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Thermodynamics and Heat-Flow Analysis by Lagrangian Methods

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Summary

New procedures for the analysis of heat flow are summarized and further applications are presented. The methods may be considered as a special case of a more general formulation of the thermodynamics of irreversible processes. This formulation is based on the introduction of a new thermodynamic potential, the concept of flow field as a state variable, and the use of Lagrangian techniques and generalized coordinates. A brief account is given of the basic principles and of the more recently developed method of "associated flow." Application to a one-dimensional problem leads to a simplified scheme for the calculation of transient temperatures in a wall by combining the Lagrangian methods with the boundary-layer techniques along the lines developed in the writer's earlier work. Another application deals with the calculation of transient temperature in a wing structure under aerodynamic heating. The concept of "effective" flange width is introduced. It represents the region of the flange which is influenced by the presence of a web.

1. Introduction

VARIATIONAL AND LAGRANGIAN METHODS in the analysis of steady and transient heat flow were the object of earlier publications.³⁻⁵ They may be considered as a particular application of more general principles of the thermodynamics of irreversible processes.^{1-3,9} Our purpose here is to give a brief outline of the theory and methods which were developed in more detail in previous publications.^{4,5} Also, we have added some new examples of application of the theory. One such example is the derivation in section 2 of an equation for the penetration depth under variable temperature at a boundary by using the boundary-layer techniques as introduced by the writer in heat-transfer problems.⁴ Another example treated in section 4 is an application of the method of associated fields to calculate the temperature in a wing structure under conditions of aerodynamic heating. The method is applied to a steady-state and a transient problem. The treatment of this problem by the method of associated fields leads naturally to the concept of an "effective" flange width. This effective width is a characteristic length representing the region of the flange which is influenced by the web.

Section 2 is a simplified outline of the general theory.⁴ Section 3 summarizes further developments of the method which introduces the concept of associated field.⁵

2. Formulation of Heat Flow by Lagrangian Coordinates

WE SHALL BRIEFLY RECALL the main results. The method is applicable to problems with temperature-dependent parameters and anisotropic properties. However, in order to simplify the presentation we shall consider the medium to be isotropic

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with temperature-independent properties. We denote by k the thermal conductivity and by c the heat capacity per unit volume. The thermal field may be defined as a vector "flow field,"

$$\mathbf{H} = \sum_{i=1}^{i} \mathbf{H}_{i} q_{i} \tag{2.1}$$

where \mathbf{H}_i are fixed configurations depending only on the location and q_i are generalized coordinates to be determined as functions of time.

More general representations of the flow field are possible and discussed later. [See Eq. (2.14)].

The temperature θ is related to **H** by the equation*

$$c\theta = -\operatorname{div} \mathbf{H} \tag{2.2}$$

This is also written

$$\theta = \sum^{\bullet} \theta_i q_i \tag{2.3}$$

with

$$c\theta_i = -\operatorname{div} \mathbf{H}_i \tag{2.4}$$

The scalar fields θ_i are now fixed-temperature distributions.

Two invariants are defined which are analogous to certain functions in Lagrangian mechanics with similar terminology, i.e., a *thermal potential*.

$$V = \frac{1}{2} \int_{\tau} c\theta^2 d\tau \tag{2.5}$$

and a dissipation function.

$$D = \frac{1}{2} \int_{\tau} \frac{1}{k} \left(\frac{\partial \mathbf{H}}{\partial t} \right)^2 d\tau + \frac{1}{2} \int_{s} \frac{1}{K} \left(\frac{\partial H_n}{\partial t} \right)^2 dS$$
(2.6)

The integral over τ is a volume integral and the integral over S is a surface integral over the boundary of τ . The quantity K is the surface heat transfer coefficient and H_n designates the component of **H** normal to the boundary and positive inward.

Again for simplicity, we assume K to be dependent only on the location. However, this assumption is not necessary and it may be time and temperature dependent. The thermal potential given in Eq. (2.5) is a particular case of a new thermodynamic potential introduced by this writer in earlier work.¹⁻³ It is different from the classical potentials and constitutes a generalization of the concept of free energy to systems with nonuniform temperatures. The dissipation function expressed in Eq. (2.6) is derived from the entropy production function introduced by Onsager⁶ and expressed by Meixner⁷ in terms of temperature gradients. For our purpose, it was reformulated in Eq. (3.4) by means of the time derivatives of the flow vector **H** (the latter being treated as a "state variable") and a term was added to include the heat transfer properties of the boundary. Finally, a generalized thermal force Q_i was defined by means of a principle of virtual work. Though the equation

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^{*}This relation expresses conservation of energy not merely as a physical condition to be satisfied by the solution of the problem but as a *constraint* to be satisfied by any variation of H.

$$Q_i \delta q_i = \int_S \theta_a \delta H_n dS \tag{2.7}$$

The temperature θ_a is that applied to the boundary *outside* the surface heat transfer film. The virtual change δq_i of the generalized coordinate is associated with a virtual normal inflow δH_n of heat through the boundary. The product $\theta_a \delta H_n$ may be interpreted as the "virtual work" of the temperature θ_a on the heat inflow, δH_n . Eq. (2.7) expresses a particular form of the concept of thermodynamic virtual work defining a generalized thermodynamic force, introduced in earlier publications.^{1, 2} The virtual work Eq. (2.7) leads to an expression for the thermal force,

$$Q_i = \int_S \theta_a \frac{\partial H_n}{\partial q_i} \, dS \tag{2.8}$$

For the particular set of coordinates defined by Eq. (2.1) this becomes

$$Q_i = \int_S \theta_a H_{in} \, dS \tag{2.9}$$

where H_{in} is the normal component of the vector \mathbf{H}_i at the boundary, with a positive direction inward. We have shown that the *n* coordinates q_i describing the flow field are functions of time and satisfy the *n* differential equations

$$(\partial V/\partial q_i) + (\partial D/\partial q_i) = Q_i$$
(2.10)

The time derivative of q_i is

$$\dot{q}_i = dq_i/dt \tag{2.11}$$

For coordinates defined by Eqs. (2.1) and (2.3), the thermal potential and the dissipation function are the quadratic forms.

$$V = (1/2) \sum_{ij}^{ij} a_{ij} q_i q_j \qquad (2.12)$$
$$D = (1/2) \sum_{ij}^{ij} b_{ij} \dot{q}_i \dot{q}_j$$

The differential Eqs. (2.10) become the linear system

$$\sum_{j=1}^{j} a_{ij} q_{j} + \sum_{j=1}^{j} b_{ij} \dot{q}_{j} = Q_{i}$$
(2.13)

The existence of normal coordinates, thermal relaxation modes and thermal impedances follows immediately from these equations.^{4,5}

It is not necessary that the generalized coordinate system be defined as a sum of fixed configurations. We may write the undetermined flow field as

$$\mathbf{H} = \mathbf{H}(x, y, z, q_1 \dots q_n, t)$$
(2.14)

where the parameters $q_1 \ldots q_n$ play the role of generalized coordinates to be determined by the same differential Eq. (2.10).

Application of these equations to one-dimensional problems is straightforward. Consider for instance the problem of a semi-infinite solid with constant parameters k and c. The boundary at y = 0 is heated to a variable temperature $\theta_0(t)$ (Fig. 1).



Fig. 1. Penetration of heat in one-dimensional flow.

We assume the temperature distribution to be parabolic and represented by

$$\begin{aligned} \theta &= \theta_0 [1 - (y/q)]^2 \quad y < q \\ \theta &= 0 \quad y > q \end{aligned}$$
 (2.15)

This approximation was introduced earlier and shown to be satisfactory.⁴ The flow field H is obtained by integrating Eq. (2.15) along the depth coordinate y, i.e.,

$$H = \int_{0}^{q-y} c\theta dy \tag{2.16}$$

Knowing H we may calculate V, D and QWe find

$$V = (1/10)c\theta_0^2 q \tag{2.17}$$

$$D = (1/2)(c^2/k)q[(13/315)\dot{q}^2\theta_0^2 + (1/21)q\theta_0\dot{q}\dot{\theta}_0 + (1/63)\dot{\theta}_0^2q^2]$$
(2.18)

$$Q = (1/3)c\theta_0^2 \tag{2.19}$$

The differential equation for the unknown "penetration depth" q is

$$(\partial V/\partial q) + (\partial D/\partial \dot{q}) = Q \qquad (2.20)$$

i.e.,

$$\dot{z} + (15/13)(\dot{\theta}_0/\theta_0)z = (147/13)(k/c)^*$$
 (2.21)

with $z = q^2$. This is a linear first-order equation the solution of which is given by the elementary integral

$$z = \frac{147}{13} \frac{k}{c} \frac{1}{F(t)} \int_0^t F(t) dt$$
 (2.22)

^{*}This equation was also derived by D. J. Johns of Cranfield College of Aeronautics (private communication)

with

$$F(t) = [\theta_0(t)]^{15/13}$$

It is interesting to examine the case where the temperature follows a power law,

$$\theta_0(t) = \alpha t^n \qquad n \ge 0 \tag{2.23}$$

We find

$$z = (147/13)(k/c)t/[(15/13)n + 1]$$
(2.24)

This result shows that the penetration depth $q = \sqrt{z}$ is independent of α and depends on the exponent *n* only through a constant factor $1/\sqrt{(15/13)n + 1}$. The exponent does not affect the square root time dependence of the penetration. This property is of importance in practical applications since an approximate knowledge of the temperature time history leads to a good approximation of the penetration depth. The latter may then be introduced in the equation as a known function of time. The case where $\theta_0 = \text{const. corresponds to } n = 0$ and yields the value (Ref. 4)

or

$$z = (147/13)(k/c)t$$

 $q = 3.36\sqrt{(kt/c)}$ (2.25)

The parabolic approximation given in Eq. (2.15) for the temperature distribution as a function of depth will generally be valid if the temperature increases or decreases monotonically. If this is not the case, we may split up the time interval into segments for which θ_0 varies monotonically, then apply Eq. (2.21) to each segment using the principle of superposition. Use can also be made of the power law solution Eq. (2.24) by dividing the time history of the temperature into segments each of which may be approximated by a power law or an additive combination of such terms including the constant value. In effecting this superposition we must, of course, avoid any procedure leading to differences of large quantities. The problem of heating of a slab of thickness l was also given as an example,⁴ for a sudden constant temperature applied to one face, while the other face is insulated.

3. Two- and Three-Dimensional Problems—Method of "Associated Fields"

The choice of the flow field **H** as the unknown thermal field presents no inconvenience in one-dimensional problems because an assumed temperature distribution leads directly to the corresponding flow field by a simple quadrature. In two or three dimensions, this is not the case since Eq. (2.2) does not determine **H** uniquely for a given temperature field θ . Of course, from a purely theoretical viewpoint, we may start by postulating an unknown vector field **H** as a two- or three-dimensional field. However, this increases the number of unknowns to be determined. Furthermore, suitable approximations for the vector field **H** are not as easy to guess as for the temperature field. An improvement is to choose a suitable temperature field

$$\theta = \sum^{i} \theta_{i} q_{i} \tag{3.1}$$

and to split the flow field into two parts

$$\mathbf{H} = \mathbf{\Theta} + \mathbf{F} \tag{3.2}$$

One part is chosen to satisfy Eq. (2.2), i.e.,

div
$$\Theta = -c\theta_i = -c\sum^{i} \theta_i q_i$$
 (3.3)

while the other is divergence-free,

$$\operatorname{div} \mathbf{F} = 0 \tag{3.4}$$

The vector field $\boldsymbol{\Theta}$ may be represented as

$$\Theta = \sum^{i} \Theta_{i} q_{i} \tag{3.5}$$

where Θ_i are fixed fields satisfying the relation

$$\operatorname{div} \Theta_i = -c\theta_i \tag{3.6}$$

However, this really displaces the problem since we still have to choose an approximation for the field \mathbf{F} which is sufficiently general to represent the correct heat flow. The difficulty may be eliminated by proceeding as follows. Let us put the field \mathbf{F} equal to,

$$\mathbf{F} = \sum \mathbf{F}_i f_i \tag{3.7}$$

with suitably chosen fixed configuration fields \mathbf{F}_i . We have called f_i the *ignorable* coordinates by analogy with similar problems in mechanics. These ignorable coordinates appear in the dissipation function but not in the thermal potential. The thermal potential V is a function of the q_i coordinates

$$V = V(q) \tag{3.8}$$

while the dissipation function depends on both \dot{q}_j and \dot{f}_j .

$$D = D(\dot{q}, \dot{f}) \tag{3.9}$$

The differential equations for the thermal field are

$$[(\partial/\partial q_i)V(q)] + [(\partial/\partial \dot{q}_i)D(\dot{q}_i\dot{f})] = Q_i$$

$$(\partial/\partial \dot{f}_j)D(\dot{q},\dot{f}) = Q_j$$
(3.10)

These equations involve coupling terms between the coordinates \dot{q}_i and \dot{f}_j . These coupling terms occur only through the dissipation function. Until now the flow configurations Θ_i are still arbitrary since the condition in Eq. (3.6) does not determine them uniquely. The problem now is to choose these flow fields *in* such a way that the coupling terms between \dot{q}_i and \dot{f}_j vanish in Eqs. (3.10). We have shown^{4,5} that this can be done provided we define Θ_i by the relation

$$\Theta_i = -k \text{ grad } \psi_i \tag{3.11}$$

where ψ_i is a "flow potential" satisfying the equation

$$\operatorname{div} \left(k \operatorname{grad} \psi_i \right) = c\theta_i \tag{3.12}$$

and the boundary condition

$$K\psi_i + k \operatorname{grad}_n \psi_i = 0 \tag{3.13}$$

By $\operatorname{grad}_n \psi_i$, we designate the normal component of the gradient of ψ_i at the boundary taken positive outward. The physical significance of Eqs. (3.12) and (3.13) is brought out by the property that ψ_i is the *steady-state temperature* in the solid for a distribution of heat sinks $c\theta_i$ per unit volume when the outside temperature θ_a is zero. The flow potential ψ_i , hence the vector field Θ_i are determined uniquely in that case by the temperature field θ_i . We have called Θ_i the flow field associated with θ_i .

For practical applications, an important feature of this associated flow field Θ_i lies in the fact that it can be determined directly from θ_i by a variational method without going through the intermediate step of evaluating the flow potential. The field Θ_i is determined by the condition that the dissipation function

$$D' = \frac{1}{2} \int_{\tau}^{\tau} \frac{1}{k} \Theta_{i}^{2} d\tau + \frac{1}{2} \int_{S}^{\tau} \frac{1}{K} \Theta_{in}^{2} dS$$
(3.14)

be a minimum under the constraint

$$\operatorname{div} \mathbf{\Theta}_i = -c\theta_i \tag{3.15}$$

The quantity Θ_{in} denotes the normal component of the vector Θ_i at the boundary. Alternate ways of solving the variational problems have been discussed⁵ by this author in previous investigations.

An expression for the associated field is readily obtained for systems where normal coordinates are known. If θ_i represents the temperature of a normal mode configuration, it will decay exponentially with time under conditions of zero external temperatures according to the law

$$\theta = \theta_i e^{-\lambda_i t} \tag{3.16}$$

We have shown⁵ that the associated flow field is given by

$$\Theta_i = k/\lambda_i \operatorname{grad} \theta_i \tag{3.17}$$

It is interesting to remember that the ignorable coordinates may be considered as combination of thermal modes with zero relaxation constant ($\lambda_i = 0$). Therefore, the decoupling of the associated field with the ignorable coordinates is closely related to the orthogenality condition of the thermal modes. The property, in fact, suggests the more general decoupling procedure here outlined.

An important aspect of the method of associated fields is brought out by considering a steady-state problem. In this case, the time derivatives disappear from the differential equations and the problem is reduced to solving algebraic equations. We notice that once we have determined the associated fields, the *steady-state* temperature may be immediately derived for any distribution of boundary temperatures. In particular, let us represent the temperature field by

$$\theta(P) = \int_{\tau} \theta(P') \delta(P, P') d\tau(P')$$
(3.18)

where $\delta(P, P')$ is the space Dirac function. This gives the temperature at P by an integration over all points P' of the volume τ . This representation may be considered as a particular case of the summation Eq. (3.1) where $\theta(P')d\tau(P')$ plays the role of a generalized coordinate q_i while $\delta(P, P')$ represents the fixed field configuration θ_i . The field associated with the fixed field $\delta(P, P')$ is given by the same Eq. (3.12) where the right-hand side is now a concentrated heat sink $c\delta(P, P')$ with total inflow equal to c and located at point P. The associated flow field is the steady heat flow due to this concentrated sink. We may denote it by $\Theta(P, P')$. We note that all temperatures are uncoupled coordinates since the thermal potential is the sum of all the local thermal potentials. Furthermore, if we consider a steadystate, all time derivatives vanish. Therefore, there is also no coupling through the dissipation function. Hence, the temperature at any given point and its value at all the other locations are uncoupled. It follows that the temperature at any point may be determined immediately and independently from the rest of the field by evaluating the virtual work of the externally applied temperature on the normal component $\delta\theta \Theta_n(P, P')$ of the virtual flow at the boundary. This particular method of solving the steady-state problem turns out to be identical with the classical method of Green's function. When applied to the unsteady state the method leads to a generalization of the Green's function solution. This point has been discussed in more detail elsewhere.⁵ It was also pointed out that the concept of associated flow field is closely related to certain reciprocity properties in analogy similar properties connected with Green's function. In conclusion, the main features of the method of associated flow field may be summarized as follows.

1. It decouples the ignorable coordinates and the temperature field thereby decreasing considerably the number of unknowns in two- and three-dimensional problems.

2. It introduces in the coordinates features which already contain basic properties of the physical system. The coordinates themselves, therefore, represent already a partial solution of the problem. This amounts to a two-step process of solution, one being the determination of the associated flow, followed by the integration of the differential equations for the coordinates. This two-step solution further reduces the number of coordinates required.

3. The associated flow field embodies in the coordinate system certain basic reciprocity properties of thermal systems and takes advantage of them in the formulation of steady-state and transient problems.

4. The procedure leads to the classical Green's function method as a particular case but is much more general than the latter. It also contributes new expression for Green's function itself.

5. Temperatures in one part of a thermal system may be determined independently from those in another region. Furthermore the temperature is obtained immediately for any distribution of externally applied temperature without having to repeat the calculation for each case.



Fig. 2. Distribution of temperature and associated flow in wing structure for the stationary antisymmetric case.

The method of associated flow fields is not restricted to the particular choice of coordinates represented by a superposition of temperature fields according to the expression in Eq. (3.1). We could also have represented the temperature field in the more general form.

$$\theta = \theta(q_1, q_2 \dots q_n, x, y, z, t)$$
(3.19)

where the parameters q_i play the role of generalized coordinates. The method is also applicable to cases where the surface heat-transfer coefficient depends on the time. In this case, the associated flow field contains the time as a parameter. With certain limitations, the method also extends the nonlinear case of temperature-dependent parameters. These various extensions were treated previously in some detail.⁵

Finally, the method is really part of an even more general theory which covers thermoelasticity and contains thermal flow as a particular case. A discussion of reciprocity relations⁸ for thermoelasticity has been given previously. These relations are related to the aforementioned reciprocity properties for the more restricted domain of purely thermal phenomena.

For a general outline of the thermodynamics, the reader is also referred to a recent publication. 9

4. Examples of Associated Fields-Concept of "Effective" Flange Width

WE SHALL PRESENT two examples of the use of associated fields in the heat-flow analysis of flight structures. Consider a portion of a wing structure whose cross section is represented in Fig. 2. This example was originally treated by Pohle and Oliver¹⁰ and later was also used as an example.⁴ The boundary-layer heat-transfer coefficient is denoted by K. The material of the structure is characterized by a constant thermal conductivity k and a heat capacity c per unit volume.

We shall first evaluate the steady-state temperature in the web due to given adiabatic temperatures of the fluid along the flange. Because of the symmetry of the structure, the web temperature may be separated into a symmetric and antisymmetric temperature distribution about the midpoint M of the web. If the adiabatic temperatures on the top and bottom flanges are antisymmetric only, the antisymmetric web temperature will be produced. It is this case which we shall now compute by applying the method of associated fields. The temperature along the web has a linear distribution

$$\theta = \theta_1(y/l) \tag{4.1}$$

where θ_1 is the unknown coordinate. It represents the temperature at the point A. In order to calculate the associated flow field, we must distribute sinks along the web of intensity $c\theta$ per unit volume. This will produce a flow as yet unknown and characterized by a vector H_1 at point A and a vector H_3 at point M. Because of symmetry, we consider the flow through a web of thickness a_1 which is half the actual thickness. There is also a flow H_f in the flange and a corresponding flow H_b through the boundary layer. We assume that the flow through the boundary layer is distributed uniformly over a half width b_e of the flange. The flow H_f through the flange is then linearly distributed. The length b_e plays the role of an "effective width" and is determined by a principle of minimum dissipation. This is done by expressing the dissipation D' given by expression Eq. (3.14) applied to the flange and boundary layer. For a given value of H_1 , this dissipation is a function of b_e only. The condition that it be a minimum yields the value of b_e . We find

$$b_e = \sqrt{(3ak/K)} \tag{4.2}$$

where a is the flange thickness. It is found that under this minimum condition, the dissipation in the flange and boundary layer are equal. On the one hand, the dissipation in the flange and boundary layer now depends only on H_1 . On the other hand, the flow in the web is completely determined by H_1 since the distribution of sinks is given. Minimizing the total dissipation in flange web and boundary layer determines H_1 , hence, the associate field. The value H_b of this associated field at the boundary is completely determined by only one unknown, i.e., θ_1 . We find

$$H_b = (a_1/b_e)1/[2(a_1b_e/al) + 3]lc\theta_1$$
(4.3)

The thermal potential for the half web is

$$V = \frac{1}{2} c a_1 \int_0^l \theta^2 dy = \frac{1}{6} c a_1 l \theta_1^2$$
(4.4)

We assume that the adiabatic temperature of the boundary layer on the top flange is θ_0 and $-\theta_0$ on the bottom. The temperature distribution is antisymmetric with respect to point M at the midweb. The temperature is zero at that point and the calculation may be restricted to the upper half of the physical system. The thermodynamic virtual work of the temperature θ_0 on the upper flange is

$$Q\delta\theta_1 = b_e\theta_0\delta H_b \tag{4.5}$$

From Eq. (4.3)

$$\delta H_b = (a_1/b_e) lc / [(2a_1b_e/al) + 3] \delta \theta_1$$
(4.6)

hence, the thermal force is

$$Q = a_1 lc / [(2a_1 b_e / al) + 3]\theta_0$$
(4.7)

Since we are dealing with a steady state, the equation to apply here is

$$\partial V / \partial \theta_1 = Q \tag{4.8}$$

Substituting the values for V and Q, this equation yields the temperature θ_1

$$\theta_1 = \theta_0 1 / [(2a_1 b_c / 3al) + 1]$$
(4.9)

The assumption that H_b is distributed uniformly over the effective width b_e is, of course, an approximation which is satisfactory only if the applied temperature b_0 does not vary appreciably along the flange. A more accurate picture is obtained if the distribution of H_b along the flange is determined by applying the principle of minimum dissipation to the distribution itself. This leads to a differential equation for H_b . The result is an exponential distribution expressed by

$$H_b = \sqrt{3}(a_1 H_1 / b_e) \exp\left(-x(\sqrt{3} / b_e)\right)$$
(4.10)

(x = distance from the web.) We then proceed exactly as in the foregoing. By using the exact distribution of H_b , it is possible to evaluate immediately the temperature explicitly in terms of any arbitrary distribution of external temperature along the flange, and calculations do not have to be repeated for different distributions of external temperatures.

The method of associated flow fields may also be used to evaluate the transient temperatures. As an example, we choose the same geometry as aforementioned. We shall consider the symmetric case where a uniform external temperature θ_0 is applied equally to the top and bottom flange. The temperatures are now symmetric with respect to the point M at the midweb. In calculating this case, we may assume that we are dealing with a finite section of the structure with a flange width equal to the effective width b_e as shown in Fig. 3. The temperature field in the web is assumed to have a parabolic distribution determined by the temperature θ_3 at the midweb M and by the temperature θ_1 at the joint A. In the flange, the temperature distribution is assumed to vary linearly from θ_1 to θ_2 (Fig. 3). The



Fig. 3. Effective portion of wing structure and assumed temperatures for the transient symmetric case.

temperature θ_2 is assumed to be the same as in an infinite flange without the web. This temperature is then determined by the differential equation

$$ac\theta_2 + K\theta_2 = K\theta_0(t) \tag{4.11}$$

If the external temperature is

$$\theta_0(t) = \begin{cases} 0 & t < 0\\ \theta_0 & t > 0 \end{cases}$$

$$(4.12)$$

the solution is

$$\theta_2 = \theta_0 (1 - e^{-t/\tau}) \tag{4.13}$$

with

 $\tau = ac/K$

The temperature field is completely determined by the two unknowns θ_1 and θ_3 . They may be considered as the two generalized coordinates. We calculate the flow field associated with this temperature field and apply the general theory. This leads to two simultaneous equations.

$$A_{11}\theta_1 + A_{13}\theta_3 + \tau(B_{11}\theta_1 + B_{13}\theta_3) = f_1 A_{13}\theta_1 + A_{33}\theta_3 + \tau(B_{13}\dot{\theta}_1 + B_{33}\dot{\theta}_3) = f_2$$

$$(4.14)$$

The coefficients A_{ij} and B_{ij} are numerical nondimensional quantities determined by the geometry and the physical parameters. On the right-hand side are given functions of time.

$$f_1 = C_{10}\theta_0 + C_{12}\theta_2 + \tau C_{12}'\dot{\theta}_2$$

$$f_2 = C_{20}\theta_0 + cC_{22}\dot{\theta}_2$$
(4.15)

The coefficients are again nondimensional quantities depending on the geometry and physical parameters.

We have solved these equations numerically for the example treated by Pohle and Oliver.¹⁰ We have also discussed the same example⁴ previously. The external temperature θ_0 is assumed to be a sudden increase to a constant value as given by Eq. (4.12). The temperature θ_1 plotted as a function of time cannot be distinguished from the exact value as calculated by Pohle and Oliver.¹⁰

This is not the case, however, for the temperature θ_3 which shows a departure from the correct value during the transit time⁴ from A to M. The values, in fact, come out negative during this phase. This is due to our assumption of a parabolic distribution extending all the way across the web, an assumption which is, of course, not valid during the penetration phase. However, this does not affect the correctness of the joint temperature at A. There is a mathematical reason for this which we shall not discuss here since it would carry the discussion further than intended. We shall be contented to indicate the result and note that the important temperature to determine in these problems is the temperature θ_1 at the joint. Once this temperature is known, it becomes quite easy to evaluate the time history in the web. A variety of methods can be employed. We can either use the procedure which was introduced in the earlier work.⁴ This procedure consists in taking the temperature θ_1 as the imposed wall temperature of a one-dimensional problem and applying Duhomel's integral. Another approach which can be used for the penetration phase is to apply the differential Eq. (2.21) derived for the penetration depth where the wall temperature is a given function of time.

The procedures which we have discussed on a simple example may be applied to complex three-dimensional structures. The concept of effective flange width makes it possible to treat parts of the structure as separate entities.

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⁶ Onsager, L., *Reciprocal Relations in Irreversible Processes I*, The Physical Review, Vol. 37, No. 4, pp. 405-26, February 15, 1931.

⁷ Meixner, J., Zur Thermodynamik der Thermodiffusion, Annalen der Physik, Ser. 5, Vol. 39, No. 5, pp. 334–55, 1941.

⁸ Biot, M. A., New Thermomechanical Reciprocity Relations with Application to Thermal Stress Analysis, Cornell Aeronautical Laboratories Report SA-987-S-7, June 1958; also Journal of the Aero/Space Sciences, Vol. 26, No. 7, pp. 401–08, July 1959.

⁹ Biot, M. A., *Linear Thermodynamics and the Mechanics of Solids*, Proceedings of the Third U.S. National Congress of Applied Mechanics (Published by A.S.M.E.), 1958.

THERMODYNAMICS AND HEAT-FLOW ANALYSIS BY LAGRANGIAN METHODS

¹⁰ Pohle, F. V., and Oliver, H., *Temperature Distribution and Thermal Stresses in a Model of a Supersonic Wing*, Journal of the Aeronautical Sciences, Vol. 21, No. 1, pp. 8–16, January 1954.

Discussion

A. R. Collar: I know Prof. Biot's expertise in the field of matrix analysis, and I suspect that the fact that he has not used matrices in the presentation of his subject may mean that matrix algebra is not well suited to it; although sets of linear equations (and such quantities as ignorable coordinates) can often be conveniently expressed in terms of matrices. Would he please comment?

M. A. Biot: Matrix algebra is indeed in many cases, though not always, a most appropriate tool for handling thermodynamic problems by Lagrangian methods. I welcome this opportunity to bring out some salient points on the subject which were not mentioned in earlier work because of limitation of time and space. It is clear that the representation of a continuum in terms of discrete generalized coordinates by use of variational methods leads in many instances to a matrix formulation. Most of the standard procedures of matrix algebra are applicable. As I have pointed out⁴ earlier, iteration methods for example and orthogonality conditions may be used. However, the elimination of the ignorable coordinates in heat-flow problems by matrix algebra does not seem convenient because there is theoretically an infinite multiplicity of such coordinates. This is in contrast with vibration problems of an elastic free body where only six ignorable degrees to freedom must be eliminated. For this reason, we have preferred to introduce the concept of "associated field" which provides a way to satisfy automatically an infinite number of orthogonality conditions. In addition, the concept itself is applicable to a much larger class of problems. I would also like to mention that there are areas of overlap between the Lagrangian approach and the powerful methods developed by Kron in his network analysis. The principle of virtual work for thermodynamic systems as formulated in one of my earlier papers² is closely related to Kron's matrix method of interconnection of networks. The principle is a generalization of the old Lagrangian technique for the elimination of forces of constraints.

A. R. Collar: Since the subject-matter of his paper appears a most useful concept, and the literature is spread through a variety of journals, is Prof. Biot proposing a book on the subject? If he is, I can promise a sale of at least one copy in the United Kingdom.

M. A. Biot: The collection of this material in book form has of course been considered. Unfortunately, there is always a considerable gap between time schedules and realization. In the meantime, a special effort was made to present the heat-flow problem as self contained in Refs. 4 and 5 while Ref. 9 offers a condensation of the more general thermodynamics in monograph form.