \\ \frac{1}{L} &= t \end{aligned}\right\} \ldots\ldots [48]$$

The inverse stress-strain relations will then be

$$\left.\begin{aligned} e_{xx} &= a\sigma_{xx} + d\sigma_{yy} + g\sigma_{zz} + h\sigma \\ e_{yy} &= d\sigma_{xx} + a\sigma_{yy} + g\sigma_{zz} + h\sigma \\ e_{zz} &= g\sigma_{xx} + g\sigma_{yy} + b\sigma_{zz} + m\sigma \\ e_{yz} &= t\sigma_{yz} \\ e_{zx} &= t\sigma_{zx} \\ e_{xy} &= s\sigma_{xy} \\ \epsilon &= h\sigma_{xx} + h\sigma_{yy} + m\sigma_{zz} + n\sigma \end{aligned}\right\} \ldots\ldots [49]$$

in which there are nine coefficients but only eight independent ones because of the original relation ($P = A + 2N$).

For this material under conditions of constant pore pressure, it will be possible to measure two separate Young's moduli by

$$\left.\begin{aligned} \frac{1}{E_1} &= \frac{\partial e_{xx}}{\partial \sigma_{xx}} = \frac{\partial e_{yy}}{\partial \sigma_{yy}} = a \\ \frac{1}{E_2} &= \frac{\partial e_{zz}}{\partial \sigma_{zz}} = b \end{aligned}\right\} \ldots\ldots [50]$$

Three separate Poisson's ratio also exist and may be measured by

$$\left.\begin{aligned} -\nu_1 &= \frac{\left(\dfrac{\partial e_{yy}}{\partial \sigma_{xx}}\right)}{\left(\dfrac{\partial e_{xx}}{\partial \sigma_{xx}}\right)} = \frac{d}{a} = E_1 d \\ -\nu_2 &= \frac{\left(\dfrac{\partial e_{xx}}{\partial \sigma_{zz}}\right)}{\left(\dfrac{\partial e_{zz}}{\partial \sigma_{zz}}\right)} = \frac{g}{z} = E_2 g \\ -\nu_3 &= \frac{\left(\dfrac{\partial e_{zz}}{\partial \sigma_{xx}}\right)}{\left(\dfrac{\partial e_{xx}}{\partial \sigma_{xx}}\right)} = \frac{g}{a} = E_1 g \end{aligned}\right\} \ldots\ldots [51]$$

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[6] As pointed out in reference (1) the case where the solid matrix and the fluid are incompressible corresponds to $\alpha = 1$, $M = \infty$. In reference (1) several other coefficients are used. E, G, and ν have their conventional significance as measured with constant pore pressure. The coefficient α has the same significance as in Equations [29]. The coefficient H is given by $1/(\kappa - \delta)$. The coefficients Q and R of reference (1) should not be confused with Q and R of Equations [1]. In reference (1) Q is equivalent to M and R is given by $1/(\gamma - \delta + \kappa)$.

There are only four independent coefficients between the two moduli and the three ratios. From the last two of Equations [51]

$$\frac{\nu_2}{E_2} = \frac{\nu_3}{E_1} \quad \text{[52]}$$

The coefficients a, b, d, and g may thereby be determined by a minimum of four measurements.

Two shear moduli also may be measured by

$$\left.\begin{aligned} \frac{1}{\mu_1} &= \left(\frac{\partial e_{xy}}{\partial \sigma_{xy}}\right) = s \\ \frac{1}{\mu_2} &= \left(\frac{\partial e_{zx}}{\partial \sigma_{zx}}\right) = \left(\frac{\partial e_{yz}}{\partial \sigma_{yz}}\right) = t \end{aligned}\right\} \quad \text{[53]}$$

Because of the original relation in Equations [45] of $P = A + 2N$, the first modulus μ_1 does not represent an independent measurement and could be determined from the other measurements.

In an unjacketed compression test

$$\left.\begin{aligned} \sigma_{xx} = \sigma_{yy} = \sigma_{zz} &= -(1-f)p' \\ \sigma &= -fp' \end{aligned}\right\} \quad \text{[54]}$$

giving the relations

$$\left.\begin{aligned} e_{xx} &= [(f-1)(a+d+g) - fh]p' = -\delta_1 p' \\ e_{yy} &= [(f-1)(a+d+g) - fh]p' = -\delta_1 p' \\ e_{zz} &= [(f-1)(g+g+b) - fm]p' = -\delta_2 p' \end{aligned}\right\} \quad \text{[55]}$$

in which δ_1 represents the change in the two equal principal axes and δ_2 the change in the third principal axis of the ellipsoid of revolution formed by a unit sphere of material strained under unit pressure in an unjacketed compression test.

This measurement permits the two additional coefficients h and m to be determined.

For a final measurement the coefficient of fluid content γ may be used, giving for the fluid strain

$$\epsilon = [(f-1)(h+h+m) - fn] = -\frac{\gamma}{f}p' - (2\delta_1 + \delta_2)p' \quad \text{[56]}$$

from which the coefficient n may be determined.

These measurements give for the Matrix [47]

$$\begin{bmatrix} \left(\frac{1}{E_1}\right) & \left(\frac{-\nu_1}{E_1}\right) & \left(\frac{-\nu_3}{E_1}\right) & \left[\frac{\delta_1}{f} - \left(\frac{1-f}{f}\right)\left(\frac{1-\nu_1-\nu_3}{E_1}\right)\right] \\ \cdots & \left(\frac{1}{E_1}\right) & \left(\frac{-\nu_2}{E_2}\right) & \left[\frac{\delta_1}{f} - \left(\frac{1-f}{f}\right)\left(\frac{1-\nu_1-\nu_3}{E_1}\right)\right] \\ \cdots & \cdots & \left(\frac{-1}{E_2}\right) & \left[\frac{\delta_2}{f} - \left(\frac{1-f}{f}\right)\left(\frac{1-2\nu_2}{E_2}\right)\right] \\ \cdots & \cdots & \cdots & \left[\frac{\gamma}{f_2} - \left(\frac{1-2f}{f}\right)\left(\frac{2\delta_1+\delta_2}{f}\right) + \left(\frac{1-f}{f}\right)^2\left(\frac{1-2\nu_2}{E_2} + \frac{2(1-\nu_1-\nu_3)}{E_1}\right)\right] \end{bmatrix} \quad \text{[57]}$$

and the two relations

$$\left. s = \frac{1}{\mu_1}; \quad t = \frac{1}{\mu_2} \right\} \quad \text{[58]}$$

The individual values of the coefficients in the Matrix [46] may be found by the inversion of the Matrix [57].

The ten measured values

$$\left.\begin{matrix} E_1\ \nu_1\ \mu_1\ \delta_1\ \gamma \\ E_2\ \nu_2\ \mu_2\ \delta_2 \\ \nu_3 \end{matrix}\right\} \quad \text{[59]}$$

have been used. Only eight of these are independent since

$$\frac{\nu_2}{E_2} = \frac{\nu_3}{E_1}$$

and since μ_1 may be determined from the relation $P = A + 2N$.

An alternate form of the coefficients of course could be easily written in terms of the variables p and ζ instead of σ and ϵ. These new coefficients will not involve the porosity f.

5 Complete Anisotropy

The stress-strain relations may be expressed in matrix form by (2)

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \\ \sigma \end{bmatrix} = \begin{bmatrix} C_{11}C_{12}C_{13}C_{14}C_{15}C_{16}C_{17} \\ C_{22}C_{23}C_{24}C_{25}C_{26}C_{27} \\ C_{33}C_{34}C_{35}C_{36}C_{37} \\ C_{44}C_{45}C_{46}C_{47} \\ C_{55}C_{56}C_{57} \\ C_{66}C_{67} \\ C_{77} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{zx} \\ e_{xy} \\ \epsilon \end{bmatrix} \quad \text{[60]}$$

in which $C_{ij} = C_{ji}$.

Eliminating ϵ by means of the last equation

$$\left.\begin{aligned} \sigma_{xx} &= \left(C_{11} - \frac{C_{17}C_{17}}{C_{77}}\right)e_{xx} + \left(C_{12} - \frac{C_{17}C_{27}}{C_{77}}\right)e_{yy} + \ldots + \frac{C_{17}}{C_{77}}\sigma \\ \sigma_{yy} &= \left(C_{12} - \frac{C_{17}C_{27}}{C_{77}}\right)e_{xx} + \left(C_{22} - \frac{C_{27}C_{27}}{C_{77}}\right)e_{yy} + \ldots + \frac{C_{27}}{C_{77}}\sigma \\ \sigma_{zz} &= \ldots \\ \sigma_{yz} &= \ldots \\ \sigma_{zx} &= \ldots \\ \sigma_{xy} &= \ldots \end{aligned}\right\} \quad \text{[61]}$$

in which the 21 terms

$$\left(C_{11} - \frac{C_{17}C_{17}}{C_{77}}\right) \cdots\cdots \left(C_{66} - \frac{C_{67}C_{67}}{C_{77}}\right) \quad \text{[62]}$$

are the classical coefficients of anisotropic elasticity, as would be measured with a constant pore pressure ($\sigma = 0$).

Seven additional measured quantities are then required to fix the properties of the system. Six of these may be obtained from an unjacketed compressibility test. An original unit sphere of the porous material will be deformed into a triaxial ellipsoid. For the general case in which the principal axes of the ellipsoid are not parallel to the co-ordinate axes, the six coefficients of strain as a function of pressure may be measured as

$$\left.\begin{aligned}\delta_{xx} &= -\frac{e_{xx}}{p'}\\ \delta_{yy} &= -\frac{e_{yy}}{p'}\\ \delta_{zz} &= -\frac{e_{zz}}{p'}\\ \delta_{yz} &= -\frac{e_{yz}}{p'}\\ \delta_{zx} &= -\frac{e_{zx}}{p'}\\ \delta_{xy} &= -\frac{e_{xy}}{p'}\end{aligned}\right\} \quad [63]$$

Alternatively it would be possible to measure the mutually perpendicular principal strains of the specimen

$$\left.\begin{aligned}\delta_1 &= -\frac{e_1}{p'}\\ \delta_2 &= -\frac{e_2}{p'}\\ \delta_3 &= -\frac{e_3}{p'}\end{aligned}\right\} \quad [64]$$

and three angles describing their orientation and to express the co-ordinate strains in terms of these.

For the special case in which the principal axes of the ellipsoid are parallel to the co-ordinate axes, the coefficients δ_{xx}, δ_{yy}, and δ_{zz} are given directly by the principal strains and coefficients δ_{yz}, δ_{zx}, and δ_{xy} are zero.

For a final measurement the coefficient of fluid content γ may again be used, permitting the fluid dilatation in the unjacketed compressibility test to be given by

$$\left.\begin{aligned}\epsilon &= -\frac{\gamma}{f}p' - (\delta_1 + \delta_2 + \delta_3)p'\\ &\text{or}\\ \epsilon &= -\frac{\gamma}{f}p' - (\delta_{xx} + \delta_{yy} + \delta_{zz})p'\end{aligned}\right\} \quad [65]$$

Therefore the strains in the unjacketed test may be expressed in terms of seven measured coefficients and the fluid pressure. The stresses again will be

$$\left.\begin{aligned}\sigma_{xx} = \sigma_{yy} = \sigma_{zz} &= -(1-f)p'\\ \sigma_{yz} = \sigma_{zx} = \sigma_{xy} &= 0\\ \sigma &= -fp'\end{aligned}\right\} \quad [66]$$

and by substitution into Equations [60] seven equations relating the original coefficients and the measured coefficients will be obtained. These relations together with the 21 relations between the Expressions [62] and the classical coefficients permit the original coefficients to be determined. Again we may remark that a different set of coefficients not involving explicitly the porosity can be derived easily by introducing the variables p and ζ.

6 Elastic Coefficients for Incremental Deformations of a Prestressed Material

In the foregoing we have assumed the strains to be small and linearly related to the stresses. We will now examine how the relations between stress and strain may be expressed if the porous solid-fluid system is still elastic but nonlinear. In particular we are interested to know if the results of the foregoing sections are directly applicable to the linearized problem when we consider small incremental stresses and strains in the vicinity of a prestressed condition.

We shall restrict ourselves to the case where the prestress is isotropic. We consider an initial state 1 which is unstressed. The state of prestress denoted as 2 results from the application of isotropic stresses in the fluid and the solid. We consider the forces acting on the solid portion of the material per unit area of the bulk material in state 2. This being an isotropic state of stress these forces are represented by the matrix

$$\begin{bmatrix}\sigma' & 0 & 0\\ 0 & \sigma' & 0\\ 0 & 0 & \sigma'\end{bmatrix} \quad [67]$$

Similarly the forces acting on the fluid portion per unit area of bulk material are represented by

$$\begin{bmatrix}\sigma & 0 & 0\\ 0 & \sigma & 0\\ 0 & 0 & \sigma\end{bmatrix} \quad [68]$$

The prestressed state 2 is now considered as a new initial state and small incremental deformations e_{ij} are superimposed leading to state 3. The incremental strain tensor is defined in terms of the incremental displacements $\bar{u}$ and $\bar{U}$ of the solid and the fluid measured from state 2 as origin. The incremental strains give rise to a new state of stress obtained by adding incremental components $\Delta\sigma_{ij}$ and $\Delta\sigma$. We represent the solid stress in state 3 by forces acting on the solid portion of the material per unit area in state 2. They are

$$\begin{bmatrix}\sigma' + \Delta\sigma_{xx} & \Delta\sigma_{xy} & \Delta\sigma_{zx}\\ \Delta\sigma_{xy} & \sigma' + \Delta\sigma_{yy} & \Delta\sigma_{yz}\\ \Delta\sigma_{zx} & \Delta\sigma_{yz} & \sigma' + \Delta\sigma_{zz}\end{bmatrix} \quad [69]$$

The forces acting on the fluid in state 3 per unit area in state 2 are

$$\begin{bmatrix}\sigma + \Delta\sigma & 0 & 0\\ 0 & \sigma + \Delta\sigma & 0\\ 0 & 0 & \sigma + \Delta\sigma\end{bmatrix} \quad [70]$$

We now assume that the incremental stresses and strains are small enough so that they are related linearly. The material in state 2 remains isotropic and therefore the incremental stress-strain relations are isotropic. This requires that they be of the form

$$\left.\begin{aligned}\Delta\sigma_{xx} &= 2Ne_{xx} + Ae + Q\epsilon\\ \Delta\sigma_{yy} &= 2Ne_{yy} + Ae + Q\epsilon\\ \Delta\sigma_{zz} &= 2Ne_{zz} + Ae + Q\epsilon\\ \Delta\sigma_{yz} &= Ne_{yz}\\ \Delta\sigma_{zx} &= Ne_{zx}\\ \Delta\sigma_{xy} &= Ne_{xy}\\ \Delta\sigma &= Q'e + R\epsilon\end{aligned}\right\} \quad [71]$$

We see that based on isotropy alone there are five elastic coefficients for incremental stresses and strains. These coefficients are functions of the initial stresses σ and σ'. In order to simplify the writing we may without loss of generality consider an incremental stress which is also isotropic. We put

$$\left.\begin{aligned} \Delta\sigma_{xx} &= \Delta\sigma_{yy} = \Delta\sigma_{zz} = \Delta\sigma' \\ e_{xx} &= e_{yy} = e_{zz} = \frac{1}{3} e \\ \Delta\sigma_{yz} &= \Delta\sigma_{zx} = \Delta\sigma_{xy} = 0 \\ e_{yz} &= e_{zx} = e_{xy} = 0 \end{aligned}\right\} \ldots\ldots\ldots [72]$$

Equations [71] become

$$\left.\begin{aligned} \Delta\sigma' &= \left(\frac{2}{3} N + A\right) e + Q\epsilon \\ \Delta\sigma &= Q'e + R\epsilon \end{aligned}\right\} \ldots\ldots\ldots [73]$$

We assume the existence of an elastic potential energy for incremental deformations in the vicinity of the prestressed state 2. If W denotes this potential energy per unit volume of the material in state 2 we may write

$$dW = (\sigma' + \Delta\sigma')de + (\sigma + \Delta\sigma)d\epsilon \ldots\ldots\ldots [74]$$

This being an exact differential we have

$$\frac{\partial}{\partial\epsilon}(\Delta\sigma') = \frac{\partial}{\partial e}(\Delta\sigma) \ldots\ldots\ldots\ldots\ldots [75]$$

hence from Equations [73]

$$Q = Q' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [76]$$

The matrix of elastic coefficients in Equations [71] is therefore symmetric and there are only four distinct coefficients.

We must bear in mind that the quantities $\Delta\sigma'$ and $\Delta\sigma$ are not the forces acting per unit area of the final deformed state 3 but per unit area of the prestressed state 2. Therefore, they are not represented by actual fluid pressures in a test but are related to them through the incremental deformations. Let us imagine a jacketed test with an initial state of stress such that the fluid pressure inside the jacket is p, while a fluid pressure p' is applied outside. Our purpose here is to point out an important difference between the test with prestress and the test without prestress. In the case of prestress, it is not as simple to evaluate the quantities $\Delta\sigma'$ and $\Delta\sigma$ in terms of the fluid pressure as for the case without prestress. This can be seen as follows. Denoting by f the porosity in the prestressed state 2 we may still write as before

$$\left.\begin{aligned} \sigma' + \sigma &= -p' \\ \sigma &= -fp \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots [77]$$

If we now apply incremental fluid pressures $\Delta p'$ and Δp, these fluid pressures are applied to changing areas. The new area for p' is $(1 + {}^1/_3 e)^2$ and for p it is $(f + \Delta f)(1 + {}^1/_3 e)^2$. We note by Δf the increment of the porosity factor. Hence

$$\left.\begin{aligned} \sigma' + \sigma + \Delta\sigma' + \Delta\sigma &= -(p' + \Delta p')\left(1 + \frac{1}{3} e\right)^2 \\ \sigma + \Delta\sigma &= -(p + \Delta p)(f + \Delta f) \left(1 + \frac{1}{3} e\right)^2 \end{aligned}\right\} \ldots [78]$$

Retaining only incremental terms of the first order, Equations [78] are written

$$\left.\begin{aligned} \Delta\sigma' + \Delta\sigma &= -\Delta p - \frac{2}{3} p'e \\ \Delta\sigma &= -f\Delta p - p\Delta f - \frac{2}{3} fpe \end{aligned}\right\} \ldots\ldots [79]$$

We see that $\Delta\sigma'$ and $\Delta\sigma$ are not exactly equal to $(1 - f)\Delta p'$ and $-f\Delta p$. The correction terms, in general, will be small but nevertheless will vanish only if the initial stress is zero.

With these limitations the results of the previous sections may be applied to incremental deformations except for a reservation regarding the validity of Expression [24] for the coefficient of fluid content γ. We have remarked that, for the unstressed initial state, this expression holds if the material of the porous matrix is homogeneous and isotropic. This is a sufficient condition only if $p = p'$, i.e., if the initial pressures on the solid and on the pores are the same. If these initial pressures are not equal, then equal increments of these pressures may produce a change in porosity thereby invalidating Relation [24].

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