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General Theorems on the Equivalence of Group Velocity and Energy Transport

M. A. Biot*
Shell Development Company, New York, New York
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It is shown that under very general conditions there is a rigorous identity between the group velocity and the velocity of energy transport in nonhomogeneous media with or without anomalous dispersion. The medium is assumed nondissipative and the parameters such as density, rigidity, dielectric constant, etc., vary from point to point but are independent of the Cartesian coordinate lying in the direction of the mode propagation. This covers propagation in any type of wave guide, surface waves, propagation in stratified media, etc. The identity is established for fluids, for isotropic and anisotropic solids with or without pre-stress, and for electromagnetic waves. Anomalous dispersion is assumed to result from hidden coordinates such as electron oscillators. A new variational formulation of field theory is introduced. An interesting application is to wave propagation in an electron gas and it is shown that such wave propagation obeys the relativistic Schrödinger equation for a mass particle.

1. INTRODUCTION

THE purpose of the present analysis is to show that there is a rigorous identity under very general conditions between the group velocity as defined kinematically and the velocity of transport of energy. The kinematic definition of group velocity uses the concept of stationary phase, and the velocity of energy propagation is defined as the energy flux divided by the energy density. The general conditions under which the identity of the two definitions is shown to be true are of two types.

One condition is of a geometrical nature. We assume that the parameters defining the heterogeneous system are functions of only two Cartesian coordinates and independent of the third, and we shall consider propagation modes along this third coordinate. Such systems generally act as wave guides and as specific examples we may cite the propagation of acoustic waves in a rod or tube composed of concentric layers of fluid and solid, seismic waves in a solid with continuous distribution of density and rigidity depending on depth, all sorts of surface waves, and analogous systems for electromagnetic waves with layered dielectric or perfectly conducting materials.

The other conditions are of a physical nature. We assume that there is no change in entropy, i.e., no dissipation or scattering. We include the case of anomalous dispersion, i.e., those for which the parameters such as the elastic or dielectric constant are functions of the frequency, provided the macroscopic effect may be represented by the dynamical behavior of certain hidden degrees of freedom such as the motion of undamped electron oscillators continuously distributed. The radiation and Coulomb interaction of the electrons are taken into account but their distance compared to the wavelength is assumed small enough so that scattering is negligible.

Under those conditions we will show that the identity of the group velocity and the velocity of propagation of energy is quite rigorous and general for heterogeneous media with frequency-dependent parameters, i.e., for cases where the dispersion of the propagation modes is both geometrical and anomalous in origin.

The general procedure to establish this identity is outlined in Sec. 2. It is shown in Sec. 3 how it can be applied to modes of propagation in a fluid. The purpose of treating first such a simple example is to clarify the procedure. The theorem is then extended in Sec. 4 to elastic waves in heterogeneous solids, including the most general case of anisotropy. Wave propagation in an elastic continuum which is in an initial state of pre-stress is considered in Sec. 5. The state of pre-stress may be heterogeneous under the same conditions as the physical parameters, i.e., it must be independent of the

* Consultant.

coordinate lying along the propagating mode. A particular example of this is that of a gas such as the atmosphere. This also applies to surface gravity waves in an incompressible fluid where the identity of group velocity and energy transport is well known.[1] The theorem is extended in Sec. 6 to the case of anomalous dispersion of elastic waves and the proof is illustrated on the simple model of a string under tension coupled elastically to another lying alongside without tension. It is found that such a model exhibits all the features of anomalous dispersion in the most general cases, including that of electromagnetic waves. Although the example is formulated for only one resonant frequency, the proof is clearly valid for any number of such frequencies. It is pointed out that for the theorem to apply to the case of anomalous dispersion it is essential to include the energy of all the hidden degrees of freedom in the definition of the energy density. Finally, it is shown in Sec. 7 how the theorem extends to electromagnetic waves in heterogeneous systems with or without anomalous dispersion. The variation of dielectric constant with frequency is assumed to result from the motion of electric charges as harmonic oscillators. Coulomb interaction and radiation due to the motion of the charges are taken into account. An interesting application is the particular case of waves propagating in an electron gas or plasma. These plane waves are found to satisfy the relativistic Schrödinger equation and to behave as de Broglie waves for mass particles. The frequency of the plasma oscillations corresponds to the frequency of the rest mass of the particle. The variational formulation of the field problem which we have introduced is different from the current one in field theory and appears to be of physical significance beyond the mere formalism.

The relation between the transport of energy and the group velocity was pointed out simultaneously by Reynolds[2] and Lord Rayleigh.[3] In the case of propagation of electromagnetic waves in a homogeneous medium this relation was the object of a detailed investigation by Brillouin,[4–6] who showed that the identity between group velocity and the velocity of transport of energy holds for media with anomalous dispersion in the frequency range where dissipation is negligible. More recently, Broer[7] has shown the identity to be valid for waves in one-dimensional conservative systems obeying a certain type of propagation equation. The treatment however does not apply to inhomogeneous or discontinuous media. With respect to anomalous dispersion it is restricted by the type of propagation equation which is assumed. The present analysis generalizes Brillouin's and Broer's results to nonhomogeneous media of the general class defined above for both elastic and electromagnetic waves. It furthermore establishes that for any type of anomalous dispersion due to hidden coordinates the identity is rigorous provided there is no dissipation.

An important application of the identity of group velocity and the velocity of propagation of energy has been suggested by Tolstoy[8,9] with reference to the numerical calculation of the group velocity. He points out that if the amplitude of a mode of propagation is known the energy flux and energy density may be easily calculated, thereby giving directly the value of the group velocity and avoiding the usual rather inaccurate process of calculating derivatives of the phase velocity plotted against frequency. The present theorems establish the validity of the procedure in a wide varity of cases.

2. OUTLINE OF THE GENERAL METHOD

Let us reformulate a general and well-known principle of eigenvalue perturbations. Consider a dynamical system defined by two positive definite forms V and T' defining the potential and kinetic energies, respectively, in terms of general coordinates q. We write

$$\begin{aligned} 2V &= \sum_{ij} a_{ij} q_i q_j, \\ 2T' &= \sum_{ij} m_{ij} \dot{q}_i \dot{q}_j. \end{aligned} \tag{2.1}$$

For harmonic motion of angular frequency ω Lagrange's equations are written

$$\partial V/\partial q_i = \omega^2 \partial T/\partial q_i, \tag{2.2}$$

where T is the function T' in which $\dot{q}_i$ is replaced by q_i. These equations are mathematically identical with the extremum condition

$$\delta V = 0 \tag{2.3}$$

for all possible variations δq_i under the constraint

$$T = \text{const.} \tag{2.4}$$

If we multiply Eq. (2.2) by q_i and then add them together, we find that for any characteristic solution of (2.2) we have

$$\omega_n^2 = V_n / T_n, \tag{2.5}$$

where V_n, T_n and ω_n are values corresponding to a characteristic solution. If we now vary the coordinates q_n, maintaining T constant but at the same time varying also the coefficients a_{ij}, we may write

$$\delta\omega_n^2 = (\delta_a V_n + \delta_q V_n)/T_n, \tag{2.6}$$

[1] A. Sommerfeld, *Mechanics of Deformable Bodies* (Academic Press, Inc., New York, 1950), p. 189.

[2] O. Reynolds, *Papers on Mathematical and Physical Subjects* (Cambridge University Press, Cambridge, 1901), Vol. 1, p. 198.

[3] Lord Rayleigh, *Scientific Papers* (Cambridge University Press, Cambridge, 1901), Vol. 1, p. 322.

[4] L. Brillouin, *Propagation des Ondes Electromagnetiques dans les Milieux Materials* (Congrès International d'Electricite, Paris, 1932), Vol. II, Ser. 1, pp. 739–788.

[5] L. Brillouin, *Wave Propagation in Periodic Structures* (Dover Publications, New York, 1946).

[6] L. Brillouin and M. Parodi, *Propagation des Ondes dans les Milieux Periodiques* (Masson and Dunod, Paris, 1956).

[7] L. J. F. Broer, Appl. Sci. Research **A2**, 329 (1951).

[8] I. Tolstoy, J. Acoust. Soc. **27**, 897 (1955).

[9] I. Tolstoy, J. Acoust. Soc. **28**, 1182 (1956).

where δ_a and δ_q indicate variations due to a_{ij} and q_i, respectively. However, because of (2.3) the variation $\delta_q V_n$ vanishes, we derive

$$\delta\omega_n^2=\delta_a V_n/T_n. \tag{2.7}$$

The variation of the eigenvalue depends only on that of the coefficients a_{ij}. This is the fundamental property which will be used hereafter in evaluating the group velocity. If the coefficients a_{ij} depend on a single parameter k, we may write (2.7) as

$$\frac{d\omega_n}{dk}=\frac{1}{2\omega_n T_n}\frac{dV_n}{dk}. \tag{2.8}$$

From (2.5) we may also write

$$E=\omega_n^2 T_n+V_n=2\omega_n^2 T_n. \tag{2.9}$$

This represents the total energy stored in the oscillation. Hence (2.8) becomes

$$\frac{d\omega_n}{dk}=\frac{\omega_n}{E}\frac{dV_n}{dk}. \tag{2.10}$$

This is the basic formula we shall apply in the analysis below. We shall consider a certain mode of wave propagation as giving rise to a standing wave pattern depending on the wave number k as a parameter. We shall omit the subscript n. The group velocity associated with the particular mode is

$$v_g=\frac{d\omega}{dk}=\frac{\omega}{E}\frac{dV}{dk}. \tag{2.11}$$

We propose to show in the following section that the group velocity is equal to the energy flux G divided by an energy density E of the wave. We shall identify E with the energy density per unit distance and give a proof that in all cases considered the group velocity is given by

$$v_g=G/E, \tag{2.12}$$

where G is the energy flux across a fixed plane per unit time. As we can see by comparison with (2.11), we shall have to prove that the energy flux is equal to

$$G=\omega dV/dk. \tag{2.13}$$

Also, we shall have to verify that T is independent of k; otherwise expression (2.8) would not be valid.

In the evaluation of the energy flux we shall also need a well-known expression for the average value of the product of two quantities varying sinusoidally in time with the same frequency and represented by two complex vectors. If the two representative vectors are Z_1 and Z_2, the corresponding real quantities as a function of time are[10]

$$\begin{aligned} z_1&=\tfrac{1}{2}(Z_1e^{i\omega t}+Z_1^*e^{-i\omega t}),\\ z_2&=\tfrac{1}{2}(Z_2e^{i\omega t}+Z_2^*e^{-i\omega t}). \end{aligned} \tag{2.14}$$

The average time integral of the product is

$$P=\frac{\omega}{2\pi}\int_0^{2\pi/\omega} z_1z_2dt=\tfrac{1}{4}(Z_1Z_2^*+Z_2Z_1^*) \tag{2.15}$$

or

$$P=\tfrac{1}{2}\,\mathrm{Re}(Z_1Z_2^*). \tag{2.16}$$

We shall apply this expression repeatedly in evaluating the energy flux.

3. GROUP VELOCITY IN A COMPRESSIBLE FLUID

We shall first consider the case of a compressible fluid. The bulk modulus $\lambda(y,z)$ and mass density $\rho(y,z)$ are assumed independent of x. Wave modes of amplitudes proportional to $\exp[\pm ikx+i\omega t]$ are propagated in this system with a dispersive phase velocity $v=\omega/k$. Two trains of waves propagating in opposite directions produce a standing wave pattern. Omitting the time factor, $\exp(i\omega t)$, the displacement components in the x, y, z directions are

$$\begin{aligned} u_1&=u_1'(y,z)\sin kx,\\ u_2&=u_2'(y,z)\cos kx,\\ u_3&=u_3'(y,z)\cos kx. \end{aligned} \tag{3.1}$$

These are all real quantities. The problem is thus reduced to that of a vibrating slab of fluid lying between rigid plane boundaries parallel to the yz plane and intersecting the x axis at $x=0$, $x=\pi/k$. The restraint that the mode is sinusoidal along x reduces the originally three-dimensional problem to a two-dimensional one in the yz plane. The boundary conditions in this plane are of course determined by the original problem. These may be on finite boundaries or at infinity.

The hydrostatic stress (negative pressure) is

$$\sigma=\lambda e, \tag{3.2}$$

with

$$e=\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z}=\left(ku_1'+\frac{\partial u_2'}{\partial y}+\frac{\partial u_3'}{\partial z}\right)\cos kx.$$

The elastic energy density is

$$W=\tfrac{1}{2}\sigma e=\tfrac{1}{2}\lambda e^2. \tag{3.3}$$

Its average value along x is

$$W_a=\frac{k}{2\pi}\int_0^{2\pi/k} Wdx=\tfrac{1}{4}\lambda\left(ku_1'+\frac{\partial u_2'}{\partial y}+\frac{\partial u_3'}{\partial z}\right)^2. \tag{3.4}$$

The average total elastic energy of the slab per unit

[10] Throughout the paper the symbol * indicates the complex conjugate and Re "the real part of."

thickness is

$$V=\int\int W_a dydz. \tag{3.5}$$

We see that V is effectively a function of k. We also verify that T which corresponds to the kinetic energy is independent of k. By averaging the kinetic energy along x, we find a value of T per unit length:

$$T=\frac{1}{4}\int\int \rho(u_1'^2+u_2'^2+u_3'^2)dydz. \tag{3.6}$$

The problem of determining the modes of propagation is an eigenvalue problem expressed by the condition that V is stationary, i.e.,

$$\delta V=0 \tag{3.7}$$

for $T=$const. We may therefore proceed exactly as outlined in the previous section. The parameter k appears in the coefficients of V. Following the general outline in Sec. 2, we must evaluate

$$\frac{dV}{dk}=\int\int\frac{dW_a}{dk}dydz, \tag{3.8}$$

with

$$\frac{dW_a}{dk}=\tfrac{1}{2}\lambda u_1'\left(ku_1'+\frac{\partial u_2'}{\partial y}+\frac{\partial u_3'}{\partial z}\right). \tag{3.9}$$

In order to relate this expression to the energy flux, it is convenient to express it in a different form as follows.

Instead of real displacements as given by (3.1), we introduce the complex amplitude fields corresponding to the wave propagation in the negative and positive x direction. They are respectively:

$$\begin{aligned} u_i&=U_i(y,z)\exp(ikx),\\ u_i^*&=U_i^*(y,z)\exp(-ikx). \end{aligned} \tag{3.10}$$

Multiplying these expressions by the time factor $\exp(i\omega t)$ shows that they represent waves moving in opposite directions. The standing wave pattern of (3.1) is there represented by $\frac{1}{2}(u_i+u_i^*)$. From (3.2) and (3.3), we may write

$$e=\left[ikU_1+\frac{\partial U_2}{\partial y}+\frac{\partial U_3}{\partial z}\right]\exp(ikx) \tag{3.11}$$

and

$$W=\tfrac{1}{4}\lambda(e+e^*)^2. \tag{3.12}$$

The average along x is

$$W_a=\frac{k}{2\pi}\int_0^{2\pi/k}\tfrac{1}{8}\lambda(e+e^*)^2dx=\tfrac{1}{4}\lambda ee^*, \tag{3.13}$$

and we derive

$$\frac{dW_a}{dk}=\tfrac{1}{4}\lambda i[u_1e^*-u_1^*e]. \tag{3.14}$$

Introducing the stress $\sigma=\lambda e$ and forming V from (3.5), we obtain

$$\omega\frac{dV}{dk}=\int\int\frac{i\omega}{4}(u_1\sigma^*-u_1^*\sigma)dydz. \tag{3.15}$$

The integrand is the product of two complex quantities. If we refer to expression (2.15), we see that it represents the time average of the product of the stress σ by the fluid velocity $i\omega u_1$, in the x direction. In other words, the quantity

$$G=\omega dV/dk \tag{3.16}$$

in (3.15) represents the energy flowing per unit time across a plane perpendicular to the x axis. Hence we have established that for the present case the group velocity is given by relation (2.12).

With reference to the sign of the group velocity as calculated from $d\omega/dk$, it should be noted that it is ambiguous, since only ω^2 is determined as a function of k^2. However, once the sign of the group velocity is chosen, that of the phase velocity, which can be positive or negative, is determined, unless the group velocity is zero. This latter case corresponds to standing waves which must be excited through local disturbances since energy input cannot occur through propagation.

4. GROUP VELOCITY IN THE ELASTIC SOLID

Next consider an elastic solid with mass density $\rho(y,z)$ and elastic moduli $\lambda(y,z)$ and $\mu(y,z)$, both independent of x. The compressible fluid is a particular case of the present one since it corresponds to an isotropic medium with vanishing shear modulus. Any combined fluid solid system such as a fluid in a pipe or a layer of fluid on an elastic medium, etc., fall in the present case. It is convenient to denote the coordinates xyz by $x_1x_2x_3$, respectively.

The displacement field of the solid, as in the case of the fluid, is represented by the complex vector u_i given in (3.10). This vector and its complex conjugate represent waves propagating in opposite directions. The corresponding complex strain components and stresses are

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{4.1}$$

$$\sigma_{ij}=2\mu e_{ij}+\delta_{ij}\lambda e, \tag{4.2}$$

with

$$e=\sum_i e_{ii}.$$

The elastic potential energy density when u_i is real is

$$W=\tfrac{1}{2}\sum_{ij}\sigma_{ij}e_{ij}=\mu\sum_{ij}e_{ij}^2+\tfrac{1}{2}\lambda e^2. \tag{4.3}$$

When u_i is complex, the energy density of the standing wave pattern must be written

$$W=\tfrac{1}{4}\mu\sum_{ij}(e_{ij}+e_{ij}^*)^2+\tfrac{1}{8}\lambda(e+e^*)^2. \tag{4.4}$$

We derive for the average value along x_1.

$$W_a=\frac{k}{2\pi}\int_0^{2\pi/k}Wdx_1=\tfrac{1}{2}\mu\sum_{ij}e_{ij}e_{ij}^*+\tfrac{1}{4}\lambda ee^*. \quad (4.5)$$

The invariant V is then obtained by the surface integral (3.5), where W_a is expressed by (4.5). The value of T is the same as for the fluid, i.e., is given by (3.6) and is independent of k. We must now evaluate dW_a/dk.

We note that e is given by (3.11) and that

$$\begin{aligned}e_{11}&=ikU_1\exp(ikx_1),\\ e_{12}&=\tfrac{1}{2}(ikU_2+\partial U_1/\partial x_2)\exp(ikx_1),\\ e_{13}&=\tfrac{1}{2}(ikU_3+\partial U_1/\partial x_3)\exp(ikx_1).\end{aligned} \quad (4.6)$$

These are the only components of e_{ij} which contain k as a coefficient. That part of $\sum_{ij}e_{ij}e_{ij}^*$ which contains k is therefore

$$e_{11}e_{11}^*+2e_{12}e_{12}^*+2e_{13}e_{13}^*. \quad (4.7)$$

Hence

$$\begin{aligned}dW_a/dk=&\tfrac{1}{2}\mu i(u_1e_{11}^*-u_1^*e_{11})+\tfrac{1}{2}\mu i(u_2e_{12}^*-u_2^*e_{12})\\ &+\tfrac{1}{2}\mu i(u_3e_{13}^*-u_3^*e_{13})+\tfrac{1}{4}\lambda i(u_1e^*-u_1^*e).\end{aligned} \quad (4.8)$$

From (4.2) we see that this may be written

$$dW_a/dk=\tfrac{1}{4}i\sum_j(u_j\sigma_{1j}^*-u_j^*\sigma_{1j}). \quad (4.9)$$

The stress component σ_{1j} is acting in the jth direction on an element of surface perpendicular to the direction of propagation x_1. The product $\frac{1}{4}i\omega(u_j\sigma_{1j}^*-u_j^*\sigma_{1j})$ represents the average product of the stress σ_{1j} by the velocity $i\omega u_j$, and hence the power input of this component.

Therefore, we may write the total energy flux across the yz plane, as

$$G=\omega\frac{dV}{dk}=\tfrac{1}{4}i\omega\int\int\sum_j(u_j\sigma_{ij}^*-u_j^*\sigma_{ij})dydz. \quad (4.10)$$

This proves the formula (2.12) for the group velocity in the isotropic solid.

These conclusions are easily extended to the anisotropic solid.

In order to show this we must introduce a more compact mathematical language. With the usual summation convention, the stress is related to the strain by

$$\sigma_{ij}=C_{ij}^{\mu\nu}\partial u_\mu/\partial x_\nu. \quad (4.11)$$

In this expression u_i is the displacement field. There are twenty-one distinct elastic moduli satisfying the symmetry relations

$$C_{ij}^{\mu\nu}=C_{ji}^{\mu\nu}=C_{ij}^{\nu\mu}=C_{\mu\nu}^{ij}. \quad (4.12)$$

The strain energy density is

$$W=\tfrac{1}{2}C_{ij}^{\mu\nu}\frac{\partial u_\mu}{\partial x_\nu}\frac{\partial u_i}{\partial x_j}. \quad (4.13)$$

The elastic moduli and the mass density are assumed to be functions of the two coordinates x_2, x_3, only. A wave propagating in the medium is thus described again by the complex vector u_i. In this case the elastic energy density of the standing wave pattern is

$$W=\tfrac{1}{8}C_{ij}^{\mu\nu}\frac{\partial}{\partial x_\nu}(u_\mu+u_\mu^*)\frac{\partial}{\partial x_j}(u_i+u_i^*). \quad (4.14)$$

We have in this case

$$W_a=\frac{k}{2\pi}\int_0^{2\pi/k}Wdx_1=\tfrac{1}{4}C_{ij}^{\mu\nu}\frac{\partial u_\mu}{\partial x_\nu}\frac{\partial u_i^*}{\partial x_j}+\text{complex conjugate}. \quad (4.15)$$

We note that, from (3.10),

$$\begin{aligned}\partial u_i/\partial x_1&=ikU_i\exp(ikx_1)=iku_i,\\ \partial u_i/\partial x_2&=\partial U_i/\partial x_2\exp(ikx_1),\\ \partial u_i/\partial x_3&=\partial U_i/\partial x_3\exp(ikx_1).\end{aligned} \quad (4.16)$$

Hence,

$$\frac{dW_a}{dk}=\tfrac{1}{4}i\left(C_{ij}^{\mu1}u_\mu\frac{\partial u_i^*}{\partial x_j}-C_{i1}^{\mu\nu}\frac{\partial u_\mu}{\partial x_\nu}u_i^*\right). \quad (4.17)$$

Denoting by σ_{ij} the stress field associated with the displacement u_i, we derive from (4.11)

$$\begin{aligned}C_{ij}^{\mu1}\partial u_i^*/\partial x_j&=C_{\mu1}^{ij}\partial u_i^*/\partial x_j=\sigma_{\mu1}^*,\\ C_{i1}^{\mu\nu}\partial u_\mu/\partial x_\nu&=\sigma_{i1}.\end{aligned} \quad (4.18)$$

We finally write

$$dW_a/dk=\tfrac{1}{4}i(u_\mu\sigma_{\mu1}^*-u_i^*\sigma_{i1}). \quad (4.19)$$

Again it is seen that $\omega dW_a/dk$ is the power input of the stress acting on the plane $x_1=\text{const}$. Hence relation (2.13) is verified and therefore also expression (2.12) for the group velocity.

5. WAVES IN AN ELASTIC CONTINUUM UNDER PRE-STRESS

A state of pre-stress modifies the wave propagation.[11] A general theory of elasticity for a body under initial stress was developed by Biot.[11–13] The state of initial stress is denoted by S_{ij}. A perturbation produces an incremental stress defined by the symmetric tensor t_{ij}. This is not a stress in the usual sense since it refers to areas as measured immediately before incremental deformation. It is related to the incremental strain by the same relation (4.11) as above:

$$t_{ij}=C_{ij}^{\mu\nu}\partial u_\mu/\partial x_\nu, \quad (5.1)$$

with the same properties (4.12) for $C_{ij}^{\mu\nu}$.

[11] M. A. Biot, J. Appl. Phys. 2, 522 (1940).
[12] M. A. Biot, Phil. Mag. 27, 468 (1939).
[13] M. A. Biot, Z. angew. Math. Mech. 20, 89 (1940).

The incremental force acting at a boundary per *unit initial area* was found to be

$$\Delta F_\mu = [t_{i\mu} + S_{i\nu}\omega_\nu{}^\mu + \tfrac{1}{2}S_{i\nu}e_{\mu\nu} - \tfrac{1}{2}S_{\mu\nu}e_{\nu i}]a_i, \quad (5.2)$$

where a_i is the unit normal to the initial boundary and

$$\omega_\nu{}^\mu = \frac{1}{2}\left(\frac{\partial u_\mu}{\partial x_\nu} - \frac{\partial u_\nu}{\partial x_\mu}\right), \qquad e_{\mu\nu} = \frac{1}{2}\left(\frac{\partial u_\mu}{\partial x_\nu} + \frac{\partial u_\nu}{\partial x_\mu}\right). \quad (5.3)$$

It was shown[12,13] that the theory of elasticity under prestress may be treated by variational methods provided the energy density be replaced by

$$W = \tfrac{1}{2}C_{ij}{}^{\mu\nu}\frac{\partial u_\mu}{\partial x_\nu}\frac{\partial u_i}{\partial x_j} + \tfrac{1}{2}S_{\mu\nu}(\omega_\mu{}^i e_{\nu i} + \omega_\nu{}^i e_{\mu i} + \tfrac{1}{2}\omega_\mu{}^i\omega_\nu{}^i). \quad (5.4)$$

This may also be written

$$W = \tfrac{1}{2}C_{ij}{}^{\mu\nu}\frac{\partial u_\mu}{\partial x_\nu}\frac{\partial u_i}{\partial x_j} + \tfrac{1}{8}S_{\mu\nu}\left(3\frac{\partial u_i}{\partial x^\mu}\frac{\partial u_i}{\partial x^\nu} - \frac{\partial u_\mu}{\partial x_i}\frac{\partial u_\nu}{\partial x_i} - \frac{\partial u_\mu}{\partial x_i}\frac{\partial u_i}{\partial x_\nu} - \frac{\partial u_i}{\partial x_\mu}\frac{\partial u_\nu}{\partial x_i}\right). \quad (5.5)$$

We consider a medium with elastic moduli $C_{ij}{}^{\mu\nu}$ and initial stress $S_{\mu\nu}$, both functions only of the coordinates x_2, x_3, and a wave mode of the type (3.10) propagating in this medium. Expressions (5.4) and (5.5) for the energy densities are valid for real displacements u_i. In order to express the energy of the standing wave pattern in terms of the complex propagation field, we must replace u_i by $\frac{1}{2}(u_i + u_i^*)$. In averaging the energy along the coordinate x_1, only those terms remain which contain complex conjugate quantities. Hence

$$W_a = \frac{k}{2\pi}\int_0^{2\pi/k} W dx_1 = \tfrac{1}{4}C_{ij}{}^{\mu\nu}\frac{\partial u_\mu}{\partial x_\nu}\frac{\partial u_i^*}{\partial x_j} + \tfrac{1}{16}S_{\mu\nu}\left[3\frac{\partial u_i}{\partial x_\mu}\frac{\partial u_i^*}{\partial x_\nu} - \frac{\partial u_\mu}{\partial x_i}\frac{\partial u_\nu^*}{\partial x_i} - \frac{\partial u_\mu}{\partial x_i}\frac{\partial u_i^*}{\partial x_\nu} - \frac{\partial u_\mu^*}{\partial x_i}\frac{\partial u_i}{\partial x_\nu}\right] + \text{complex conjugate}. \quad (5.6)$$

We must also evaluate the derivative of this expression with respect to k. We find

$$dW_a/dk = \tfrac{1}{4}iu_\mu t_{\mu i}{}^* + \tfrac{1}{16}i\left[3S_{1\nu}u_i\frac{\partial u_i^*}{\partial x_\nu} - S_{\mu\nu}u_\mu\left(\frac{\partial u_\nu^*}{\partial x_1} + \frac{\partial u_1^*}{\partial x_\nu}\right) - S_{\mu 1}u_i\frac{\partial u_\mu^*}{\partial x_i}\right] + \text{complex conjugate}. \quad (5.7)$$

By referring to relation (5.2) for the case $a_i = (1,0,0)$, i.e., for a boundary parallel to the x_2x_3 plane, we may write

$$dW_a/dk = \tfrac{1}{4}i[u_\mu \Delta F_\mu{}^* - u_\mu{}^* \Delta F_\mu]. \quad (5.8)$$

The quantity $\omega dW_a/dk$ is the power input per unit area in the initial x_2x_3 plane of the incremental force ΔF_μ acting on this plane. Hence again we have demonstrated the formula (2.12) for the group velocity.

This applies, of course, not only to elastic solids but to waves in a gas under initial distributed pressure such as waves propagating in a horizontal direction in the atmosphere, and also to the limiting case where the waves are due to the initial stress alone such as surface waves in the ocean.

6. ANOMALOUS DISPERSION

We have not introduced any dependence of the material constants λ, μ, ρ on the frequency. Such a dependence introduces what is known as anomalous dispersion. The previous treatment may be extended to this case when the anomalous dispersion is a consequence of the existence of hidden degrees of freedom without dissipation. This is best illustrated by a simple example.

Consider a string under tension F and of mass density ρ_1 per unit length along its coordinate x. It is coupled elastically to another parallel string without tension, of mass density ρ_2 per unit length. The deflections u_1 and u_2 of the strings for harmonic motion satisfy the equations

$$\begin{aligned} Fd^2u_1/dx^2 - r(u_1 - u_2) + \omega^2\rho_1 u_1 &= 0, \\ r(u_1 - u_2) + \omega^2\rho_2 u_2 &= 0, \end{aligned} \quad (6.1)$$

where r is an elastic coupling coefficient between the strings. Eliminating u_2, we find

$$F\frac{d^2u_1}{dx^2} + \left[1 + \frac{\rho_2}{\rho_1}\left(\frac{1}{1 - \omega^2/\omega_0{}^2}\right)\right]\omega^2\rho_1 u_1 = 0, \quad (6.2)$$

with $\omega_0{}^2 = r/\rho_2$. This shows that the string behaves as if its mass ρ_1 varied with the frequency as the bracketed factor in the equation. Consider standing waves of amplitude distribution

$$u_1 = u_1' \sin kx, \quad u_2 = u_2' \sin kx. \quad (6.3)$$

Substitution in (6.2) gives

$$Fk^2 = \left[1 + \frac{\rho_2}{\rho_1}\left(\frac{1}{1 - \omega^2/\omega_0{}^2}\right)\right]\omega^2\rho_1. \quad (6.4)$$

There is a cutoff for a range of frequencies in the vicinity of ω_0 for which the bracket is negative. The phase velocity is ω/k and the group velocity v_g is found by calculating $d\omega/dk$ from the above relation. We find[14]

$$v_g=\frac{d\omega}{dk}=\frac{F_k}{\omega[\rho_1+\rho_2/(1-\omega^2/\omega_0^2)^2]}. \qquad (6.5)$$

We may, however, proceed as we have done in the previous sections. The value of V is found by averaging the value of the elastic energy per unit length. It is given by

$$2V=\frac{k}{\pi}\int_0^{\pi/k}\left[F\left(\frac{du_1}{dx}\right)^2+r(u_1-u_2)^2\right]dx$$
$$=\tfrac{1}{2}[Fk^2u_1'^2+r(u_1-u_2)^2]. \qquad (6.6)$$

The value of T is found by averaging the total kinetic energy per unit length. We find

$$2T=\tfrac{1}{2}(\rho_1u_1'^2+\rho_2u_2'^2). \qquad (6.7)$$

Proceeding as before, we evaluate

$$dV/dk=\tfrac{1}{2}Fku_1'^2. \qquad (6.8)$$

It is easily seen that $\frac{1}{2}\omega Fku_1'^2$ is the power input G into the string, since Fku_1' is the normal component of the string tension and $\omega u_1'$ the normal velocity. Hence

$$\omega dV/dk=G. \qquad (6.9)$$

Applying (2.11), we finally get

$$v_g=G/(2\omega^2T)=G/E. \qquad (6.10)$$

The group velocity is again the power input divided by the energy density E. It is important to point out that the energy density in this case must include the energy of both strings, i.e., if we consider the string under tension as the observed system the energy must also include that of the "hidden" degree of freedom represented by the coupled string. It easily verified that expression (6.10) for the group velocity gives the same value as (6.5) derived from the kinematic definition.

The procedure illustrated here on a simple example is obviously quite a general one and may be used to prove the identity of the kinematic and energy definition of the group velocity whenever the anomalous dispersion is caused by nondissipative hidden degrees of freedom in any number.

7. GROUP VELOCITY OF ELECTROMAGNETIC WAVES

In the absence of charge and current, Maxwell's equations for periodic phenomena proportional to the factor $\exp(i\omega t)$ are

$$\begin{aligned}\text{curl}\mathbf{E}&=-(i\omega/c)\mathbf{H},\\ \text{curl}\mathbf{H}&=i(K\omega/c)\mathbf{E}.\end{aligned} \qquad (7.1)$$

We assume first that the dielectric constant K is independent of the frequency and a function only of the coordinates. With a vector potential $\mathbf{A}$ defined by

$$\begin{aligned}\mathbf{E}&=-(i\omega/c)\mathbf{A},\\ \mathbf{H}&=\text{curl}\mathbf{A},\end{aligned} \qquad (7.2)$$

Maxwell's Eqs. (7.1) reduce to

$$\text{curl curl}\mathbf{A}-K(\omega^2/c^2)\mathbf{A}=0. \qquad (7.3)$$

We introduce the two invariants

$$V=\frac{1}{8\pi}\int\int\int(\text{curl}\mathbf{A})^2d\tau, \qquad (7.4)$$

$$T=\frac{1}{8\pi}\int\int\int\frac{K}{c^2}\mathbf{A}^2d\tau. \qquad (7.5)$$

These quantities correspond to the magnetic and electric energies. It is easily verified that Eq. (7.3) is equivalent to the extremum principle,

$$\delta V-\omega^2\delta T=0, \qquad (7.6)$$

for all arbitrary variations $\delta\mathbf{A}$ which vanish at the boundary. This is also equivalent to the bound extremum

$$\delta V=0, \quad \text{with} \quad T=\text{const}. \qquad (7.7)$$

This constitutes a variational formulation of the eigenvalue problem for any standing waves. We may, therefore, apply the general considerations of Sec. 1 to this case.

We now consider a dielectric system such that K is function only of x_2 and x_3. The standing waves are represented by a vector potential $\frac{1}{2}(\mathbf{A}+\mathbf{A}^*)$, where

$$\mathbf{A}=\mathbf{a}(x_2x_3)\exp(ikx_1) \qquad (7.8)$$

represents a mode of propagation in the negative direction of x_1. In analogy with the mechanical problem, we average the energy density in the x_1 direction. We may write a relation such as (3.5) with

$$W_a=\frac{k}{2\pi}\int_0^{(2\pi/k)}\frac{1}{8\pi}[\text{curl}(\mathbf{A}+\mathbf{A}^*)]^2dx_1. \qquad (7.9)$$

This may be written

$$W_a=\frac{1}{16\pi}\text{curl}\mathbf{A}\ \text{curl}\mathbf{A}^*. \qquad (7.10)$$

Components of $\mathbf{a}$ are denoted by a_i and those of $\mathbf{A}$ by A_i. The components of curl$\mathbf{A}$ are:

$$\begin{aligned}\text{curl}_1\mathbf{A}&=\left(\frac{\partial a_3}{\partial x_2}-\frac{\partial a_2}{\partial x_3}\right)\exp(ikx_1),\\ \text{curl}_2\mathbf{A}&=\left(\frac{\partial a_1}{\partial x_3}-ika_3\right)\exp(ikx_1),\\ \text{curl}_3\mathbf{A}&=\left(ika_2-\frac{\partial a_1}{\partial x_2}\right)\exp(ikx_1).\end{aligned} \qquad (7.11)$$

[14] Note that the group velocity is zero at the lower cut-off frequency ω_0 while it does not vanish at the upper cutoff.

As before, we must evaluate dW_a/dk. This gives

$$\frac{dW_a}{dk}=\frac{i}{8\pi}[A_2\,\mathrm{curl}_3\mathbf{A}^*-A_3\,\mathrm{curl}_2\mathbf{A}^*]+\text{complex conjugate.} \tag{7.12}$$

This can also be written

$$dW_a/dk=-(c/4\pi)\,\mathrm{Re}\tfrac{1}{2}[\mathbf{E}\times\mathbf{H}^*]_1. \tag{7.13}$$

The subscript 1 denotes the x_1 component of the Poynting vector.

This represents the average value of the energy flux in the negative x_1 direction.[15] Hence

$$\omega\frac{dV}{dk}=\omega\int\int\frac{dW_a}{dk}dydz=G \tag{7.14}$$

represents the energy flow across a plane perpendicular to the axis. This establishes relation (2.12) for the group velocity.

Note that we have assumed the dielectric constant K to be a function of two coordinates so that the above consideration applies to any layered material. They may be concentric layers or plane layers of any arbitrary distribution. It is also easily seen that the theorem is applicable if some of the layers are perfect conductors.

Let us now extend this to the case where the dielectric constant is not only a function of the coordinates but also of the frequency, i.e., to the case of anomalous dispersion. We shall, however, assume as above that the anomalous dispersion arises from nondissipative hidden coordinates. Such is the case, for instance, if the wave propagates in a medium where electrons are continuously distributed as harmonic oscillators. It is assumed that they are attached to fixed positive charges of equal magnitude and that the charges of opposite sign coincide in the undisturbed state. The displacement of the electrons from their equilibrium is elastically restrained and the application of an electric field produces a dipole. The natural frequency of an isolated electron oscillator of mass m is denoted by ω_0. If n is the number of electrons per unit volume in the undisturbed state, then $\rho=-ne$ is the electrical charge density in this initial state. We denote the electron displacement field by $\mathbf{u}$. Because of the appearance of first-order time derivatives, it is convenient to use the time differential operator $p=\partial/\partial t$ or $i\omega$. Equations for the coupled mechanical and electromagnetic fields are

$$\begin{aligned}&\mathrm{curl}\mathbf{E}=-(p/c)\mathbf{H},\\&\mathrm{curl}\mathbf{H}=(p/c)\mathbf{E}+(4\pi/c)\rho p\mathbf{u},\\&\mathrm{div}\mathbf{H}=0,\\&nm(p^2+\omega_0^2)\mathbf{u}=\rho\mathbf{E}.\end{aligned} \tag{7.15}$$

We shall consider the modes of propagation in such a system, assumed to behave as a continuum, for the general case where the two constants ρ and ω_0 are functions of two coordinates x_2, x_3. The propagation is considered along x_1. The first and third equations are satisfied by introducing a vector potential following Eqs. (7.2). This leads to the two equations

$$\begin{aligned}&\frac{1}{4\pi}\,\mathrm{curl}\;\mathrm{curl}\mathbf{A}+\frac{p^2}{4\pi c^2}\mathbf{A}-\frac{\rho}{c}p\mathbf{u}=0,\\&nm(p^2+\omega_0^2)\mathbf{u}+\frac{\rho}{c}p\mathbf{A}=0.\end{aligned} \tag{7.16}$$

Eliminating $\mathbf{u}$, we find

$$\mathrm{curl}\;\mathrm{curl}\mathbf{A}+\frac{p^2}{c^2}\left[1+\frac{\omega_p^2}{p^2+\omega_0^2}\right]\mathbf{A}=0. \tag{7.17}$$

The constant ω_p^2 is

$$\omega_p^2=4\pi ne^2/m, \tag{7.18}$$

where ω_p represents the "plasma" frequency of an electron gas of charge density ρ.[16,17] Note that Eq. (7.17) does not represent a simple eigenvalue problem since we are dealing with a nonhomogeneous medium, i.e., the equivalent dielectric constant is at the same time a function of the frequency and of the two coordinates x_2, x_3. The interaction of matter and radiation in equations (7.16) is represented by the antisymmetric coupling terms $-(\rho/c)p\mathbf{u}$ and $(\rho/c)p\mathbf{A}$ of the gyrostatic type. However, we may represent the equations by another equivalent system using another vector ξ as auxiliary variable. We write

$$\begin{aligned}&\mathrm{curl}\;\mathrm{curl}\mathbf{A}+\frac{p^2}{c^2}\mathbf{A}+\frac{\omega_p^2}{c^2}(\mathbf{A}-\xi)=0,\\&\frac{p^2}{c^2}\frac{\omega_p^2}{\omega_0^2}\xi-\frac{\omega_p^2}{c^2}(\mathbf{A}-\xi)=0.\end{aligned} \tag{7.19}$$

By eliminating ξ between these equations, we obtain Eq. (7.17). The eigenvalues are, therefore, the same as for the original Eqs. (7.16). The value of ξ is related to $\mathbf{u}$ and, by comparing the second equations of each system, we find

$$\mathbf{u}=p(e/mc\omega_0^2)\xi. \tag{7.20}$$

The system (7.19) contains only p^2 and represents standing waves of real amplitude $\mathbf{A}$ and ξ, all in phase. Relation (7.20) shows that $\mathbf{u}$ is 90° out of phase with ξ as it should be. There obviously is a group of linear transformations of $\mathbf{A}$ and $\mathbf{u}$ which yields the same eigenvalue of ω. The particular combination $\mathbf{A}$, ξ is chosen because it suits our purpose here. Note that the transformation is not Hermitian.

[15] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, New York, 1939), p. 137.

[16] I. Langmuir, Phys. Rev. 33, 954 (1929).

[17] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956).

It is readily seen that the variational form of Eqs. (7.19) is obtained as in the previous problems by putting

$$V=\frac{1}{8\pi}\int\int\int\left[(\mathrm{curl}\,\mathbf{A})^2+\frac{\omega_p{}^2}{c^2}(\mathbf{A}-\xi)^2\right]d\tau,$$
$$T=\frac{1}{8\pi}\int\int\int\frac{1}{c^2}\left(\mathbf{A}^2+\frac{\omega_p{}^2}{\omega_0{}^2}\xi^2\right)d\tau. \tag{7.21}$$

Then, proceeding exactly as for (7.14), it is shown that $\omega dV/dk$ is the energy flux. However, in the present case we must also show that $2\omega^2T$ is the energy density. From the general properties outlined in Sec. 2, we write $\omega^2T=V$; hence

$$2\omega^2T=V+\omega^2T. \tag{7.22}$$

Now if we consider only absolute amplitudes, we may write (7.18) as

$$\xi=(mc\omega_0{}^2/e\omega)\mathbf{u}. \tag{7.23}$$

Substituting this value into V and T in Eq. (7.21) and using (7.19), we find

$$V+\omega^2T=\frac{1}{8\pi}\int\int\int\left[(\mathrm{curl}\,\mathbf{A})^2+\frac{\omega^2}{c^2}\mathbf{A}^2\right]d\tau$$
$$+\frac{1}{2}\int\int\int nm(\omega^2+\omega_0{}^2)\mathbf{u}^2d\tau. \tag{7.24}$$

The first line is the energy of the electromagnetic field and the second line the mechanical energy stored in the oscillators. Expression (2.12) for the group velocity is thereby established.

It is of considerable interest to compare Eqs. (7.17) and (7.19) with Eqs. (6.1) and (6.2) for the propagation of waves in the string with dispersion. In fact, for propagation in a homogeneous medium they become identical. This indicates that the propagation in the string constitutes a model for a very general class of dispersive phenomena. In fact, the analogy may be extended easily much further to include propagation in a medium with boundaries, i.e., with geometric dispersion by adding additional features such as lateral elastic restraint.

Another interesting point is brought out by considering the particular case of wave propagation in an electron gas. This case is obtained by canceling the elastic restraint of the electrons, i.e., putting $\omega_0=0$. Equation (7.19) for the propagation then becomes

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{A}+(1/c^2)(p^2+\omega_p{}^2)\mathbf{A}=0. \tag{7.25}$$

The vector $\mathbf{A}$ may be separated into a solenoidal part and a gradient. The first represents wave propagation and the second the plasma oscillations.

Consideration of a plane wave with $\mathbf{A}$ parallel to one axis and proportional to $\exp(ikx+i\omega t)$ leads to the relation:

$$-c^2k^2+\omega^2=\omega_p{}^2. \tag{7.26}$$

For infinite wavelength, $k=0$, the frequency becomes equal to the plasma frequency ω_p. The propagation is dispersive. Differentiating (7.26), we find that the phase and group velocities v_p and v_g satisfy the relation

$$v_pv_g=c^2. \tag{7.27}$$

This is the same relation as that satisfied in a perfect wave guide for modes of propagation of reflected waves. It is also the same as for de Broglie waves of the free mass particle. As in the latter case, Eq. (7.27) is also a direct consequence of the Lorentz transformation. This points to an analogy between plane waves in an electron gas and the wave mechanical behavior of mass particles. Putting $p^2=-\omega^2=-4\pi^2m^2c^4/h^2$ and $\omega_p{}^2=4\pi^2m_0{}^2c^4/h^2$, Eq. (7.25) becomes the Schrödinger equation for a particle of relativistic mass m and rest mass m_0. The plasma frequency ω_p is nothing but the characteristic frequency associated with the rest mass m_0. The analogy may be carried out much further and may very possibly have deeper implications, but we shall not elaborate on it beyond these few remarks, since it reaches beyond the scope of the present article. It is also of interest to point out that the mechanical model of the string under tension with lateral elastic restraint analyzed in Sec. 6 is a model for the relativistic Schrödinger equation of the mass particle.