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(Proceedings of an I.U.T.A.M. Symposium held in Warsaw, Sept. 1958)
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PAPER 2

STABILITY PROBLEMS OF INHOMOGENEOUS VISCOELASTIC MEDIA

M. A. BIOT (NEW YORK)

Abstract

A SYSTEMATIC theory has been developed for the stability of viscoelastic media under prestress. Use is made of linear thermodynamics and of general equations for the deformation of a prestressed medium established by the author. The theory is applied to the analysis of sinusoidal folding of stratified viscoelastic media in compression with continuous and discontinuous inhomogeneity. The influence of gravity is included. Some applications and preliminary model test data are presented.

The phenomenon of buckling in compression of an elastic inhomogeneous medium under initial stress is well known. It is illustrated by the instability of a longitudinally compressed elastic plate embedded in a medium of lower rigidity. Under a critical load buckling appears suddenly in the form of a sinusoidal folding of a certain wave length. Such sudden buckling will, of course, not occur in general if the solids involved are viscoelastic.

The interest in studying the stability problem of viscoelastic media lies partly in some technological applications connected with the use of polymers or with the thermomechanics of metallic structures. Our main interest, however, was directed to geodynamics in an attempt to furnish the beginnings of a quantitative approach to problems of deformation of the earth's crust. A systematic program to this effect was initiated about ten years ago and embodies several distinct areas of investigation. One such investigation was the development of general stress-strain relations derived from the thermodynamics of irreversible processes [1]. We also had to establish a theoretical approach for the treatment of stability problems in viscoelastic media [2]. This was applied to the gradual clarification of the qualitative as well as quantitative nature of the phenomena. Along with the theoretical work a model testing program has also been initiated as an intermediate check toward its application to full-scale geophysical configurations.

The static stability of a purely elastic plate under axial compression lying on a heavy fluid was analyzed many years ago by Smoluchowski [3] in the context of geophysics. A similar analysis was also carried out by Goldstein [4]. Some instability cases of an incompressible inhomogeneous viscous fluid in a gravity field were analyzed by Chandrasekhar [5] and by Hide [6].

The stability of an elastic plate embedded in an infinitely extended elastic medium was analyzed by Gough, Elam and De Bruyne [7], Bijlaard [8], [9], and Van der Neut [10].

The general relation between stress σ_{ij} and strain e_{ij} in a linear viscoelastic medium for small deformations in the vicinity of an equilibrium state was found to be

$$\sigma_{ij} = Z_{ij}^{*\mu\nu} e_{\mu\nu} \tag{1}$$

with an operator

$$Z_{ij}^{*\mu\nu} = \int_0^\infty \frac{p}{p+r} D_{ij}^{\mu\nu}(r)\mathrm{d}r + D_{ij}^{\mu\nu} + pD_{ij}'^{\mu\nu}, \tag{2}$$

where $p = \mathrm{d}/\mathrm{d}t$.

This relation was derived from the thermodynamics of irreversible processes [1]. The operators $Z_{ij}^{*\mu\nu}$ constitute 6×6 matrix, symmetric with respect to the main diagonal with twenty-one distinct elements. The nature of the expression [2] was discussed in detail in references [1] and [11]. If we are dealing with a medium which is in equilibrium under an initial stress, S_{ij} we have shown [12] that the relations between the incremental stresses and strains must be of the type

$$\sigma_{ij} = (Z_{ij}^{*\mu\nu} + B_{ij}^{\mu\nu})e_{\mu\nu}, \tag{3}$$

where

$$B_{ij}^{\mu\nu} = \delta_{\mu\nu} S_{ij} \tag{4}$$

depends on the initial stress. This is generally *not a symmetric tensor*, i.e.

$$B_{ij}^{\mu\nu} \neq B_{\mu\nu}^{ij} \tag{5}$$

except if the initial state is a *hydrostatic stress*. The presence of the term $B_{ij}^{\mu\nu}$ is in consequence of Onsager's relations. In most practical cases it will be justified to neglect it and write the incremental stress-strain relations in the form (1). For an isotopic medium they become

$$\sigma_{ij} = 2Q^* e_{ij} + \delta_{ij} R^* e \tag{6}$$

with the dilatation

$$e = \delta_{ij} e_{ij}. \tag{7}$$

The case of an incompressible fluid with Newtonian viscosity is found by putting

$$R^* = \infty, \quad e = 0, \quad R^* e = \sigma, \quad Q^* = \eta p, \tag{8}$$

where η is the coefficient of viscosity. Relations (6) become

$$\sigma_{ij} - \delta_{ij}\sigma = 2\eta p e_{ij}. \tag{9}$$

In stability problems we are looking for solutions such that all quantities are proportional to an increasing exponential e^{pt}. In this case all operators become algebraic quantities where p is the coefficient in the exponent.

For the case of a layered viscoelastic medium and a compression parallel with the layer the simplest problem of stability is that of a layer embedded in an infinite medium or lying on the surface of a half-space. A longitudinal compression acts in the layer. This problem has been treated quite simply by analogy with the corresponding elastic problem [13]. The stress-strain relations are formulated operationally in the same form as in the elastic case but the elastic moduli are replaced by operators in conformity with a general «correspondence rule». This correspondence rule leads to an immediate solution by using the known equations for the elastic case. For instance, consider the case of a layer of thickness h embedded in an infinite medium (Fig. 1). A compression P acts in the layer. We first neglect the compression P, which may exist in the surrounding medium, and write the equation for the deflection w of an elastic plate under an axial compression P and embedded in an infinite elastic medium. We assume a sinusoidal deflection, i.e. that w is proportional to $\cos lx$ where x lies in the direction of the compression. This yields a relation between the buckling wavelength and load P for the elastic case. We now introduce the correspondence rule and replace the Lamé constant λ and μ by their corresponding operators R^* and Q^*. The relation obtained is [13].

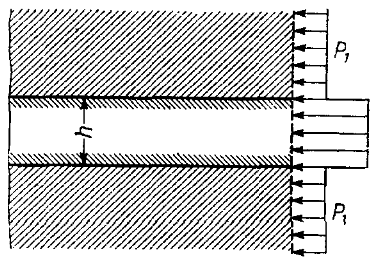

FIG. 1. Layer embedded in an infinite medium

$$\frac{1}{12} B^*(p) l^2 h^2 + \frac{B_1^*(p)}{lh} = P, \tag{10}$$

where the operator

$$B^*(p) = \frac{4Q^*(Q^* + R^*)}{2Q^* + R^*} \tag{11}$$

determines the viscoelastic properties of the layer and an identical expression $B_1^*(p)$ with a subscript refers to the properties of the surrounding medium.

The viscoelastic case differs from the elastic case by the appearance of gradual folding. Relation (10) shows there is a wavelength for which the folding amplitude has a maximum rate of growth, i.e. for which p is maximum. We call it the *dominant wavelength* L_d. The general case has been discussed in reference [13] and simple expressions established for the dominant wavelength. In particular, if the layer and the surrounding medium are incompressible and purely viscous the dominant wavelength is

$$L_d = 2\pi h \sqrt[3]{\frac{\eta}{6\eta_1}} \tag{12}$$

where η is the viscosity coefficient of the layer and η_1 that of the surrounding medium. This wavelength is *independent of the compression P*. For an elastic layer of Young's modulus E and Poisson's ratio ν in a viscous medium the dominant wavelength is

$$L_d = \pi h \sqrt{\frac{E}{(1-\nu^2)P}} \,. \tag{13}$$

In this case the dominant wavelength depends on the load but is independent of the viscosity. The latter affects only the rate of growth of the folds. We also discussed other cases such as the viscous layer in an elastic medium and the case of two Maxwell materials. It is found that the ratio of relaxation times of the two materials is an important parameter. In equation (10) we have assumed that there is perfect slip at the interface between the layer and the surrounding medium. We have also investigated [12] the effect of adherence at the interface and found that its influence on the dominant wavelength is small.

In addition to the dominant wavelength another important feature is the amplitude of folding. For the instability to be significant the rate of growth of the folding must be sufficiently high compared to the over-all compression rate. The theory shows that in the case of purely viscous solids this requires the viscosity of the layer to be at least of the order of sixty times that of the surrounding medium [12], [16].

The next phase involved is the development of a rigorous theory which treated the layer as a two-dimensional continuum instead of using the approximate plate theory. Furthermore, the influence of the initial stress in the surrounding medium cannot rigorously be neglected. The rigorous treatment was accomplished by applying equations developed earlier by this writer in a series of pre-war publications for an elastic continuum under initial stress [14], [15]. The equations are immediately applicable to viscoelastic media [2]. Results indicate the limits of applicability of the previous approximate results. As a by-product of this investigation we also solved the problem of stability of a viscoelastic half-space under a compression parallel with the surface. The surface is found to be unstable. This is offered as an explication for the wrinkles which appear at the surface of a compressed solid in the plastic range. This exact theory of folding instability was carried out in reference [16]. Perfect slip has been assumed at the interface. The equilibrium equations for the incremental stress field in a continuum under a state of uniform initial compression P parallel with the x direction are

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} - P \frac{\partial \omega}{\partial y} &= 0, \\ \frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} - P \frac{\partial \omega}{\partial x} &= 0, \end{aligned} \tag{14}$$

where ω is the local rotation of the continuum and σ_{ij} is the incremental stress of relations (6). The field must satisfy these equilibrium equations and suitable boundary conditions. We have first applied the theory to a viscoelastic homogeneous half-space with a uniform compression P parallel with the surface. The surface is also loaded with a sinusoidally distributed load of magnitude $q_0 \cos lx$ per unit area. The x axis is located at the surface and is parallel with the compression load P. The surface deflection is also sinusoidal, $v_0 = V \cos lx$. We derive the relation between load and deflection

$$V = \frac{2}{lB^* \Phi^*} q_0 . \tag{15}$$

This is an operational relation where B^* is defined by (11). The influence of the compression P is contained in the operator Φ^*. The factor $1/\Phi^*$ plays the role of an amplification factor; it is unity in the absence of compression and larger than unity if the compression is present. For an incompressible medium we may write

$$\Phi^* = \frac{1}{\zeta}\left[\sqrt{\frac{1-\zeta}{1+\zeta}}\,(1+\zeta)^2 - 1\right], \tag{16}$$

where

$$\zeta = \frac{P}{2Q^*} . \tag{17}$$

The surface is unstable for $\Phi^* = 0$. In that case a deflection may occur in the absence of surface load ($q_0 = 0$). Rationalizing the equation $\Phi^* = 0$ yields a cubic in ζ. It has a positive real root, $\zeta_1 = 1/1.192$. The two other roots are complex and are spurious because they do not correspond to a solution which remains finite at infinite depth. This bears some anology with the theory of Rayleigh waves. For an incompressible purely viscous medium of viscosity η we may write $Q^* = \eta p$. In that case the root ζ_1 corresponds to exponentially increasing surface corrugations proportional to the factor $e^{p_1 t}$, where

$$p_1 = 1.192 \frac{P}{2\eta}. \tag{18}$$

Note that all wavelengths increase at the same rate so that initial surface wrinkles are simply magnified at an exponential rate. There is no dominant wavelength. The amplification however is small and considerable compression is necessary for the wrinkles to become apparent.

Having solved the problem of the half-space we may consider that of the layer of thickness h (Fig. 1). The latter is now under a compression P while the surrounding medium is under a compression P_1. The exact solution for the stability problem of this system leads to the relation

$$\frac{Q_1^* \Phi_1^*}{Q^*} = \frac{1}{\zeta} \tanh \gamma - \frac{(1+\zeta)^2}{\zeta} k \tanh k\gamma \tag{20}$$

with

$$k = \sqrt{\frac{1-\zeta}{1+\zeta}}, \quad \gamma \doteq \frac{1}{2} lh. \tag{21}$$

The operators Q_1^* and Φ_1^* refer to the embedding medium, the latter containing P_1. It can be shown that the effect of the compression P_1 in the embedding medium is usually negligible and we may put $\Phi_1^* = 1$. If in addition the materials are purely viscous we may write

$$\frac{Q_1^*}{Q^*} = \frac{\eta_1}{\eta} \tag{22}$$

and (20) becomes a relation between γ, ζ and η_1/η. This is plotted in Fig. 2 for three values of the viscosity ratio η/η_1. The parameter γ is a non-dimensional quantity proportional to the inverse wavelength of the folding. Minima of ζ correspond to the dominant wavelength. Values of ζ derived from the approximate equation (10) are shown by the dotted lines. The approximate theory seems therefore satisfactory for practical purposes.

From the viewpoint of geodynamics the influence of gravity on the stability is of paramount interest. This was investigated for the compressed layer at the surface of a half-space and for the layer embedded between top and bottom media of different densities and viscoelastic properties [17]. The effect of gravity is embodied in a parameter which involves the ratio of the horizontal compression to the gravity force.

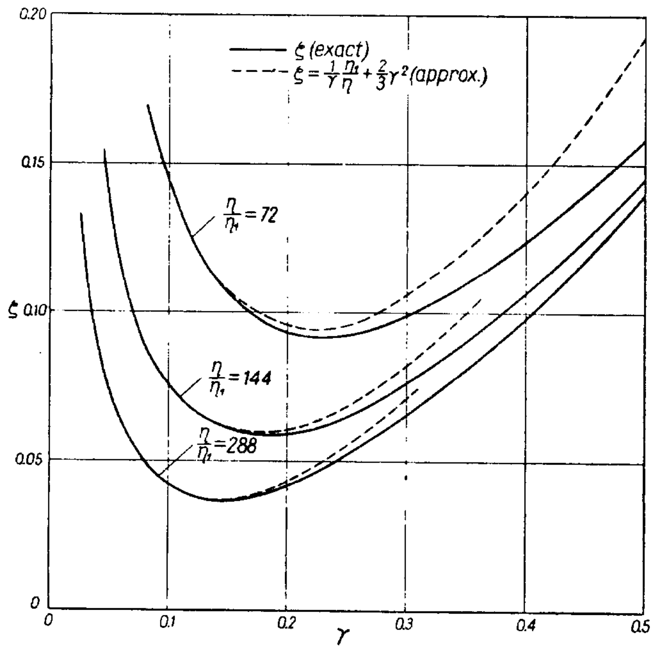

FIG. 2. Comparison of exact and approximate theories

For a layer of thickness h under a horizontal compression P and lying on top of a half-space of mass density ϱ_1 equation (10) is replaced by

$$\frac{1}{3}Q^*h^3l^* - Phl^2 + 2Q_1^*l + \varrho_1 g = 0. \tag{23}$$

The two media are assumed incompressible with Q^* referring to the layer and Q_1^* to the medium.

The influence of gravity is contained in the term $\varrho_1 g$ where g is the acceleration of gravity.

We assume again two purely viscous media of viscosities η and η_1. In Figure 3 we have plotted the variable $l_d h = 2\eta h/L_d$ containing the dominant wavelength L_d as a function of η_1/η and $\varrho_1 gh/P$. The latter parameter measures the effect of gravity.

The case of a layer embedded between two media of different densities was also investigated [17]. For the case where the top material is denser than the bottom one the phenomenon is complicated by the fact that a spontaneous instability arises due to gravity alone in the absence of horizontal compression.

We have also treated the case of the inhomogeneous half-space. The material is incompressible and viscoelastic, of mass density ϱ. The free surface is horizontal and the weight per unit volume is ϱg. The stress-strain relations are written

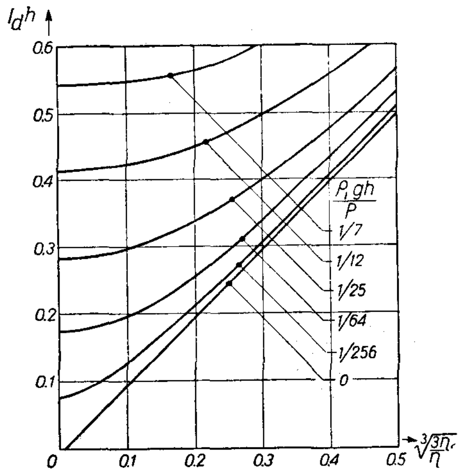

FIG. 3. Effect of gravity and viscosity ratio on the dominant wavelength

$$\sigma_{ij} - \delta_{ij}\sigma = 2Q^* e_{ij} \tag{24}$$

where Q^* is an operator of the type (2).

The case where Q^* is a general viscoelasticity operator may be treated but we shall assume for simplicity that we are dealing with a purely viscous solid whose coefficient of viscosity η decreases exponentially with the depth y

$$\eta = \eta_0 e^{-ry}. \tag{25}$$

Then

$$Q^* = \eta_0 p e^{-ry}. \tag{26}$$

Originally the material is in equilibrium under gravity with a hydrostatic pressure ϱgy proportional to the depth. We then squeeze the half-space horizontally at a uniform rate. This superimposes a horizontal compressive stress P whose value is P_0 at the surface and also decreases exponentially with depth

$$P = P_0 e^{-ry}. \tag{27}$$

This state of stress is found to be unstable and the surface tends to develop sinusoidal folds with a dominant wavelength depending on the non-dimensional parameter

$$G = \frac{\varrho g}{p_0 r}. \tag{28}$$

The relationship between the dominant wavelength and G is found by proceeding as in the previous problems. The dominant wavelength is determined by the relation

$$L_d = \frac{2\pi}{r\delta}, \tag{29}$$

where δ is a function of G which has been determined numerically. When plotting this relationship it is found that it may be approximated by

$$\delta = 2{,}2G^{6/10}. \tag{30}$$

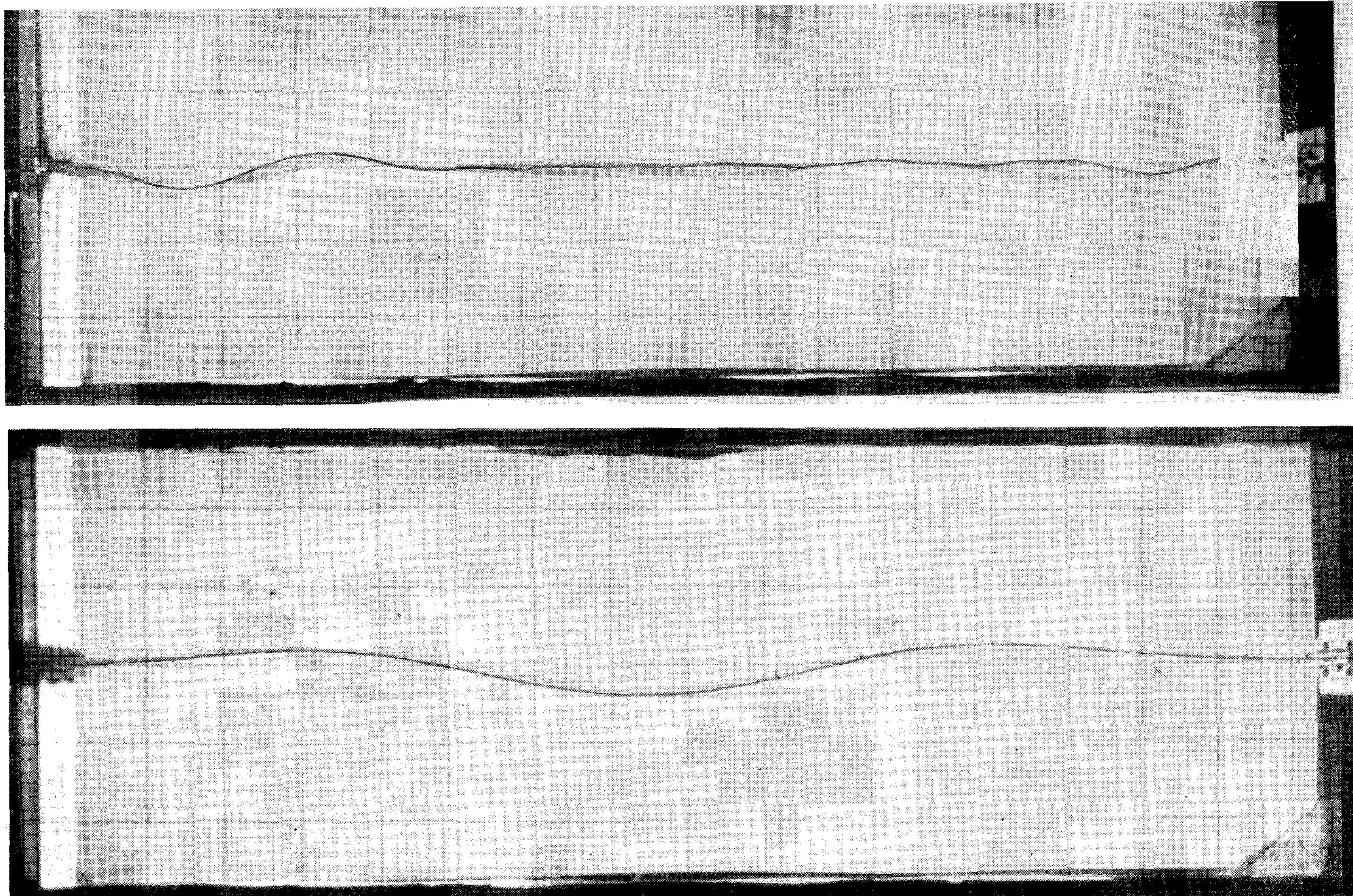

FIG. 4. Model tests, for the case of an elastic layer in a viscous medium

When $G = 0$ i.e. without the effect of gravity the dominant wavelength tends to an infinite value. The results are also applicable to an incompressible solid whose properties are defined by an operator of the type

$$Q^* = \bar{e}^{ry}\left[\int_0^\infty \frac{p}{p+r} Q(r)\, dr + Q + pQ'\right]. \tag{31}$$

Some of the most interesting applications of the present theory are in the field of geophysics and geology. It leads for the first time to a quantitative analysis of folding of stratified sedimentary rock. Elastic constants of rock do not vary greatly in order of magnitude from one type of material to another. The elastic moduli are in the range of 10^{11} to ^{12} dynes/cm^2. On the other hand viscosity properties for slow deformations cover a range of from 10^{14} to 10^{22} poise (c.g.s.) for the viscosity coefficients. The range of values is even wider if we take the temperaeture into account. A layer of viscous rock embedded in a medium of viscosity a thousand times smaller would develop under compression a folding whose wavelength is given by expression (12) and is independent of the compression. The wavelength in this case will be 34 times the thickness of the layer. If the layer is elastic the wavelength depends on the load according to expression (13). Preliminary model tests[1] have been carried out for this case and the folding is shown in Fig. 4. A careful quantitative evaluation has not yet been made but it was found that the wavelength is proportional to the thickness and inversely proportional to the square of the compressive load as predicted by equation (13).

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