# New Thermomechanical Reciprocity Relations With Application to Thermal Stress Analysis†

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SUMMARY

Based on the variational formulation of linear thermodynamics as developed previously by the writer, thermomechanical reciprocity relations are discussed which lead to new methods of analysis of thermal stresses. These reciprocity relations are quite different from the usual ones derived from the analogy of thermal loading with a combination of surface and body-force distribution. The results are applicable to stationary and transient temperatures in elastic and viscoelastic structures. The methods are entirely variational and do not require the evaluation of the temperature field. The stresses at one point are expressed directly in terms of any arbitrary distribution temperatures applied externally, including the effect of surface heat-transfer layer. The concepts and procedures are illustrated on a simple example. The relation is pointed out between the reciprocity property and the generalization of Castigliano's principle to thermomechanics.

## (1) INTRODUCTION

In the past, problems of thermal stresses in elastic systems have been treated by neglecting the reciprocal coupling between the temperature and deformation. What is meant by reciprocal coupling is the fact that a change of temperature produces a deformation and in turn a deformation produces a change of temperature. Classical thermodynamics shows that one effect cannot occur without the other. This coupling leads to the well-known phenomenon of thermoelastic dissipation in elastic solids. The coupling is important from the standpoint of the physicist and in certain specific technological applications such as electronics, but in the theory of structures its order of magnitude is not significant.

However the coupling does acquire importance in thermal stress analysis as a concept because, as we will show, it leads to entirely new methods of calculation. We have shown[2] that the most general thermoelastic system obeys variational principles. These variational principles were derived by introducing the concept of entropy displacement and thermoelastic potential and by expressing a dissipation function in terms of the time derivatives of the entropy displacement. From these principles we have derived a formulation of thermoelasticity where the mechanical and thermal variables play identical roles. The boundary forces and temperatures then obey *thermomechanical* reciprocity relations. In the case of stationary temperatures, this is analogous to Maxwell's reciprocity relations for the forces acting on an elastic structure. Furthermore, the reciprocity properties apply to transient temperatures. Attention is called to the essential difference between the reciprocity relations discussed in this paper and the usual ones derived from the analogy of thermal loading with distributed surface and body forces or its equivalent variational form which uses the *isothermal* free energy. The new viewpoint presented here leads to influence coefficients and influence functions which are truly of a *hybrid thermomechanical* character—i.e., they belong to both thermodynamics and mechanics. By their use it becomes possible to predict the stresses of one point due to any application of temperature at the boundary applied either directly to the solid or through a surface heat-transfer layer. Advantages of the method are many. The procedure makes use entirely of variational methods and *by-passes the evaluation of the temperature field itself.* The process is error-smoothing, eliminates the evaluation of eventual local temperature singularities, and requires only one calculation for all possible cases of applied temperatures at the boundary. The procedure may also be used to calculate thermal stresses in visco-elastic systems.

In Section (2) we review our previous results on the variational formulation of thermoelasticity. The resulting reciprocity relations for stationary temperatures are discussed in Section (3), along with some

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variational procedures for the evaluation of the thermomechanical influence coefficients. Section (4) discusses the use of the reciprocity relations in thermal stress analysis for stationary temperatures, and Section (5) extends the same methods for transient thermal stresses in elastic and viscoelastic structures. For the purpose of illustration of the concepts and procedure, we have treated a simple example in Section (6).

## (2) Variational Formulation of Thermoelasticity

In previous work[2–5] we have described the state of a thermoelastic system by two vector fields $\boldsymbol{u}$ and $\boldsymbol{S}$ of components $u_i$ and $S_i$, respectively. The vector $\boldsymbol{u}$ is the geometrical displacement of the medium. The vector $\boldsymbol{S}$, which we have called the entropy displacement or entropy flow, is defined so that its time derivative is

$$(\partial \boldsymbol{S}/\partial t) = (1/T_r)\ (\partial \boldsymbol{H}/\partial t) \tag{2.1}$$

where $\partial \boldsymbol{H}/\partial t$ is the rate of heat flow. The symbol $T_r$ denotes a constant reference temperature which represents the uniform temperature acquired by the system when it is in a state of equilibrium in the presence of a large heat reservoir at the same temperature $T_r$. The terminology "entropy displacement" for $\boldsymbol{S}$ is of course only justified if the local temperatures $\theta + T_r$ are such that $\theta << T_r$. The theory, being linear, is valid when this inequality is not fulfilled, but there is no need of changing the terminology. The thermoelastic properties of the system are then completely defined[2–5] by a thermoelastic potential

$$V = \iiint_\tau [W + (1/2)\ (T_r/c)\ (\beta_{ij}e_{ij} + \text{div } \boldsymbol{S})^2] d\tau \tag{2.2}$$

and a dissipation function

$$D = (1/2)T_r \iiint_\tau \lambda_{ij}(\partial S_i/\partial t)\ (\partial S_j/\partial t) d\tau + (1/2)T_r \iint_A (1/K)\ (\partial S_n/\partial t)^2 dA \tag{2.3}$$

The integrals are extended to the volume $\tau$ and the boundary surface $A$. The symbols are

$e_{ij}$ = $(1/2)\ [(\partial u_i/\partial x_j) + (\partial u_j/\partial x_j)]$ strain components
$C_{\mu v}{}^{i\gamma}$ = isothermal elastic moduli
$W$ = $(1/2)C_{\mu v}{}^{ij}e_{ij}e_{\mu v}$, isothermal strain energy
$K$ = coefficient of surface heat transfer
$[\lambda_{ij}]$ = $[k_{ij}]^{-1}$ = thermal resistivity matrix—i.e., inverse of the thermal conductivity $(k_{ij})$
$S_n$ = normal component of $\boldsymbol{S}$ at the boundary
$c$ = specific heat per unit volume at zero strain

The $\beta_{ij}$'s are coefficients which appear in the equations of state for the stress $\sigma_{\mu v}$,

$$\sigma_{\mu v} = C_{\mu v}{}^{ij}e_{ij} - \beta_{\mu v}\theta \tag{2.4}$$

where $\theta$ is the local excess temperature above the equilibrium value $T_r$. The field $\boldsymbol{S}$, the strain $e_{ij}$, and the excess temperature $\theta$ are related by the equation

$$\text{div } \boldsymbol{S} = -(c\theta/T_r) - \beta_{ij}e_{ij} \tag{2.5}$$

We now disturb the equilibrium of the system by applying forces $\boldsymbol{F}$ per unit area at the boundary and external temperatures $\theta_a$. These temperatures may be applied at the boundary *outside* the surface heat-transfer layer and not directly to the solid. They may also of course be applied directly to the solid, which amounts to putting $K = \infty$.
We have shown that the system responds as if generalized thermomechanical forces $Q_i$ were applied to it, and that these forces are defined by a method of virtual work. This virtual work is

$$Q_i\delta q_i = \iint_A (\boldsymbol{F}\cdot\delta\boldsymbol{u} + \theta_a\delta S_n) dA \tag{2.6}$$

where $S_n$ is the normal component of $\boldsymbol{S}$ at the boundary chosen *positive inward*. The thermoelastic system is represented by the generalized coordinates $q_i$, i.e., we have put

$$\boldsymbol{u} = \boldsymbol{u}_i q_i, \qquad \boldsymbol{S} = \boldsymbol{S}_i q_i \tag{2.7}$$

where $\boldsymbol{u}_i$ and $\boldsymbol{S}_i$ are vector fields of fixed configuration. The thermoelastic potential and the dissipation function then become

$$V = (1/2)a_{ij}q_iq_j, \qquad D = (1/2)b_{ij}\dot{q}_i\dot{q}_j \tag{2.8}$$

Applying variational methods, we have derived the following differential equations:

$$(\partial V/\partial q_i) + (\partial D/\partial \dot{q}_i) = Q_i \tag{2.9}$$

or

$$a_{ij}q_j + b_{ij}\dot{q}_j = Q_i$$

These equations are a particular case of the more general treatment of linear thermodynamics which we have introduced in references 1 and 2 and displayed in more detail in reference 5. The quadratic forms $V$ and $D$ are positive definite because of their physical nature.

## (3) Reciprocity Relations for the Thermoelastic Admittance

It is of interest to consider the system as a "black box," forces being applied to a small number of "observed" coordinates $q_i$. The other coordinates which may be very large or even infinite in number are unobserved and the conjugate forces applied to them are zero. We may solve the system (2.9) for the observed coordinates in terms of the corresponding forces. We have shown that the solution is

$$q_i = A_{ij}{}^*Q_j \tag{3.1}$$

where $A_{ij}{}^* = A_{ji}{}^*$ is a symmetric matrix representing the thermoelastic admittance of the system. In operational form this admittance is

$$A_{ij}{}^* = \sum^s [C_{ij}{}^{(s)}/(p + \lambda_s)] + C_{ij} \tag{3.2}$$

This was given a rigorous proof in reference 1. The symbol $p$ refers to the time derivative $p = d/dt$ which in case of harmonic functions of time with angular frequency $\omega$ becomes $p = i\omega$. Relations (3.1) may of course also be considered as relations between Laplace transforms of $q_i$ and $Q_j$. Consider now two points $P$ and $M$ at the boundary of a thermoelastic system. A temperature $\theta_P$ applied externally at $P$ over a unit area produces at $M$ a displacement $u_M$ in a certain direction.† We may assimilate the subscripts $j$ and $i$ in expressions (3.1) and (3.2) with the points $P$ and $M$ and the quantities $Q_j$ and $q_i$ with $\theta_P$ and $u_M$. We shall assume that the temperatures are varying with time at an infinitely slow rate, so that at every moment the temperatures are the same as in a steady state. This assumption being equivalent to that of zero frequency we may put $p = 0$ in expression (3.2). Hence, we may write

$$u_M = A_{MP}\theta_P \qquad (3.3)$$

with an "influence coefficient"

$$A_{MP} = \sum^{s} (C_{ij}^{(s)}/\lambda_s) + C_{ij} \qquad (3.4)$$

We note that, if $C_{ij}^{(s)} \neq 0$, the corresponding $\lambda_s$ cannot be zero, since a finite temperature must produce a finite displacement. Conversely, a force $F_M$ applied at $M$ in the direction $u_M$ at an infinitely slow rate will produce per unit area at $P$ an inflow of entropy $S_P$, which may be written

$$S_P = A_{PM}F_M \qquad (3.5)$$

As a consequence of the symmetry of the admittance matrix (3.2), we may write

$$A_{MP} = A_{PM} \qquad (3.6)$$

This *reciprocity relation* is the extension to mixed thermomechanical variables of the analogous Maxwell's relations between forces and displacements at different locations of an elastic system. The temperature $\theta_P$ plays the role of $\theta_a$, and we have already pointed out that this indicates the temperature either outside the surface transfer layer or at the solid boundary itself.

For practical purposes an interesting feature of the coefficient $A_{PM}$ is the possibility of establishing its value by a relatively simple procedure, as follows. Since the force is applied gradually and very slowly the deformation may be considered isothermal ($\theta = 0$). However, a flow of entropy is produced because each element of volume due to its deformation exudes a certain amount of heat. This can be seen from Eq. (2.5), which for isothermal deformations, i.e., $\theta = 0$, becomes

$$\operatorname{div} \boldsymbol{S} = -\beta_{ij}e_{ij} \qquad (3.7)$$

In this equation $e_{ij}$ is the strain produced by the application of the force $F_M$. The force $F_M$ and the vector $\boldsymbol{S}$ may be assumed to vary linearly with time at a very slow rate. The time derivative of $\boldsymbol{S}$ is proportional to the rate of heat flow and, according to the general principles developed earlier,[6] must satisfy a principle of minimum dissipation—i.e., its spatial distribution must minimize the dissipation function as given by Eq. (2.3). Since we are dealing with a stationary state, $\boldsymbol{S}$ may be assumed to vary linearly with time and we may replace $\partial \boldsymbol{S}/\partial t$ by $\boldsymbol{S}$ in the dissipation function $D$. *The problem is then to minimize*

$$D' = (1/2) \iiint_\tau \lambda_{ij}S_iS_jd\tau + (1/2) \iint_A (1/K)\,(S^n)^2dA \qquad (3.8)$$

under the constraint (3.7). To do this we may either minimize absolutely

$$D' + \lambda \iiint_\tau (\operatorname{div} \boldsymbol{S} + \beta_{ij}e_{ij})d\tau \qquad (3.9)$$

with the Lagrangian multiplier $\lambda$, or, preferably, introduce any field $\boldsymbol{S}^*$ which satisfies Eq. (3.7), then put

$$\boldsymbol{S} = \boldsymbol{S}^+ + \boldsymbol{S}^* \qquad (3.10)$$

where $\boldsymbol{S}^+$ is an unknown field with conservative flow—i.e.,

$$\operatorname{div} \boldsymbol{S}^+ = 0 \qquad (3.11)$$

*We then find the field $\boldsymbol{S}^+$ which minimizes $D'$ absolutely.* Identical procedures have been discussed by this writer in more detail in connection with heat flow analysis problems.[4] It may be convenient to express the right-hand side of Eq. (3.7) in terms of the stresses due to $F_M$. This will be the case in particular if such stresses are statically determined directly from $F_M$. Consider a material under zero stress $\sigma_{\mu\nu} = 0$. Relation (2.4) becomes

$$C_{\mu\nu}{}^{ij}e_{ij} = \beta_{\mu\nu}\theta \qquad (3.12)$$

With thermal dilation coefficient $\alpha_{ij}$ we also have

$$e_{ij} = \alpha_{ij}\theta \qquad (3.13)$$

hence,

$$C_{\mu\nu}{}^{ij}\alpha_{ij} = \beta_{\mu\nu} \qquad (3.14)$$

If now $e_{\mu\nu}$ is the strain due to the force $F_M$ for isothermal deformation ($\theta = 0$), the corresponding stress is

$$\sigma_{ij} = C_{ij}^{\mu\nu}\,e_{\mu\nu} \qquad (3.15)$$

Multiplying Eq. (3.14) by $e_{\mu\nu}$ and taking Eq. (3.15) into account we derive

$$\sigma_{ij}\alpha_{ij} = \beta_{\mu\nu}e_{\mu\nu} \qquad (3.16)$$

We may therefore write Eq. (3.7) as

$$\operatorname{div} \boldsymbol{S} = -\sigma_{ij}\alpha_{ij} \qquad (3.17)$$

In the particular case of an isotropic medium, the stress-strain relations are

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e - \beta\theta)\delta_{ij} \qquad (3.18)$$

where $e = e_{xx} + e_{yy} + e_{zz}$ is the dilation, and $\mu$, $\lambda$, the Lamé constants. The coefficients $\beta_{ij}$ are reduced to

$$\beta_{ij} = \delta_{ij}\beta \qquad (3.19)$$

and $\beta$ may be expressed in terms of the linear coefficient $\alpha_l$ of thermal dilation as

† Note that $u_M$ has the dimension of a displacement per unit area hence of (length)$^{-1}$.

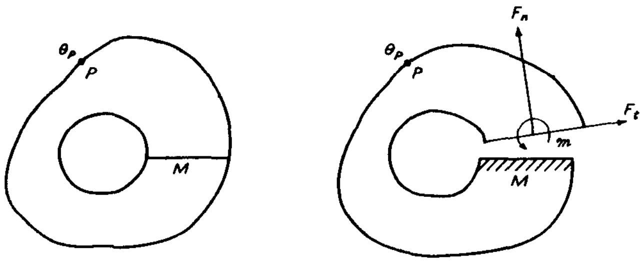


FIG. 1. Thermal stress at $M$ due to temperature $\theta P$ applied at $P$.

$$\beta = \alpha_l(2\mu + 3\lambda) \tag{3.20}$$

Relation (3.7) becomes

$$\operatorname{div} \boldsymbol{S} = -\beta e \tag{3.21}$$

where $e$ is the dilation produced by the force $F_M$. This equation may also be written in terms of the stress invariant $\sigma = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$ due to $F_M$ by applying Eq. (3.17). We find

$$\operatorname{div} \boldsymbol{S} = -\alpha_l \sigma \tag{3.22}$$

This expression will be convenient in problems where the stress field due to $F_M$ is statically determined.

The expression for $D'$ also simplifies to

$$D' = (1/2) \iiint_\tau (1/k)\boldsymbol{S}^2 d\tau + (1/2) \iint_A (1/K)S_n^2 dA \tag{3.23}$$

where $k$ is the thermal conductivity.

Eq. (3.17) may be considered as generalizing the method of the associated flow field of reference 4 to thermoelasticity by associating an entropy flow with a given stress configuration.

## (4) Application to Thermal Stress Analysis for Stationary Temperatures

Application of the reciprocity relations to thermal stress analysis may be made as follows: Consider a structure as shown in Fig. 1 and suppose we wanted to determine the stress at a plane cross section $M$ due to the application of a temperature $\theta_P$ per unit area at point $P$ of the boundary. As mentioned above, this temperature may be applied directly to the solid boundary or through a surface heat-transfer layer. The stress at $M$ is represented by a normal force $F_n$, a tangential force $F_t$, and a moment $\mathfrak{M}$. We cut the structure along the plane $M$ and apply such forces very slowly—i.e., isothermally at the cut. The linear and angular displacements at the cut associated with the forces $F_n$, $F_t$, and $\mathfrak{M}$ are, respectively, $u_n$, $u_t$, and $\alpha$. They may be written

$$\left.\begin{aligned} u_n &= C_{nn}F_n + C_{nt}F_t + C_{n\alpha}\mathfrak{M} \\ u_t &= C_{tn}F_n + C_{tt}F_t + C_{t\alpha}\mathfrak{M} \\ \alpha &= C_{\alpha n}F_n + C_{\alpha t}F_t + C_{\alpha\alpha}\mathfrak{M} \end{aligned}\right\} \tag{4.1}$$

The matrix of coefficients is, of course, symmetric, and relations (4.1) may be conveniently expressed by applying Castigliano's principle—i.e.,

$$u_n = \partial U/\partial F_n, \quad u_t = \partial U/\partial F_t, \quad \alpha = \partial U/\partial \mathfrak{M} \tag{4.2}$$

where $U(F_n, F_t, \mathfrak{M})$ is a quadratic form representing the isothermal strain energy. Up to now we have followed the well-known classical procedure. We now suppose that we have evaluated for each of the forces $F_n$, $F_t$, and $\mathfrak{M}$, the entropy flow, and, in particular, the normal inward flow of entropy per unit area at point $P$. For each force component this entropy inflow at $P$ may be written

$$\left.\begin{aligned} S_P^{(n)} &= A_{PM}^{(n)}F_n \\ S_P^{(t)} &= A_{PM}^{(t)}F_t \\ S_P^{(\alpha)} &= A_{PM}^{(\alpha)}\mathfrak{M} \end{aligned}\right\} \tag{4.3}$$

In evaluating the flow field $S$ we must assume that it remains *continuous across the gap at* $\mathfrak{M}$. The calculation is conveniently carried out by a variational procedure as outlined in the previous section. We are now in a position to apply the reciprocity relations (3.6). This means that we may evaluate immediately the displacements produced at the gap by a temperature $\theta_P$ applied at point $P$ per unit area. We write

$$u_n = A_{MP}^{(n)}\theta_P, \quad u_t = A_{MP}^{(t)}\theta_P, \quad \alpha = A_{MP}^{(\alpha)}\theta_P \tag{4.4}$$

The influence coefficients in these relations are already determined by Eqs. (4.3) since from the reciprocity relations (3.6) we find

$$A_{MP}^{(n)} = A_{PM}^{(n)}, \quad A_{MP}^{(t)} = A_{PM}^{(t)}, \quad A_{MP}^{(\alpha)} = A_{PM}^{(\alpha)} \tag{4.5}$$

Thermal stresses due to the temperature $\theta_P$ correspond to forces $F_n$, $F_t$, $\mathfrak{M}$ which, at the gap, produce displacements which are equal and opposite in sign to those Eqs. (4.4) produced by $\theta_P$. Hence the thermal stresses at $M$ are given by the equations

$$\left.\begin{aligned} A_{PM}^{(n)}\theta_P &= -(\partial U/\partial F_n) \\ A_{PM}^{(t)}\theta_P &= -(\partial U/\partial F_t) \\ A_{PM}^{(\alpha)}\theta_P &= -(\partial U/\partial \mathfrak{M}) \end{aligned}\right\} \tag{4.6}$$

The reader will note the advantages of this procedure over the classical method. They are as follows:

(a) The method does not require the knowledge of the temperature field in the body and by-passes completely the necessity of calculating this temperature distribution.

(b) The thermomechanical influence coefficients $A_{PM}^{(n)}$, $A_{PM}^{(t)}$, $A_{PM}^{(\alpha)}$, appearing in Eqs. (4.6) are determined for all points $P$ of the boundary by a single calculation. The calculation does not have to be repeated for every new distribution of boundary temperature.

(c) The thermomechanical influence coefficients are determined by a variational method which amounts to minimizing the dissipation function. This function includes the effect of any surface heat-transfer layer with a transfer coefficient which may be dependent on the location.

(d) The method avoids the evaluation of complicated temperature fields which may arise due to local effects near the points of application of the boundary temperatures, and provides a "smoothing out" of these effects.

Needless to say, the method is applicable to very complex structures and is not at all restricted to the particular configuration of Fig. 1 which is used here to explain the method. As shown by this writer some years ago,[7, 8] the connectivity of a body introduces essential differences in the thermoelastic properties. In problems of two-dimensional stress, for example, we have shown that an homogeneous isotropic body without holes—i.e., simply connected—exhibits no thermal stresses under arbitrary stationary temperatures at the boundary. If there are holes, a cut such as exhibited in Fig. 1 will render the body simply connected and thermal stresses will disappear. They will then be due entirely to the forces required to close the gap at the cut. Application of these theorems to the photoelastic determination of thermal stresses was discussed in reference 9.

The reciprocity relations are equivalent to a generalization of Castigliano's principle for thermomechanical phenomena, if we introduce the concept of associated field as developed in reference 4. The thermoelastic potential may be written

$$V = (1/2)\ (u_n'F_n + u_t'F_t + \alpha'\mathfrak{M} + S_P\theta_P) \quad (4.7)$$

where

$$\left.\begin{aligned} u_n' &= \partial U/\partial F_n + A_{MP}^{(n)}\theta_P \\ u_t' &= \partial U/\partial F_t + A_{MP}^{(t)}\theta_P \\ \alpha &= \partial U/\partial \mathfrak{M} + A_{MP}^{(\alpha)}\theta_P \\ S_P &= A_{PM}^{(n)}F_n + A_{PM}^{(t)}F_t + \\ &\qquad A_{PM}^{(\alpha)}\mathfrak{M} + A_{PP}^{(\theta)}\theta_P \end{aligned}\right\} \quad (4.8)$$

The quantity $S_P$ or total entropy inflow at $P$ is the sum of expression (4.3) and an additional term $A_{PP}\theta_P$ which is the inflow due to $\theta_P$ itself after elimination of the ignorable coordinates. This elimination may be accomplished by introducing associated flow fields as developed in reference 4. The quantities $u_n'$, $u_t'$, and $\alpha'$ are the total displacements at $M$ obtained by adding Eqs. (4.2) and (4.4). Because of reciprocity and the properties of quadratic forms we may write

$$u_n' = \partial V/\partial F_n, \quad u_t' = \partial V/\partial F_t, \quad \alpha' = \partial V/\partial \mathfrak{M}, \quad S_P = \partial V/\partial \theta_P \quad (4.9)$$

This is a generalized form of Castigliano's principle. Putting

$$u_n' = u_t' = \alpha' = 0 \quad (4.10)$$

is equivalent to Eqs. (4.6) and determines the thermal stresses. Eqs. (4.10) also state that the thermoelastic potential is a minimum when considered as a function of a self-equilibrating stress field.

## (5) Extension to Transient Stresses and to Viscoelasticity

Until now we have only considered the application of reciprocity relations to the case of stationary temperatures. The next question is to extend the procedure to the analysis of thermal stresses in an elastic system for the case of transient temperatures. Since the admittance matrix $A_{ij}^*$ is symmetric, it is clear that a similar reciprocity property as Eq. (3.6) exists for transients. Considering again point $P$ and $M$, and a temperature $\theta_P(t)$ function of the time $t$ applied at $P$, we may write, for the displacement at $M$, the operational relation

$$u_M(t) = A_{MP}^*\theta_P(t) \quad (5.1)$$

In this expression $A_{MP}^*$ is an operator of the type (3.2). This operator corresponds of course to an indicial cross-admittance function $A_{MP}(t)$. This admittance function is obtained as the displacement $u_M$ at $M$ corresponding to the sudden application of a unit constant temperature at $P$. We put

$$\theta_P(t) = 1(t) \quad (5.2)$$

where $1(t)$ is the Heaviside step function

$$1(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \quad (5.3)$$

The indicial admittance is

$$A_{MP}(t) = A_{MP}^*\ 1(t) \quad (5.4)$$

Conversely, for the entropy inflow at $P$ due to a force $F_M(t)$ at $M$, we may also write the operational relation

$$S_P(t) = A_{PM}^*F_M(t) \quad (5.5)$$

The corresponding indicial admittance $A_{PM}(t)$ equal to the entropy inflow at $P$ due to the sudden application of a unit force $F_M$ at $M$ is

$$A_{PM}(t) = A_{PM}^*\ 1(t) \quad (5.6)$$

The symmetry of the admittance matrix (3.2) leads to

$$A_{MP}^* = A_{PM}^* \quad (5.7)$$

and, hence, to

$$A_{MP}(t) = A_{PM}(t) \quad (5.8)$$

This is a reciprocity relation for the indicial thermomechanical admittance functions.

Application of these results to thermal stress analysis requires the evaluation of the indicial admittance $A_{PM}(t)$. This requires the evaluation of the transient entropy flow field due to the sudden application of a unit force at $M$. Calculation of this field may be achieved fairly simply by an approximate method. The general thermoelastic theory[2, 5] establishes a relation between the temperature $\theta$ and the deformation, which is

$$(\partial/\partial x_i)\ [k_{ij}(\partial\theta/\partial x_j)] = c(\partial\theta/\partial t) + T_r\beta_{ij}(\partial e_{ij}/\partial t) \quad (5.9)$$

The deformation $e_{ij}$ is due to the suddenly applied unit force at $M$. Since we are dealing with a thermoelastic system, if we wish to be exact the deformation $e_{ij}$ must be computed along with the coupled temperature field and will be a function of time. Actually, however, the coupling between the deformation and the temperature is very small, and it is justified to assume that the deformations $e_{ij}$ are the same as if they were isothermal. At this point, we make use of a scalar concept already introduced in reference 3, which we called a flow potential. In this case, we define it as

$$\psi = (1/T_r) \int_0^t \theta \, dt \tag{5.10}$$

Eq. (5.9) may then be written

$$(\partial/\partial x_i) \, [k_{ij}(\partial\psi/\partial x_j)] = c(\partial\psi/\partial t) + \beta_{ij}e_{ij} \tag{5.11}$$

Because of Eq. (3.16), we may also write $\beta_{ij}e_{ij} = \alpha_{ij}\sigma_{ij}$ where $\sigma_{ij}$ is the stress due to $F_M$.

The scalar $\psi$ plays the role of a temperature field with distributed heat sinks $\beta_{ij}e_{ij} = \alpha_{ij}\sigma_{ij}$ per unit volume. The entropy displacement is then given by

$$S_i = -k_{ij}(\partial\psi/\partial x_j) \tag{5.12}$$

Actually it is not necessary to evaluate the scalar $\psi$, and the field $S_i$ may be computed directly by variational methods as developed in considerable detail in references 3 and 4 in connection with purely thermal problems. The problem here is the same as the evaluation of the heat flow, represented by $\mathbf{S}$ when a sudden distribution of heat sinks $\beta_{ij}e_{ij}$ is applied throughout the body. This can be done by considering the problem as one of thermal relaxation, by splitting $\mathbf{S}$ into two parts,

$$\mathbf{S} = \mathbf{S}_t + \mathbf{S}_z \tag{5.13}$$

where $\mathbf{S}_0$ is the steady-state flow corresponding to the sources, while $\mathbf{S}_t$ is a transient field without sources determined by the initial condition $\mathbf{S}_t = -\mathbf{S}_0$ and vanishing for $t = \infty$. The steady-state field $\mathbf{S}_0$ has already been considered above in connection with stationary problems and, as is shown, may also be evaluated by variational methods.

We now go back to the more specific configuration of Fig. 1. By the method just outlined we may evaluate the three admittance functions $A_{MP}^{(n)}(t)$, $A_{MP}^{(t)}(t)$, $A_{MP}^{(\alpha)}(t)$, which are associated with the normal tangential and angular displacement at the cut $M$. A time varying temperature $\theta_P(t)$ applied at $P$ per unit area produces at $M$ displacements given by

$$\left.\begin{aligned}
u_n(t) &= A_{MP}^{(n)}(t)\theta_P(0) + \int_0^t A_{MP}^{(n)}\,(t-\tau)\,(d\theta_P/d\tau)d\tau \\
u_t(t) &= A_{MP}^{(t)}(t)\,\theta_P(0) + \int_0^t A_{MP}^{(t)}(t-\tau)\,(d\theta_P/d\tau)d\tau \\
\alpha(t) &= A_{MP}^{(\alpha)}(t)\,\theta_P(0) + \int_0^t A_{MP}^{(\alpha)}\,(t-\tau)\,(d\theta_P/d\tau)d\tau
\end{aligned}\right\} \tag{5.14}$$

If, again, we neglect the temperature changes produced by the forces $F_n$, $F_t$, $\mathfrak{M}$ applied at $M$, the forces necessary to close the gap are the same as in Section (4) for isothermal deformation. Using the same isothermal strain energy $U$ we write

$$\left.\begin{aligned}
A_{MP}^{(n)}(t)\theta_P(0) + \int_0^t A_{MP}^{(n)}(t-\tau) \times (d\theta_P/d\tau)d\tau &= -(\partial U/\partial F_n) \\
A_{MP}^{(t)}(t)\theta_P(0) + \int_0^t A_{MP}^{(t)}\,(t-\tau) \times (d\theta_P/d\tau)d\tau &= -(\partial U/\partial F_t) \\
A_{MP}^{(\alpha)}(t)\theta_P(0) + \int_0^t A_{MP}^{(\alpha)}(t-\tau) \times (d\theta_P/d\tau)d\tau &= -(\partial U/\partial \mathfrak{M})
\end{aligned}\right\} \tag{5.15}$$

These three equations yield the thermal stress at $M$ as a function of time.

The reciprocity relations and the above methods are readily extended to what might be called linear thermoviscoelasticity—i.e., the problem of thermal stresses in a viscoelastic material such that the operational moduli are independent of the temperature. In such a case the stress-strain relations are formally identical with Eqs. (2.4) where the coefficients $C_{\mu\nu}{}^{ij}$, $\beta_{ij}$, are replaced by the operators $C_{\mu\nu}{}^{*ij}$, $\beta_{ij}{}^*$. The operators $C_{\mu\nu}{}^{*ij}$ and $\beta_{ij}{}^*$ are related to the coefficient of thermal dilation by equations similar to (3.14); i.e.,

$$C_{\mu\nu}{}^{*ij}\alpha_{ij} = \beta_{\mu\nu}{}^* \tag{5.16}$$

The requirement that the operators be temperature-independent strongly restricts the applicability of thermoviscoelasticity since most viscoelastic materials show deformation rates which are very sensitive to temperature changes. We shall therefore only briefly indicate how the method may be extended to this case.

The validity of the reciprocity relations (5.8) for the thermomechanical admittance operator in viscoelastic media may be immediately derived from either the correspondence rule or the general thermodynamic principles as formulated in earlier publications. The admittance functions are again computed by solving Eq. (5.11) for $\psi$ or $\mathbf{S}$. The sinks in that equation due to the application of the force $F_M$ are $\beta_{ij}{}^*e_{ij} = \alpha_{ij}\sigma_{ij}$. We see that if $\sigma_{ij}$ is statically determined they are constant. The admittance functions thus calculated are related to the stresses at $M$ by equations formally identical with (5.15) except for the fact that the coefficients on the right-hand side are operators. The reader familiar with operational or Laplace transform methods will have no difficulty in evaluating these stresses by standard procedures.

Castigliano's principle may be extended in operational form to the transient thermoelastic case by writing an expression identical in form with (4.7) except that the coefficients are replaced by the corresponding operators. The same generalization holds for thermoviscoelasticity or any other more complex field of application of irreversible thermodynamics. It also corresponds to a generalization of methods of complementary energy to thermoelasticity and thermoviscoelasticity.

## (6) Example

We shall consider the problem of stationary thermal stresses in a thin circular cylinder of radius $a$ and thickness $h$ (see Fig. 2). The deformation is restrained along the axis, and the problem is one of two-dimensional strain. The two-dimensional stress-strain relations are

$$\left.\begin{aligned} \sigma_{xx} &= 2\mu e_{xx} + \lambda e - \beta\theta \\ \sigma_{yy} &= 2\mu e_{yy} + \lambda e - \beta\theta \\ \sigma_{xy} &= 2\mu e_{xy} \end{aligned}\right\} \tag{6.1}$$

with the dilation $e = e_{xx} + e_{yy}$ and Lamé constants $\lambda$ and $\mu$. The stress $\sigma_{zz}$ in a direction parallel with the axis is

$$\sigma_{zz} = \lambda e - \beta\theta \tag{6.2}$$

Following the procedure outlined above, we cut the cylinder at $M$ (Fig. 2). Forces and moments applied at $M$ produce a bending moment $\mathfrak{M}_P$ at point $P$.

$$\mathfrak{M}_P = \mathfrak{M} + F_n a(1 - \cos\varphi) + F_t a \sin\varphi \tag{6.3}$$

The forces are expressed per unit length along the axis. The strain energy due to these forces is

$$\left.\begin{aligned} U &= (1/2)[(1 - \nu^2)/E]\,(12/h^3)a \int_0^{2\pi} \mathfrak{M}_P{}^2 d\varphi \\ U &= (1/2)R[2\mathfrak{M}^2 + 3a^2F_n{}^2 + a^2F_t{}^3 + 4a\mathfrak{M}F_n] \end{aligned}\right\} \tag{6.4}$$

with

$$\begin{aligned} R &= [(1 - \nu^2)/E]\,(12/h^3)\pi a \\ E &= \text{Young's modulus} \\ \nu &= \text{Poisson's ratio} \end{aligned}$$

Castigliano's principle expressed by Eqs. (4.2) yields the linear and angular displacements at $M$.

$$\left.\begin{aligned} u_n &= 3a^2RF_n + 2aR\mathfrak{M} \\ u_t &= Ra^2F_t \\ \alpha &= 2aRF_n + 2R\mathfrak{M} \end{aligned}\right\} \tag{6.5}$$

We must now calculate the displacements at $M$ due to temperatures at the cylindrical boundary by applying the thermomechanical reciprocity relations. The forces at $M$ produce a dilation $e$ and a corresponding entropy flow $S$ which we assume to be perpendicular to the circumference. The region around point $P$ is represented in Fig. 3. The flow $S$ across the thickness must satisfy

$$\partial S/\partial y = -\beta e \tag{6.6}$$

where the dilation $e$ is linearly distributed and given by†

$$\left.\begin{aligned} e &= -Dy \\ D &= [(1 + \nu)\,(1 - 2\nu)/E]\,(12/h^3)\mathfrak{M}_P \end{aligned}\right\} \tag{6.7}$$

Integrating Eq. (6.6),

$$S = (1/2)\beta D y^2 + C \tag{6.8}$$

where $C$ is a constant of integration to be determined by a principle of minimum dissipation—i.e., by minimizing expression (3.23). If there is no heat-transfer surface layer ($K = \infty$) this results in the equation

$$(\partial/\partial C) \int_{-(h/2)}^{+(h/2)} S^2 dy = 0 \tag{6.9}$$

† We shall neglect here that part of the deformation which is due to other forces than the bending moment $\mathfrak{M}_P$.

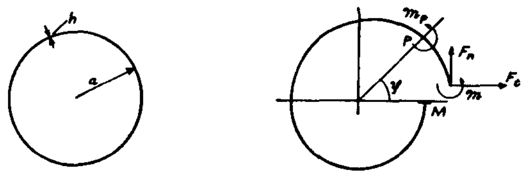


FIG. 2. Stresses in a thin circular cylinder.

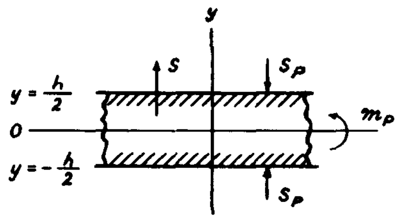


FIG. 3. Section across the thickness at $P$.

from which we determine $C$. We find

$$S = (1/2)\beta D[y^2 - (b^2/12)] \tag{6.10}$$

The inward component $S_P$ of $S$ at the inside boundary ($y = -h/2$) is

$$\left.\begin{aligned} S_P &= (1/12)\beta D h^2 \\ S_P &= [\beta(1 + \nu)\,(1 - 2\nu)/Eh]\mathfrak{M}_P \end{aligned}\right\} \tag{6.11}$$

It is easily shown that

$$\gamma = (\beta/E)\,(1 + \nu)\,(1 - 2\nu) \tag{6.12}$$

represents the coefficient of thermal dilation for two-dimensional strain; hence, we may write

$$S_P = \gamma(\mathfrak{M}_P/h) \tag{6.13}$$

For each force component at $M$ we derive from Eq. (6.3)

$$\left.\begin{aligned} S_P{}^{(n)} &= (\gamma a/h)\,(1 - \cos\varphi)F_n \\ S_P{}^{(t)} &= (\gamma a/h)\,F_t \sin\varphi \\ S_P{}^{(\alpha)} &= (\gamma/h)\mathfrak{M} \end{aligned}\right\} \tag{6.14}$$

These equations correspond to relations (4.3). The thermomechanical influence coefficients are

$$\left.\begin{aligned} A_{MP}{}^{(n)} &= (\gamma a/h)\,(1 - \cos\varphi) \\ A_{MP}{}^{(t)} &= (\gamma a/h)\sin\varphi \\ A_{MP}{}^{(\alpha)} &= \gamma/h \end{aligned}\right\} \tag{6.15}$$

We shall apply relations (4.6) to the case of an arbitrary distribution of temperatures $\theta_P$ along the inside circumference. On the left-hand side we must therefore integrate the temperatures over all points $P$. The right-hand side is given by relations (6.5). We find

$$\left.\begin{aligned} -\int_0^{2\pi} A_{MP}{}^{(n)}\theta_P a\,d\varphi &= 3a^2RF_n + 2aR\mathfrak{M} \\ -\int_0^{2\pi} A_{MP}{}^{(t)}\theta_P a\,d\varphi &= Ra^2F_t \\ -\int_0^{2\pi} A_{MP}{}^{(\alpha)}\theta_P a\,d\varphi &= 2aRF_n + 2R\mathfrak{M} \end{aligned}\right\} \tag{6.16}$$

If the temperature is uniform, i.e., $\theta_P = \theta_0$, we find

$F_t = F_n = 0$ and

$$\mathfrak{M} = -(\gamma\theta_0/h)\ [E/(1 - \nu^2)]_P(h^3/12) \quad (6.17)$$

The thermal stress is a pure moment. For $\theta_P = \theta_0 \sin\varphi$, we find $F_n = \mathfrak{M} = 0$ and

$$F_t = -(\gamma\theta_0/ah)\ [E/(1 - \nu^2)]\ (h^3/12) \quad (6.18)$$

For $\theta_P$ proportional to $\sin n\varphi$ or $\cos n\varphi$ where ($n$ = integer) $n > 1$ the thermal stress vanishes. These results are in agreement with our general theorem of reference 8 on two-dimensional thermal stresses.

If, instead of applying the temperature $\theta_P$ directly to the solid boundary, we apply it through a layer of heat-transfer coefficient $K$ we must minimize the expression

$$D' = (1/2k)\int_{-(h/2)}^{+(h/2)} S^2\, dy + (1/K)S_P{}^2 \quad (6.19)$$

with

$$\left.\begin{aligned} k &= \text{heat conductivity of the solid} \\ S &= (1/2)\beta D y^2 + C \\ S_P &= (1/8)\beta D h^2 + C \end{aligned}\right\} \quad (6.20)$$

The factor 1/2 does not appear in the second term of Eq. (6.19) because it is the sum of two terms corresponding to the inner and outer boundaries. Minimizing $D'$ determines $C$, and we find

$$S_P = \beta D h^2/12[1 + (2k/Kh)] \quad (6.21)$$

Comparing with (6.11), we see that all calculations may be repeated, provided we multiply all thermomechanical coefficients $A_{MP}{}^{(n)}$, etc., by $1/[1 + (2k/Kh)]$.

The example presented here has been chosen for its simplicity and as an illustration of the methods. It is not intended to show the particular advantages attached to the procedure. This should come out in the treatment of more complex problems.

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