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Trapping of Acoustic Energy Near a Source Above a Submerged Elastic Plate

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The propagation of acoustic waves in a layered medium is considered for the case of an elastic plate submerged at a certain depth below the surface of a liquid-filled half-space. It is shown that there exist unattenuated modes for the plate-liquid layer system which have horizontal phase velocities greater than the sound velocity in the liquid. In spite of such greater phase velocities, radiation into the liquid below does not take place because the lower surface of the plate exhibits no vertical motion. Thus, energy can be trapped in and above the plate by vibrations which leave the underlying liquid undisturbed. It is pointed out that radiation in the horizontal direction may also be very small when the above conditions of total reflection are approximately satisfied, as is indicated by the existence of low group velocities for the modes of a free plate. Attention is called to the three-dimensional nature of such trapping, vertically by an approximate condition of no transmission across the plate, and horizontally by the simultaneous vanishing of the group velocity at nonvanishing angles of incidence. The present modes are in contrast with the usual case of total reflection and "wave guide propagation" in layered elastic media, where the horizontal phase velocity is less than the sound velocity in the liquid (lower half-space), and the signal below the plate decays exponentially with distance from the interface. Conditions under which the present modes can arise have been found and evaluated numerically. Relations between phase velocity, wave number, and ratio of fluid layer to plate thickness are presented for a lucite plate in water. These have been compared with the results of much more elaborate calculations for the point source "singing" problem.

INTRODUCTION

WE consider the case of an elastic plate of infinite extent immersed in a liquid half-space and oriented parallel to the free surface of the liquid. A point source of explosive sound is located in the liquid layer above the plate (see Fig. 1). This arrangement represents one of the simplest geometries for which sustained vibrations can be observed in the liquid layer. A simple and instructive explanation of these persistent oscillations can be given in terms of plane harmonic waves incident upon the plate from above, although a quantitative description of the phenomenon requires the solution of the complete point source problem.

It is our purpose to bring out clearly the characteristic features of this phenomenon, which is essentially different from the type of total reflection and "wave guide propagation" in layered media, usually discussed in the literature. In the latter case, each mode propagates with a phase velocity lower than the body wave velocity in the underlying half-space, and the resulting total reflection occurs for angles of incidence greater than the critical. The amplitude in the lower half-space decays exponentially with distance from the interface.

In the present case, the total reflection is of a different nature. We will show that unattenuated modes of vibration may propagate in the fluid-plate system with a phase velocity greater than the sound velocity in the underlying fluid, and the bottom side of the plate

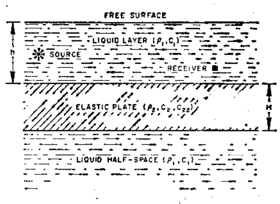

FIG. 1. The submerged plate problem.

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may exhibit no vertical motion or stress. This particular feature cannot occur in a medium consisting entirely of fluid layers; it results solely from the fact that one of the layers is a solid in which two types of body waves can coexist. It is the interplay of these two types of waves which makes the phenomenon possible.

We believe that an understanding of this comparatively simple case of a plate immersed in a liquid half-space will be of help in the investigation of more complicated problems of direct practical interest. A case in point is the so-called "singing" phenomenon,[1] which has been observed in seismic work in several offshore areas. Records taken in such areas may show persistent, slowly decaying types of oscillation, which drown out all other signals.

In order that sustained vibrations be observed in the liquid layer above the plate, it must be possible to trap energy in the layer in both vertical and horizontal directions. This can happen when (1) there is no net transmission through the plate (over one cycle) in the vertical direction; (2) the liquid layer is resonant to such totally reflected waves; (3) the transport of energy in the horizontal direction is small. These conditions, which apply to harmonic plane waves trapped in a large horizontal section of the liquid layer, are also applicable approximately to the transient point source problem, discussed elsewhere.

Now the first two conditions are applicable to the usual type of modes, whose phase velocities are less than the sound velocity in the underlying liquid, as well as to the present case, where the phase velocity exceeds the sound velocity in the liquid. The third condition of small horizontal energy transport, however, appears as an important feature in the present case of total reflection. As an example, take a case in the region near normal incidence (small wave number in the horizontal direction). For physical reasons this should also be a region of small group velocity and, therefore, slow energy transport in the horizontal direction.[2] Indeed low, if not vanishing, group velocities in the region *near* normal incidence can be inferred from an inspection of the known dispersion curves for the free plate.[3] There are also indications that low and vanishing group velocities may be observed for angles of incidence not near normal in a system consisting of a free plate with a liquid layer above it. Low horizontal energy transport may thus be expected in the case of the submerged plate when the present total reflection conditions are *approximately* satisfied.[4]

[1] K. E. Burg, M. Ewing, F. Press, and E. J. Stulken, Geophysics **16**, 594 (1951).

[2] General theorems on the equivalence of group velocity and energy transport in layered media have been established in M. A. Biot, Phys. Rev. **105**, 1129 (1957).

[3] R. D. Mindlin, "An introduction to the mathematical theory of vibrations of elastic plates," U. S. Army Signal Corps Engr. Lab., Fort Monmouth, New Jersey, (1955).

[4] This has actually been verified in a detailed analysis of the point source problem shown on Fig. 1.

It should also be noted that for the lowest usual mode, which corresponds to a bending of the plate, the group velocity vanishes at infinite wavelength. However, vibrations in this neighborhood would have a very low amplitude because they are not easily excited by a pressure source in the liquid layer,[5] and a large amount of energy would be required to move a great mass of liquid underneath the plate, where the wave motion decays exponentially with depth. In contrast, the present unattenuated modes do not excite the liquid below the plate.

As was pointed out in the preceding, the type of unattenuated modes considered here results essentially from the fact that forced vibrations may be excited in a free plate by a pressure applied to the top face in such a manner that for certain frequencies there is no vertical motion of the bottom face. To be more specific, let us apply to the top surface of a plate a normal force which is harmonic in time and in space. Forced vibrations of the same frequency and the same wavelength along the plate will be excited. These vibrations can be considered as a superposition of the two types of characteristic plate modes: symmetric and antisymmetric with respect to the central plane of the plate. For the elastic plate, the resonance frequencies of these two sets of modes are spaced quite differently from those of a fluid slab, and it is possible to adjust their phase and amplitude in such a way that by superposition the resultant vertical motion at the bottom face of the plate vanishes. Furthermore, it is always possible to find a fluid layer of appropriate thickness, so that it is in resonance with the plate under the foregoing conditions. In the following sections we treat forced vibrations of the plate and the liquid layer in turn and obtain the free vibrations of the plate-liquid layer system by impedance matching.

THE ELASTIC PLATE

We consider the case of total reflection associated with the absence of stress and vertical motion at the lower plate-liquid interface. To derive this condition, we need consider only the plate by itself (see Fig. 2). As is usual for problems of this type, we define a scalar and a vector displacement potential such that the displacement vector $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi} = -\nabla M + \nabla \times \mathbf{N}, \qquad (1)$$

where $\nabla \cdot \mathbf{N} = 0$. On account of the symmetry of our two-dimensional problem, only the y component of N

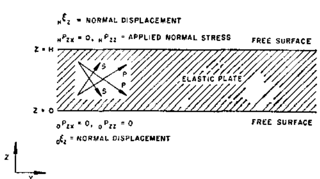

FIG. 2. The free elastic plate.

[5] M. A. Biot, and I. Tolstoy, J. Acoust. Soc. Am. **29**, 381 (1957).

exists (y is perpendicular to the plane of Fig. 2). We write this component of $\mathbf{N}$ as N. Then

$$\nabla^2 M - \frac{1}{c_2^2}\frac{\partial^2 M}{\partial t^2}=0,$$
$$\nabla^2 N - \frac{1}{c_{22}^2}\frac{\partial^2 N}{\partial t^2}=0, \tag{2}$$

where c_2 and c_{22} are the compressional and shear wave velocities, and t is the time. For plane harmonic waves traveling in the positive x direction, but in the positive and in the negative z directions, we write

$$M=(A\cos\alpha_2 kz+B\sin\alpha_2 kz)\exp[i(\omega t-kx)],$$
$$N=(C\cos\alpha_{22}kz+D\sin\alpha_{22}kz)\exp[i(\omega t-kx)]. \tag{3}$$

M corresponds to the compressional (P) waves N to the distortional (S) waves, ω is the angular frequency, k the wave number in the x direction.

$$\alpha_2=\left[\frac{c_{22}^2}{c_2^2}V^2-1\right]^{\frac{1}{2}}\quad \alpha_{22}=(V^2-1)^{\frac{1}{2}}, \tag{4}$$

where $V=\omega/(c_{22}k)$ is the phase velocity in the x direction, normalized with respect to the shear wave velocity. The normal stress P_{zz} and the shear stress P_{zx} are given by the formulas

$$P_{zz}=-\rho_2(c_2^2-2c_{22}^2)\nabla^2 M+2\rho_2 c_{22}^2\left[-\frac{\partial^2 M}{\partial z^2}+\frac{\partial^2 N}{\partial z\partial x}\right],$$
$$P_{zx}=\rho_2 c_{22}^2\left[-2\frac{\partial^2 M}{\partial x\partial z}+\frac{\partial^2 N}{\partial x^2}-\frac{\partial^2 N}{\partial z^2}\right], \tag{5}$$

where ρ_2 is the density of the plate.

We designate boundary values at the lower and upper surface of the plate of thickness H by the subscripts 0 and H. Set ${}_0P_{zx}={}_HP_{zx}={}_0P_{zz}=0$. (Such boundary conditions are of course peculiar to our problem of the submerged plate, as was indicated earlier.) We define the transfer receptance $Y(H,0)={}_0\xi_z/{}_HP_{zz}$ as the ratio of the normal displacement at the bottom of the plate to the applied normal stress at the top.[6] From the foregoing equations and boundary conditions, we derive

$$Y(H,0)=\frac{V^2H}{\rho_2 c_{22}^2\kappa}[4\alpha_2\alpha_{22}\sin\alpha_2\kappa+(V^2-2)^2\sin\alpha_{22}\kappa]/$$
$$\{[4\alpha_2\alpha_{22}\sin\alpha_2\kappa+(V^2-2)^2\sin\alpha_{22}\kappa]$$
$$\times[4\alpha_2\alpha_{22}\sin\alpha_{22}\kappa+(V^2-2)^2\sin\alpha_2\kappa]$$
$$-4\alpha_2\alpha_{22}(V^2-2)^2(\cos\alpha_2\kappa-\cos\alpha_{22}\kappa)^2\}, \tag{6}$$

[6] The "receptance," as defined here, corresponds to the more familiar "admittance," except that it describes the displacement-to-stress ratio instead of the usual particle velocity-to-stress ratio.

$\kappa=kH$ is the wave number in the x direction, normalized with respect to the plate thickness.

Evidently, there will be no transmission through the plate when $Y(H,0)=0$. This condition arises for

$$4\alpha_2\alpha_{22}\sin\alpha_2\kappa+(V^2-2)^2\sin\alpha_{22}\kappa=0, \tag{7}$$

unless

$$\cos\alpha_{22}\kappa=\cos\alpha_2\kappa, \tag{8}$$

that is, unless $\alpha_{22}\kappa=n\pi$ and $\alpha_2\kappa=l\pi$, where $n=2$, 3, $4\cdots$, $l=0$, 1, 2, $\cdots$, and $n-l$ is even. For $l\geq 1$, this is a case of degeneracy where frequencies of the symmetric and antisymmetric modes of the free plate coincide. The vertical motion of the plate bottom obviously remains indeterminate. A similar explanation applies to the point $V=2(1-c_{22}^2/c_2^2)^{\frac{1}{2}}$, the longitudinal plate velocity, where both the frequency ω and the wave number κ vanish. Other exceptional points are the limits $V=\infty$ ($\kappa=0$, $\alpha_{22}\kappa=n\pi$), where the plate vibrates in pure shear, and both surfaces move horizontally. Such motion cannot be excited by a normal force on the surface. Finally, the case $V=\sqrt{2}$ has a solution only if α_2 vanishes at the same time. This means that the plate must have a vanishing Poisson's ratio. The symmetric mode associated with this case is a longitudinal vibration of the plate with no distortion of the faces. Again this cannot be excited by normal pressures.

Equation (7) can be derived even more simply if we set ${}_0P_{zx}={}_HP_{zx}={}_0P_{zz}={}_0\xi_z=0$, and look for nontrivial solutions of this system of equations. An expression which corresponds to Eq. (7) has been given by Levi and Nagendra Nath.[7] No such condition of vanishing transfer receptance is possible when the solid plate is replaced by a liquid layer. This follows from the fact that if pressure and normal displacement both vanish on one face of the liquid layer, they must vanish everywhere in the layer, and can readily be verified with the aid of expressions analogous to Eq. (12) and (11) of the next section.

Equation (7) provides a relation between κ and V (or κ and ω). Let us consider it as an equation for κ and vary V. For $V>c_2/c_{22}$, α_2 and α_{22} are real, and there is an infinite number of roots κ. In the interval

$$2(1-c_{22}^2/c_2^2)^{\frac{1}{2}}<V<c_2/c_{22},$$

α_{22} is real and α_2 pure imaginary, and there is a finite number of roots κ. There are no real roots κ for

$$V<2(1-c_{22}^2/c_2^2)^{\frac{1}{2}}.$$

Thus solutions for real values of κ and V exist only for phase velocities in the interval $2(1-c_{22}^2/c_2^2)^{\frac{1}{2}}<V<\infty$. As mentioned previously, κ vanishes at the lower limit of the interval, which is in fact the velocity of longitudinal plate waves of infinite wavelength.[8] Similarly,

[7] F. Levi, and N. S. Nagendra Nath, Helv. Phys. Acta **11**, 408 (1939), Eq. (39).

[8] This velocity is more commonly expressed as $[E/\rho_2(1-\nu^2)]^{\frac{1}{2}}$, where E is Young's modulus and ν is Poisson's ratio.

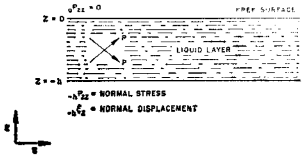

FIG. 3. The liquid layer.

κ vanishes at the upper limit of $V\to\infty$, but in such a way that $\alpha_{22}\kappa\to n\pi$, ($n=1,2,\cdots$). n may be used to identify various plate "modes;" each branch of solutions of Eq. (7) extends from infinite phase velocity to a phase velocity less than the compressional wave velocity of the plate c_2. The curves for $n=2,3$; $n=4,5$; $n=6,7$; etc., which incidentally do not represent dispersion curves of the submerged plate problem, merge at their lowest velocity, and for that reason the distinction between an even and the next higher odd branch is somewhat arbitrary. For large n, this point of lowest phase velocity approaches the compressional wave velocity.

For use in the next section, we define the receptance of the plate $Y(H,H)={}_H\xi z/{}_HP_{zz}$ as the ratio of the displacement of the top surface of the plate to the normal stress applied there. For the case of vanishing transfer receptance, $Y(H,0)=0$, we find that

$$Y(H,H)=\frac{H}{\rho_2c_{22}^2\kappa}\left[\frac{V^2\sin\alpha_{22}\kappa}{4\alpha_{22}(\cos\alpha_{22}\kappa-\cos\alpha_2\kappa)}\right].\qquad(9)$$

THE LIQUID LAYER

The pressure P in the liquid layer (see Fig. 3) is governed by the wave equation

$$\nabla^2P-\frac{1}{c_1^2}\frac{\partial^2P}{\partial t^2}=0,\qquad(10)$$

where c_1 is the sound velocity. The displacement ξ is related to the pressure by the expression

$$\frac{\partial^2\xi}{\partial t^2}=-\frac{1}{\rho_1}\nabla P,\qquad(11)$$

where ρ_1 is the density of the liquid. Again we consider plane harmonic waves traveling in the positive x direction, and in the positive and in the negative z directions.

$$P=(E\cos\alpha_1kz+F\sin\alpha_1kz)\exp[i(\omega t-kx)],$$
$$\alpha_1=\left[\frac{c_{22}^2}{c_1^2}V^2-1\right]^{\frac{1}{2}}.\qquad(12)$$

At the upper (free) boundary of the liquid layer of thickness h, the pressure $_0P$ vanishes. Therefore E must also vanish. At the lower boundary, we define the receptance $Y(-h,-h)=-({}_{-h}\xi_z/{}_{-h}P)={}_{-h}\xi_z/{}_{-h}P_{zz}$, where the normal stress P_{zz} is the negative of the fluid pressure. We find that

$$Y(-h,-h)=\frac{\alpha_1k}{\rho_1\omega^2}\cot(\alpha_1kh).\qquad(13)$$

Since the liquid layer is to be resonant to waves which are totally reflected from the plate, we must match the receptance of the layer to that of the non-transmitting plate. By equating the right-hand sides of Eqs. (9) and (13), we obtain the resonance condition of the liquid layer

$$\tan\left[\alpha_1\kappa\frac{h}{H}\right]=\frac{4\alpha_1\alpha_{22}(\cos\alpha_{22}\kappa-\cos\alpha_2\kappa)}{(\rho_1/\rho_2)V^4\sin\alpha_{22}\kappa},\qquad(14)$$

which, along with Eq. (7), gives the water layer to plate thickness ratio as a function of κ or V.

A simple ray picture of the phenomenon in the liquid layer is readily obtained. The angle of incidence θ of a ray on the plate is given by

$$\theta=\sin^{-1}[c_1/(c_{22}V)]=\cot^{-1}\alpha_1.\qquad(15)$$

With the aid of Eq. (9), we can readily show that the reflection coefficient of a ray totally reflected from the liquid-plate interface must be $\exp(i2\eta)$, where

$$\eta=\cot^{-1}\left[\frac{4\alpha_1\alpha_{22}(\cos\alpha_{22}\kappa-\cos\alpha_2\kappa)}{(\rho_1/\rho_2)V^4\sin\alpha_{22}\kappa}\right].\qquad(16)$$

The angle 2η is to be interpreted as the phase shift suffered by the pressure wave on total reflection by the plate. At the free surface, the ray again suffers total reflection with a phase shift of angle $-\pi$. It can be shown[9] that constructive interference will occur in the liquid layer if

$$2h\frac{\omega}{V}\cos\theta+2\eta=(2m+1)n\quad(n=0,1,2,\cdots)\qquad(17)$$

TABLE I. Comparison of vertically trapped modes with more exact solutions to the singing problem.

h/H	n,	m	Calculated for long-time singing Re($\bar\omega$)	U	Calculated for total reflection $\bar\omega$	U
0.378	2	0	6.22	0	6.47	0.475
0.567	1	0	3.13	0	3.71	1.02

[9] W. M. Ewing, W. S. Jardetsky, and F. Press, *Elastic Waves in Layered Media* (McGraw-Hill Book Company, Inc., New York, 1957), p. 140.

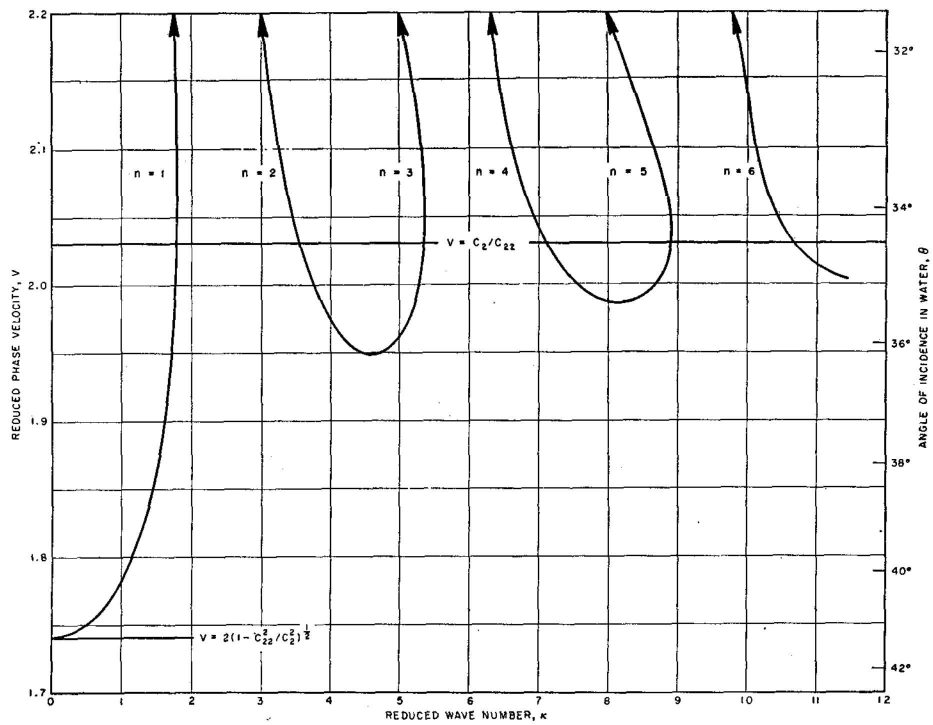

FIG. 4. Roots of Eq. (7) for total reflection from a submerged Lucite plate.

Therefore,

$$\alpha_1\kappa\frac{h}{H}=[m+\tfrac{1}{2}]\pi-\eta$$

$$=m\pi+\tan^{-1}\left[\frac{4\alpha_1\alpha_{22}(\cos\alpha_{22}\kappa-\cos\alpha_2\kappa)}{(\rho_1/\rho_2)V^4\sin\alpha_{22}\kappa}\right],\quad(18)$$

which agrees with Eq. (14).

The ratio h/H is a multiple-valued function of κ. The letter m is used to identify the various "modes" in the liquid layer. The higher modes have an increasing number of nodal planes in the layer. The values of h/H for the various modes differ by a value $m\Delta$ from the fundamental. From Eq. (18), we see that

$$\Delta=\pi/\alpha_1\kappa.\quad(19)$$

It is interesting to note the role of the medium in the lower half-space in the expressions which we have derived here. The lower medium enters into the derivation of Eq. (7) only to the extent that it may support no shear stress at the boundary of the plate. Its other properties do not enter into the foregoing expressions, but will be of importance in the complete point source problem, shown in Fig. 1.

NUMERICAL EXAMPLES

Solutions to Eqs. (7), (14), and (19) have been obtained for a lucite plate in water and are presented here as an example. A comparison of the calculated parameters with more elaborate calculations (involving complex dispersion relations) for the point source-submerged plate singing problem, as illustrated in Fig. 1, is presented in Table I. The nondimensional angular frequency is defined as $\bar{\omega}=\kappa V=\omega H/c_{22}$, and U is the horizontal group velocity normalized with respect to the shear wave velocity. Re designates the real part of a complex quantity. Although strong singing for the plate always appears to be associated with a mode of the type described here, the inverse is not necessarily true. For as we indicated earlier, the actual singing phenomenon requires low net transmission of energy vertically through the plate at vanishing horizontal group velocity.

For the numerical results presented here, the following constants, are applicable: $\rho_2/\rho_1=1.15$; $c_2:c_1:c_{22}=2{,}650:1{,}500:1{,}305$. The first six plate modes, $n=1$–6 have been evaluated. The curves for h/H apply to the lowest water mode; values for higher water modes can be obtained by the addition of integer multiples of the function Δ, the supplemental thickness ratio. Numerical results are presented in Figs. 4–9. We wish

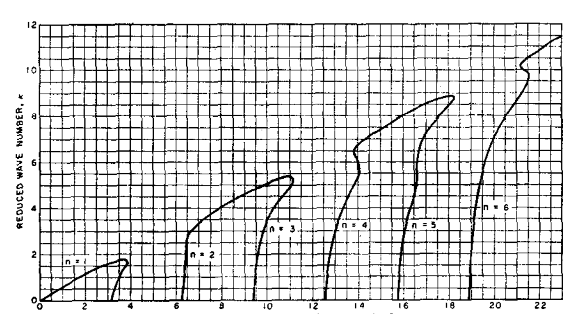

FIG. 5. Roots of Eq. (7) for total reflection from a submerged Lucite plate.

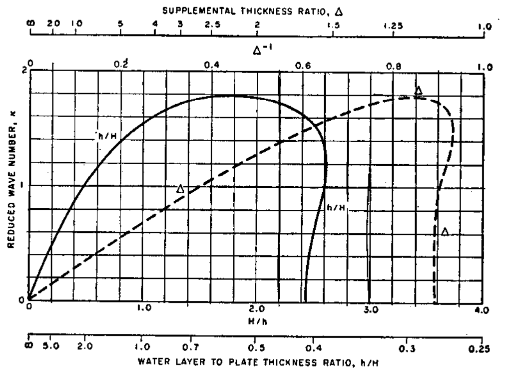

FIG. 6. Solutions of Eqs. (14) and (19) for the resonant water layer above a totally reflecting Lucite plate. First plate mode, $n=1$.

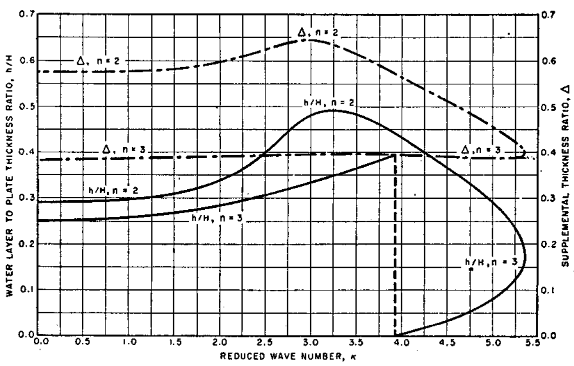

FIG. 7. Solutions of Eqs. (14) and (19) for the resonant water layer above a totally reflecting Lucite plate. Second and third plate modes, $n=2$, 3.

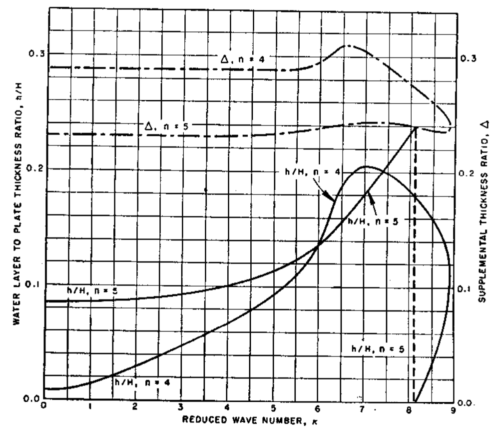

FIG. 8. Solutions of Eqs. (14) and (19) for the resonant water layer above a totally reflecting Lucite plate. Fourth and fifth plate modes, $n=4$, 5.

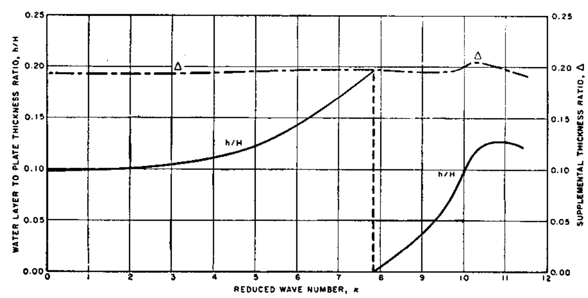

FIG. 9. Solutions of Eqs. (14) and (19) for the resonant water layer above a totally reflecting Lucite plate. Sixth plate mode, $n=6$.

to emphasize again that Figs. 4 and 5 do not represent the usual dispersion curves of the submerged plate problem of Fig. 1, but rather the locus of points where dispersion curves of complex $\bar{\omega}$ versus real κ (or complex κ versus real $\bar{\omega}$) touch the real $\bar{\omega}$ (or real κ) axis as we vary h/H.[10]

Total reflection at the plate and resonance of the water layer do not arise at points where Eq. (8) is applicable. For the cases $\alpha_2=0$, these points occur at the intersection of the even plate modes with the line $V=c_2/c_{22}$ (see Fig. 4). For the cases $\alpha_2\neq 0$, they occur at the places where $h/H=0$, i.e., where we have indicated jumps in the h/H curves of Figs. 7–9.

[10] When the sound velocity in the liquid exceeds the longitudinal plate velocity, Eq. (7) has solutions which correspond to points on the "normal mode" dispersion curves of the submerged plate problem (real $\bar{\omega}$ versus real κ).

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