THE INFLUENCE OF GRAVITY ON THE FOLDING OF A LAYERED VISCOELASTIC MEDIUM UNDER COMPRESSION

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ABSTRACT

The theory of folding of a layered viscoelastic medium under a horizontal compression parallel with the layer, as developed in previous publications (**1**, **2**, **3**),[2] is extended to take into account the influence of gravity. Characteristic features of the folding are evaluated in terms of nondimensional parameters. The influence of gravity appears through a parameter which depends on the relative magnitude of the gravity forces and the horizontal compression. Specific cases discussed are that of a layer on top of a half-space and the layer embedded between two half-spaces, with the bottom medium either denser or less dense than the top one.

1. INTRODUCTION

In references 1 and 2 we have investigated the folding due to instability when a layered viscoelastic medium is subject to a compression in a direction parallel with the layer. The layer was assumed to be embedded in an indefinite solid or lying on the surface of a half-space. The problem was analyzed from a very general viewpoint and the viscoelastic and hereditary characteristics of the material were assumed to obey the most general linear laws. In particular, conclusions are drawn for materials which obey the Onsager relations and the laws of the thermodynamics of irreversible processes as formulated in some our previous work (**7**, **8**). The folding was also evaluated for various specific combinations of viscoelastic media, and attention was given to the effect of interfacial friction at the boundary and to the evaluation of the magnitude of the folding as distinct from mere instability (**2**). In the cited work we used an approximation of the "plate theory" type to represent the folding of the layer. In reference 3 this approximation was discarded and the problem was treated by using exact equations of the stability of a prestressed continuum as developed in earlier papers (**4**, **5**, **6**, **7**). The problem of stability of the surface of a viscoelastic half-space was also treated as a particular case.

In the present analysis we are dealing with the problem of folding of a layer, either lying on top of a viscoelastic half-space, or embedded between two such half-spaces. When in addition to a compression parallel with the direction of the layer there is also a uniform gravity field perpendicular to this direction, we are therefore dealing with in-

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

stability under a combined state of prestress—the horizontal compression in the layer, and a prestress due to gravity which is assumed to be a hydrostatic field. The addition of gravity to the problem is essential of course if we have in mind some of the geological applications. We shall discuss these at a later date.

A general formulation of the problem is outlined in Section 2. From the previously established theory (4, 5, 6, 7), we derive the equations for the stability problem of a continuum under a uniform compression and a hydrostatic stress gradient due to gravity. These equations are quite general and may be applied to both compressible and incompressible media. For simplicity we have applied the theory only to incompressible media and in Section 3 we show that in this case simplifications may be introduced along with a "plate type" equation for the layer. This leads to a very simple equation for the folding of a layer lying on top of a half-space. The case of a layer and underlying medium which are incompressible and purely viscous is discussed in Section 4. The dominant wave length (that is, that whose amplitude grows at the fastest rate) is evaluated as a function of the ratio of the viscosity coefficients of the two media and a nondimensional parameter which represents the influence of gravity. This parameter is a ratio of two quantities, one is the pressure of a column of underlying material of height equal to the layer thickness and the other the compressive horizontal force. The density of the layer itself does not enter into the picture because of our assumption that the layer behaves like a thin plate of constant thickness. The relative amplitude of the folding for the dominant wave length is also evaluated. This yields an indication of the magnitude of the instability.

Section 5 discusses the case of a layer embedded between two different materials. The materials are again assumed incompressible and purely viscous. Two cases must be considered. In the first case the underlying material is denser and the effect of gravity is stabilizing. The theory is the same as for the layer lying on top of the half-space except for modified parameters which introduce the sum of the viscosities of top and bottom media and the difference of their densities. The effect of gravity tends to shorten the dominant wave length. The second case is that of an inverse density gradient, that is, the underlying material is lighter than the top one. The layer is then unstable because of gravity alone. The phenomenon then becomes somewhat more complex. For certain ranges of the parameters the combined effect of gravity and compression will increase the dominant wave length while in another range the dominant wave length will be controlled by geometric factors such as the depth of the surrounding medium. It should be kept in mind that we have not taken into consideration any inertia forces. The deformations are therefore assumed to be sufficiently slow for these forces to be negligible.

Strictly speaking, the theory is also restricted to infinitesimal deformations but in practice it is of course applicable to a large category of observable phenomena in the same sense as this is understood to be true in the applications of the classical linear Theory of Elasticity. Furthermore it is most likely that the folding wave length which appears in the incipient phase of the instability will also correspond to the characteristic feature in the later phases with large deformations. An experimental program of model tests on the folding of layered viscoelastic media has been initiated. Results are in good agreement with the theory. They will be presented in later publications.

2. GENERAL EQUATIONS FOR THE DEFORMATION OF A PRESTRESSED MEDIUM

In reference 3 we have analyzed the folding instability under compression of a layered viscoelastic medium by using the exact equations for the deformation of an anelastic continuum under prestress. These equations are the same as those which we established in references 4, 5 and 6 for the elastic continuum. The general applicability of these equations to anelastic media was discussed in reference 7. A brief rederivation of the equations was also presented in reference 3. In the previous work we did not introduce the gravity force. In the following we shall consider the prestress to result from both the lateral compressive forces on the layered material and the hydrostatic pressure gradient due to its weight. In the present section we shall write out the equations and boundary conditions for this particular state of prestress. We shall also restrict ourselves to two-dimensional deformations. As established in the work quoted above the incremental stress field of the continuum satisfies the equations

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \rho Y \omega - 2S_{12}\frac{\partial \omega}{\partial x} + (S_{11} - S_{22})\frac{\partial \omega}{\partial y} + \frac{\partial S_{11}}{\partial x} e_{yy} - \left(\frac{\partial S_{11}}{\partial y} + \frac{\partial S_{12}}{\partial x}\right) e_{xy} + \frac{\partial S_{12}}{\partial y} e_{xx} = 0 \quad (1)$$

and another similar equation obtained by permuting x and y, 1 and 2 and changing ω into $-\omega$. The coordinate system is assumed to be clockwise. In these equations S_{11}, S_{22}, S_{12} are the initial stress components in the x, y plane, s_{11}, s_{22}, s_{12} are the incremental stresses, and ρY is the y component of the body force per unit volume (ρ mass density). The displacement field u, v yields the strain components

$$e_{xx} = \frac{\partial u}{\partial x} \qquad e_{yy} = \frac{\partial v}{\partial y} \quad (2)$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

and the rotation

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{3}$$

In this representation the stress components s_{ij} are the incremental stresses at the displaced point $x + u$, $y + v$, and relative to axes rotated by an angle ω. The strain components e_{ij} are also referred to the same displaced and rotated axes. The latter are of course not distinct, in the first order, from the classical components for small strain.

Consider now a particular case of prestress, namely a combination of a horizontal compressive stress P and a hydrostatic stress due to gravity. The gravity force is assumed parallel with the y axis. In that case

$$Y = g \tag{4}$$

where g is acceleration of gravity and the initial state of stress is

$$\begin{aligned} S_{11} &= -P - \rho g y \\ S_{22} &= -\rho g y \\ S_{12} &= 0. \end{aligned} \tag{5}$$

The origin of the y direction is left arbitrary which is equivalent to adding an arbitrary uniform hydrostatic stress in order to suit the boundary conditions. Equations 1 to be satisfied by the incremental stress field then become

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \rho g \frac{\partial v}{\partial x} - P\frac{\partial \omega}{\partial y} &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - \rho g \frac{\partial u}{\partial x} - P\frac{\partial \omega}{\partial x} &= 0. \end{aligned} \tag{6}$$

The boundary conditions may be formulated using the following relations (4, 5, 6):

$$\begin{aligned} dF_x = &-[s_{12} + S_{12} - S_{22}\omega + S_{12}e_{xx} - S_{11}e_{xy}]dx \\ &+ [s_{11} + S_{11} - S_{12}\omega + S_{11}e_{yy} - S_{12}e_{xy}]dy \\ dF_y = &-[s_{22} + S_{22} + S_{12}\omega + S_{22}e_{xx} - S_{12}e_{xy}]dx \\ &+ [s_{12} + S_{12} + S_{11}\omega + S_{12}e_{yy} - S_{22}e_{xy}]dy \end{aligned} \tag{7}$$

where dF_x and dF_y are the forces acting on an element of boundary initially represented by dx, dy. The force $d\bar{F}$ is acting on matter lying on the left side of the arc dx, dy, considered as a vector differential. For the particular prestress field under consideration, Eq. 5, these relations become

$$dF_x = -\left[s_{12} + Pe_{xy} + \rho g y \frac{\partial v}{\partial x}\right] dx + \left[s_{11} - (P + \rho g y)\left(1 + \frac{\partial v}{\partial y}\right)\right] dy \quad (8)$$

$$dF_y = -\left[s_{22} - \rho g y\left(1 + \frac{\partial u}{\partial x}\right)\right] dx + \left[s_{12} - P\omega + \rho g y \frac{\partial u}{\partial y}\right] dy.$$

We must also express relations between the incremental stresses and deformations. If we assume that they are of the same form as for an unstressed medium, we write

$$s_{ij} = 2\bar{Q}e_{ij} + \delta_{ij}\bar{R}e \quad (e = e_{xx} + e_{yy}) \quad (9)$$

with the operators $\left(p = \frac{d}{dt}, \text{ time derivative}\right)$,

$$\begin{aligned} \bar{Q} &= \int_0^\infty \frac{p}{p + r} Q(r)dr + Q + Q'p \\ \bar{R} &= \int_0^\infty \frac{p}{p + r} R(r)dr + R + R'p. \end{aligned} \quad (10)$$

We have previously derived these expressions from irreversible thermodynamics (8). Strictly speaking their use for prestressed media may involve an approximation. The nature of the limitations involved was discussed in reference 2. However, in most problems these limitations are only of academic interest.

Substitution of expression 9 for the stresses s_{ij} into Eqs. 6 yields two operational equations for the displacements u and v. The problem of deformation and folding of a layer lying on a semi-infinite medium both of heavy viscoelastic material and subject to a horizontal compression is thus reduced to solving the two simultaneous equations for u and v with appropriate boundary conditions. This can be done in a way entirely analogous to the procedure followed in reference 3. The exact solution however is somewhat involved and we have preferred to follow an approximate but much simpler approach in analogy with our treatment of the problem in reference 1.

3. APPROXIMATE SOLUTION OF THE PROBLEM

A viscoelastic layer lies on a semi-infinite medium also viscoelastic of mass density ρ_1. A force of gravity g per unit mass acts on both media and the layer is subject to a horizontal compressive stress (Fig. 1). The following assumptions are introduced:

(*a*). The layer behaves like a plate and obeys the simplified equations of the plate theory.

(*b*). The hydrostatic prestress due to gravity is the only prestress in the underlying medium.

(*c*). Interfacial adherence of the layer and medium is neglected.

(*d*). Both materials are assumed incompressible.

(*e*). The weight of the layer has no effect on the folding, i.e., the layer density may be assumed equal to zero.

The justification of assumptions (*a*) and (*b*) is established by the results in reference 3 which indicate that the factors neglected are not too significant for practical purpose. The same remark holds for assumption (*c*) on the basis of the analysis carried out in reference 2 by taking into account the adherence. Assumption (*e*) is an approximation which for physical reasons may be considered to follow from (*a*) since the plate bending occurs with no variation in thickness.

It may be expected that the assumption of incompressibility does not restrict the results significantly while providing the advantage of simplicity in the mathematical treatment.

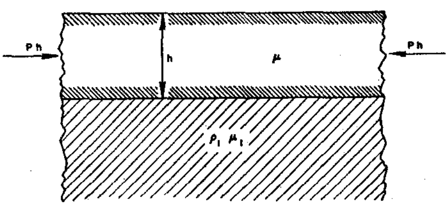

FIG. 1. Compressed layer lying on top of a heavy medium.

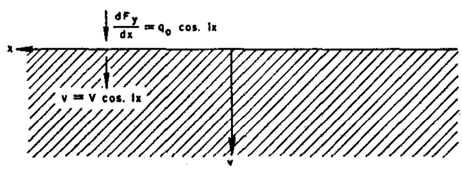

FIG. 2. Coordinates, surface forces, and deflections for the half space.

Consider first the underlying medium as an infinite half-space, the surface lying at $y = 0$, and the y axis being directed downward (Fig. 2). Since we neglect any initial horizontal compression P in the underlying medium, the equations for the displacement components u and v are obtained by putting $P = 0$ and $\rho = \rho_1$, in Eq. 6, hence,

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \rho_1 g \frac{\partial v}{\partial x} &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - \rho_1 g \frac{\partial u}{\partial x} &= 0. \end{aligned} \tag{11}$$

Since we are dealing with an incompressible material, we may write

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{12}$$

Introducing this relation in Eq. 11, they become

$$\begin{aligned}\frac{\partial}{\partial x}(s_{11}+\rho_1\rho v)+\frac{\partial s_{12}}{\partial y}&=0\\ \frac{\partial s_{12}}{\partial x}+\frac{\partial}{\partial y}(s_{22}+\rho_1 g v)&=0.\end{aligned}\tag{13}$$

The boundary conditions (8) at the surface $y = 0$, when the prestress is zero ($P = 0$), become

$$\begin{aligned}\frac{dF_x}{dx}&=-s_{12}\\ \frac{dF_y}{dx}&=-s_{22}.\end{aligned}\tag{14}$$

The operators $\bar{R}$ and $\bar{Q}$ of the stress-strain relations (9) defining the viscoelastic properties of the underlying medium are designated by $\bar{R}_1$ and $\bar{Q}_1$. For an incompressible material we put (8) $\bar{R}_1 = \infty$, $e = 0$, and

$$s = \bar{R}_1 e. \tag{15}$$

Relations (9) then become

$$s_{ij} - \delta_{ij} s = 2\bar{Q}_1 e_{ij}. \tag{16}$$

If we write

$$\begin{aligned}s'&=s+\rho_1 g v\\ s_{ij}'&=s_{ij}+\delta_{ij}\rho_1 g v,\end{aligned}\tag{17}$$

Eqs. 13 are transformed into

$$\begin{aligned}\frac{\partial s_{11}'}{\partial x}+\frac{\partial s_{12}'}{\partial y}&=0\\ \frac{\partial s_{12}'}{\partial x}+\frac{\partial s_{22}'}{\partial y}&=0.\end{aligned}\tag{18}$$

Relations (16) become

$$s_{ij}' - \delta_{ij} s' = 2\bar{Q}_1 e_{ij} \tag{19}$$

and the boundary condition is

$$\begin{aligned}\frac{dF_x}{dx}&=-s_{12}'\\ \frac{dF_y}{dx}-\rho_1 g v&=-s_{22}'.\end{aligned}\tag{20}$$

Equations 18 and 19 are those for a medium without initial stress. The medium thus behaves as though there were no initial stress except for a modified boundary condition 20 which contains an additional "buoyancy term"—$\rho_1 g v$.

The problem of surface deflection of a viscoelastic half-space under a sinusoidal vertical load distribution was solved in references 1 and 2. For the case of an incompressible material, a vertical load distribution

$$\frac{dF_y}{dx} = q_0 \cos lx \tag{21}$$

produces a normal deflection $v = V \cos lx$ (Fig. 2) related to the load by

$$q_0 = 2\bar{Q}_1 l V. \tag{22}$$

It was also found that in the absence of a horizontal force at the surface ($F_x = 0$) the horizontal displacement at the surface also vanishes for an incompressible material. In order to take gravity into account we have just shown that we must replace the boundary force $\frac{dF_y}{dx}$ by $\frac{dF_y}{dx} - \rho_1 g v$; hence we must replace q_0 by $q_0 - \rho_1 g V$. Relation 22 then becomes

$$q_0 = (2\bar{Q}_1 l + \rho_1 g) V. \tag{23}$$

Let us now consider the layer of thickness h. We assume that it bends as a viscoelastic plate under an axial compression P and a transverse vertical load,

$$q = -q_0 \cos lx \tag{24}$$

equal and opposite in sign to the load applied to the underlying medium. We have shown in reference 1 that the plate deflection v satisfies the equation

$$\bar{B}\frac{h^3}{12}\frac{d^4 v}{dx^4} + Ph\frac{d^2 v}{dx^2} = q \tag{25}$$

where $\bar{B}$ is the operator

$$\bar{B} = \frac{4\bar{Q}(\bar{Q} + \bar{R})}{2\bar{Q} + \bar{R}} \tag{26}$$

defining the viscoelastic properties of the layer. For an incompressible material $R = \infty$ and we have

$$\bar{B} = 4\bar{Q}. \tag{27}$$

With a sinusoidal deflection $v = V \cos lx$ after introducing expressions 22 and 24 for the load, Eq. 25 becomes

$$\tfrac{1}{3}\bar{Q}h^3l^4 - Phl^2 + 2\bar{Q}_1 l + \rho_1 g = 0. \tag{28}$$

This is the basic equation of the problem. We note that the mass density of the layer does not appear in this equation. This is due primarily to our assumption that the deformation of the layer obeys the plate bending equation and therefore does not undergo a change of thickness. Actually of course a more accurate treatment would show an influence of the layer density, but the correction due to this effect may be assumed to be small.

In the next section we shall discuss the significance of Eq. 28 in relation to the folding of the layer under the simultaneous action of the axial compression P and gravity.

4. FOLDING OF A VISCOUS LAYER LYING ON A HEAVY VISCOUS MEDIUM

We shall apply Eq. 28 for an incompressible medium to the particular case of a purely viscous layer under a horizontal compression P lying on top of another heavy viscous fluid of infinite depth. The viscosity coefficient of the layer is denoted by μ, that of the underlying medium by μ_1, and the mass density of the latter by ρ_1. The corresponding operators are

$$\begin{aligned}\bar{Q} &= \mu p \\ \bar{Q}_1 &= \mu_1 p.\end{aligned} \tag{29}$$

Relation 28 may then be written

$$\frac{\tfrac{1}{3}l^2h^2 + 2\dfrac{\mu_1}{\mu}\dfrac{1}{lh}}{1 - \dfrac{1}{l^2h^2}\dfrac{\rho_1 gh}{P}} = \frac{P}{\mu p}. \tag{30}$$

The physical interpretation of this relation lies in the significance of the variable p considered as an algebraic quantity. As pointed out in our previous work (1, 2, 3) if relation 30 is satisfied, any sinusoidal folding of wave length

$$L = 2\pi/l \tag{31}$$

has an amplitude increasing with time t, proportionally to the factor e^{pt}. We have called the dominant wave length L_d that which increases at the maximum rate. This dominant wave length corresponds to the minimum value of $P/\mu p$ on the right hand side of Eq. 30. This minimum depends on two parameters, μ/μ_1 and $\rho_1 gh/P$. The latter represents the influence of gravity, as the ratio of the gravity forces to the compressive load P. The relative importance of gravity is therefore dependent on the magnitude of the horizontal compression.

We note that only the density ρ_1 of the underlying medium appears in the expression for the gravity parameter. As already pointed out this is due to our assumption that the layer behaves like a plate in flexure and therefore that there is no change in thickness. The dominant wave length L_d of the folding corresponds to values of lh for which expression 30 is a minimum. Denoting by $l_d h$ this value of lh, it is related to the dominant wave length by the relation

$$\frac{L_d}{h} = \frac{2\pi}{l_d h}. \tag{32}$$

The value of $l_d h$ is plotted as a function of $\sqrt[3]{\frac{3\mu_1}{\mu}}$ in Fig. 3 for various

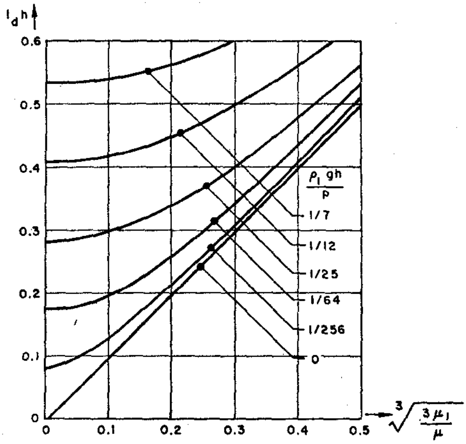

FIG. 3. The combined effect of gravity, compressive load, and viscosity on the value of $l_d h$.

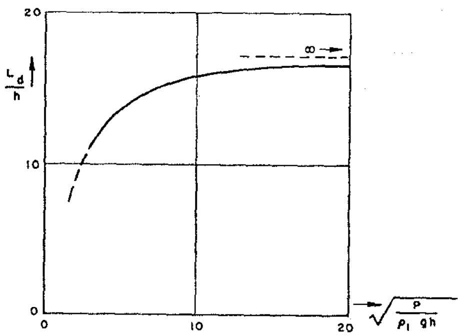

FIG. 4. The combined effect of gravity and compressive load on the dominant wave length.

values of the parameter $\rho_1 gh/P$. The case $\rho_1 gh/P = 0$ corresponds to the absence of gravity. In that case we find the straight line

$$l_d h = \sqrt[3]{\frac{3\mu_1}{\mu}}. \tag{33}$$

This is identical with the relation found in reference 1 except for the factor 3 in the present formula instead of 6. This is due to the fact that the formula of reference 1 applies to a layer embedded in a medium on both top and bottom while the present case deals with a layer lying on the surface of a semi-infinite medium. Clearly the latter case amounts to the former if we divide by two the viscosity of the embedding medium. For $\mu_1 = 0$, that is if the underlying medium is a fluid without viscosity, the problem reduces to that of folding of a viscous plate laterally restrained only by the buoyancy of the underlying fluid. This amounts to putting $\mu_1 = 0$ in Eqs. 28 and 30. We derive

$$\frac{\mu p}{P} = \frac{3}{l^2h^2} - \frac{3\rho_1 gh}{P}\frac{1}{l^4h^4}. \tag{34}$$

The maximum of this expression as a function of lh yields the value corresponding to the dominant wave length, that is,

$$l_d h = \sqrt{\frac{2\rho_1 gh}{P}}. \tag{35}$$

These are the values plotted along the vertical axis at zero abscissa in Fig. 3. From Eq. 35 we may write for the dominant wave length due to gravity alone

$$L_d = \pi h \sqrt{\frac{2P}{\rho_1 gh}}. \tag{36}$$

For example if $P = g\rho_1 gh$, that is, if the compressive load is nine times the weight of a column of the underlying material of height equal to the layer thickness we find $L_d = 13.4 \times h$, that is, the dominant wave length due to gravity alone is about thirteen times the layer thickness.

At this point it is of interest to examine the magnitude of the instability. Following a procedure identical with that used in the previous work (2, 3) let us evaluate the factor by which the amplitude of the dominant wave length is multiplied for a period of application of the compressive load P during which the layer would undergo a compression of 25 per cent.[3] This time t_1 satisfies the relation

$$Pt_1 = 4\mu\epsilon = \mu \tag{37}$$

where the compressive strain is

$$\epsilon = \tfrac{1}{4}. \tag{38}$$

We shall examine only the case when the underlying medium has no viscosity ($\mu_1 = 0$). The amplification factor is $\exp(pt_1)$ with

$$pt_1 = \frac{3}{4}\frac{P}{\rho_1 gh}. \tag{39}$$

[3] It is of course understood that the theory is not rigorously applicable to such large deformations and the figures are only given as an indication of orders of magnitude.

In Table I we have shown the values of the amplification factor in terms of the gravity parameter.

TABLE I.

$\rho_1 gh/P$	$\exp(pt_1)$
1	2.11
1/4	20
1/7	190
1/9	850

It is seen that significant amplification occurs only for $P/\rho_1 gh \gtrsim 7$. A discussion of the amplification factor in the absence of gravity was given in references 2 and 3, and it was found that significant amplification occurred only if $\mu/\mu_1 \gtrsim 70$ that is, if $\sqrt[3]{\frac{3\mu_1}{\mu}} \lesssim 0.35$. The range of the parameters shown in Fig. 3 corresponds to a domain for which the amplification factor is significant.

The combined effect of gravity and viscosity is completely represented by the diagram of Fig. 3. We notice that the effect of gravity dominates in regions where the wave length due to gravity alone is larger than that due to viscosity alone as determined from Eq. 33. For example, assuming a viscosity ratio $\mu/\mu_1 = 64$, we find $\sqrt[3]{\frac{3\mu}{\mu_1}} = 0.37$. The ratio L_d/h of dominant wave length to layer thickness for this case is plotted in Fig. 4 as a function of $\sqrt{\frac{P}{\rho_1 gh}}$. We see that contrary to the case where gravity is neglected the wave lengths depends on the compression P. For large values of P the ratio tends to $L_d/h = 17.2$. For the smaller values of the compressive load this ratio drops to about 12, as may be seen from the discussions in references 2 and 3. Below this value the magnitude of the instability is such that it loses its physical significance, as shown by the values of the amplification factor in Table I.

This means that in order to exhibit appreciable folding a very thick layer may require a compressive load beyond the physically possible range. This conclusion should be of significance in geological applications.

5. FOLDING OF A LAYER LYING BETWEEN TWO HEAVY MEDIA

In the previous section we have considered the layer to lie on top of another medium of infinite depth. We shall now consider the case where another medium of different density and viscosity lies on top of the layer. For simplicity the discussion is restricted to materials which are purely viscous and incompressible. We designate by ρ_1 and

μ_1 the mass density and viscosity of the bottom medium, and by ρ_2 and μ_2 the corresponding quantities for the medium lying on top. The layer of viscosity μ and thickness h under a horizontal compression P lies between those two media (Fig. 5).

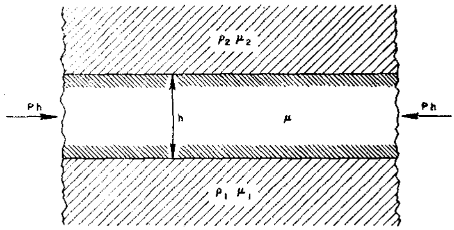

FIG. 5. Compressed layer lying between two heavy media.

We may again write Eq. 25 for the layer flexure. However in the expression of the total transverse load on the layer $q = q_0 \cos lx$ we must take into account the restraint due to top and bottom media. We therefore write in place of Eq. 23,

$$q_0 = [2(\bar{Q}_1 + \bar{Q}_2)l + (\rho_1 - \rho_2)g]V. \tag{40}$$

Since the materials are assumed purely viscous, the operators are

$$\begin{aligned} \bar{Q}_1 &= \mu_1 p \\ \bar{Q}_2 &= \mu_2 p. \end{aligned} \tag{41}$$

The sum $\bar{Q}_1 + \bar{Q}_2$ reflects the fact that the restraint of the two media on both sides of the layer is additive while $\rho_1 - \rho_2$ corresponds to the fact that the effect of gravity is subtractive. Equation 28 for the stability is therefore replaced by

$$\tfrac{1}{3}\bar{Q}h^3l^4 - Phl^2 + 2(\bar{Q}_1 + \bar{Q}_2)l + (\rho_1 - \rho_2)g = 0 \tag{42}$$

and Eq. 30 is replaced by

$$\frac{\tfrac{1}{3}l^2h^2 + 2\dfrac{\mu_1 + \mu_2}{\mu}\dfrac{1}{lh}}{1 - \dfrac{1}{l^2h^2}(\rho_1 - \rho_2)\dfrac{gh}{P}} = \frac{P}{\mu p}. \tag{43}$$

The dominant wave length is determined as before by evaluating the value of lh which minimizes this quantity. The equations are formally identical with those of the previous section except that the parameter μ_1/μ is replaced by $\dfrac{\mu_1 + \mu_2}{\mu}$ and $\rho_1 gh/P$ is replaced by $(\rho_1 - \rho_2)\dfrac{gh}{P}$.

We must distinguish two cases. In one case the bottom material is heavier than the top one, that is, $\rho_1 - \rho_2$ is positive. The dominant wave length is then determined by the same diagrams as in Figs. 3 and 4 where μ_1/μ is replaced by $(\mu_1 + \mu_2)/\mu$ and $\rho_1 gh/P$ by $(\rho_1 - \rho_2)gh/P$. In the other case the bottom material is lighter than the top one, that is, $\rho_1 - \rho_2$ is negative. This case differs essentially from the previous one in that an instability due to gravity alone occurs in the absence of any compressive stress in the layer.[4] In order to understand this more clearly let us write relation 43 in the form

$$p = \frac{Pl^2h^2 + (\rho_2 - \rho_1)gh}{lh[\frac{1}{3}\mu l^3h^3 + 2(\mu_1 + \mu_2)]}. \tag{44}$$

As pointed out above, this value of p gives a folding amplitude proportional to $\exp(pt)$ for the wave length $L = 2\pi/l$.

We see immediately that $p = \infty$, for $l = 0$, that is, the rate of folding increases indefinitely with the wave length. The reader will note that we have assumed the deformation to be very small, therefore inertia forces have been neglected. Hence, theoretically the dominant wave length is infinite. In an actual situation the surrounding medium is not infinite and the dominant wave length will be restricted by the thickness of the surrounding material. This result is of significance in geology as it is related to the formation of salt domes.

There is however the possibility of occurrence of a secondary dominant wave length due primarily to the compression P. Obviously if the effect of gravity disappears, for example if $\rho_1 = \rho_2$, there is a maximum value of p for lh equal to

$$l_d h = \sqrt[3]{\frac{3(\mu_1 + \mu_2)}{\mu}}. \tag{45}$$

This relation is found by replacing in Eq. 33 $\frac{\mu_1}{\mu}$ by $(\mu_1 + \mu_2)/\mu$. When gravity enters into the picture this maximum is displaced. This can be shown by writing Eq. 44 in the form

$$\psi = \frac{\delta^2 + \kappa}{\delta(\delta^3 + 2)} \tag{46}$$

with

$$\delta = lh\left[\frac{1}{3}\frac{\mu}{\mu_1 + \mu_2}\right]^{1/3} \tag{47}$$

[4] The stability of a stratified heavy viscous fluid has been analyzed from the viewpoint of hydrodynamics by S. Chandrasekhar (9) and R. Hide (10).

$$\kappa = \frac{(\rho_2 - \rho_1)gh}{P\left[3\left(\frac{\mu_1 + \mu_2}{\mu}\right)\right]^{2/3}} \tag{48}$$

$$\psi = \frac{P(\mu_1 + \mu_2)^{2/3}\mu^{1/3}}{\sqrt[3]{3}P}. \tag{49}$$

The problem is to find the maximum value of ψ considered as a function of δ. When we plot ψ as a function of δ we must distinguish three regions. For

$$\kappa < \tfrac{1}{2}(\tfrac{1}{4})^{2/3} = 0.198, \tag{50}$$

the plot shows a maximum and a minimum, as represented by curve 1 in Fig. 6. For

$$\kappa = \tfrac{1}{2}(\tfrac{1}{4})^{2/3}, \tag{51}$$

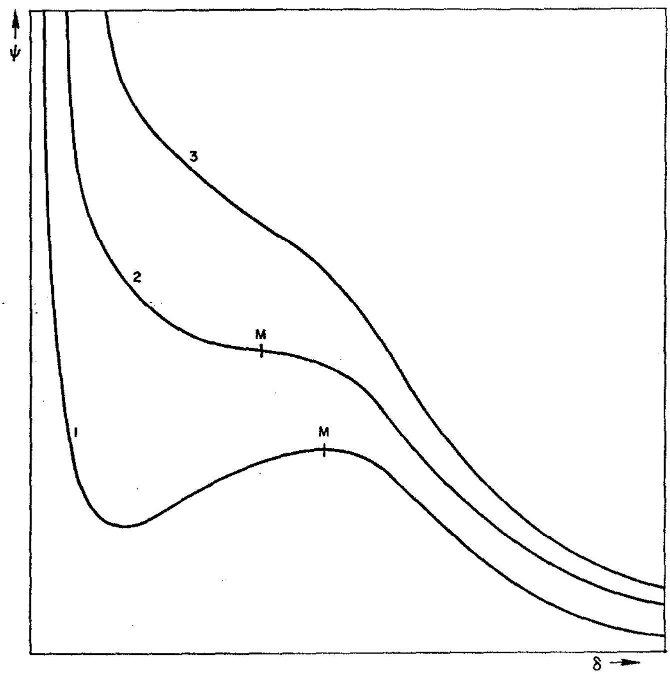

FIG. 6. Three typical curves for the dependence of ψ on δ.

the maximum and minimum coincide (curve 2) and become a horizontal inflexion point at

$$\delta = \sqrt[3]{\tfrac{1}{4}} = 0.63. \tag{52}$$

For

$$\kappa > \tfrac{1}{2}(\tfrac{1}{4})^{2/3}, \tag{25}$$

the curve has no maximum or minimum (curve 3). The point M at which the maximum of ψ occurs corresponds to a wave length of maximum rate of growth. The values of κ and δ are related at point M by the relation

$$\kappa = \frac{\delta^2(1 - \delta^3)}{2\delta^3 + 1}. \tag{54}$$

For values of κ satisfying the inequality (50) there are two positive values of δ satisfying this relation. The largest value corresponds to point M. This largest value of κ *versus* δ is plotted in Fig. 7.

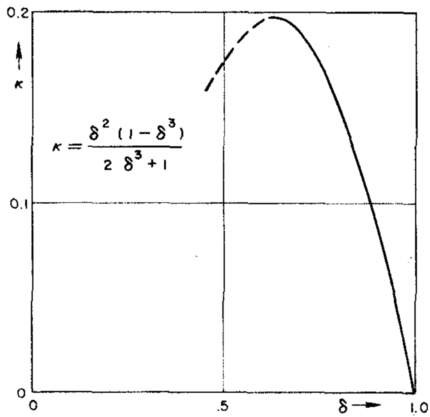

FIG. 7. Relation between κ and δ corresponding to point M of Fig. 6.

From the relationship (Eq. 54) between κ and δ we deduce the value of $l_d h$ corresponding to the dominant wave length. This is done by writing Eqs. 47 and 48 in the form

$$l_d h = \frac{\delta}{\sqrt{\kappa}}\sqrt{\frac{(\rho_2 - \rho_1)gh}{P}} \tag{55}$$

$$\sqrt[3]{\frac{3(\mu_1 + \mu_2)}{\mu}} = \frac{1}{\sqrt{\kappa}}\sqrt{\frac{(\rho_2 - \rho_1)gh}{P}}. \tag{56}$$

In these expressions κ is a function of δ through relation (54). They define a family of plots for $l_d h$ *versus* $\sqrt[3]{3(\mu_1 + \mu_2)/\mu}$ with a parameter $\sqrt{(\rho_2 - \rho_1)gh/P}$ which measures the influence of gravity. These curves are plotted in Fig. 8, in a way analogous to Fig. 3. We remember that the dominant wave length is related to $l_d h$ by Eq. 32. The interrupted line in Fig. 8 corresponds to the case where the maximum vanishes as in curve (2) of Fig. 6, hence to the disappearance of any dominant wave length. The disappearance of the dominant wave length occurs for $\kappa > 0.198$, hence for

$$\sqrt[3]{\frac{3(\mu_1 + \mu_2)}{\mu}} > 2.25 \sqrt{\frac{(\rho_2 - \rho_1)gh}{P}}. \tag{57}$$

It will be noted that in the case of an inverse density gradient the effect of gravity is to increase the dominant wave length.

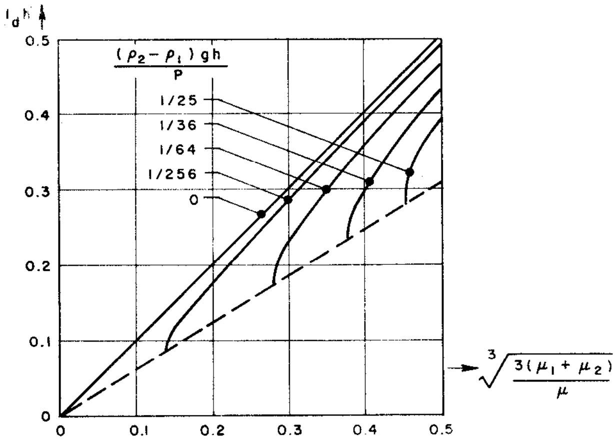

FIG. 8. Combined effect of gravity, compressive load, and viscosity, on the value of $l_d h$ for the case of an inverse gravity gradient ($\rho_2 > \rho_1$).

Finally in connection with the disappearance of the dominant wave length if the inequality (57) is satisfied we should remark that this will only be true if the surrounding medium is of infinite extent. In actuality of course the thickness will have a finite value, and if we neglect inertia forces as we have done the dominant wave length caused by the inverse gravity gradient will not be infinite but will be determined by the thicknesses of the upper and lower media and their own boundary conditions.

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