Variational and Lagrangian Thermodynamics of Thermal Convection—Fundamental Shortcomings of the Heat-Transfer Coefficient

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April 10, 1961

In two recent papers we have discussed the shortcomings of the concept of boundary-layer heat-transfer coefficient and developed some new approaches to thermal flow analyses which do not make use of this concept and are based on the variational and Lagrangian thermodynamics introduced earlier by this writer.

The first paper,[1] which deals more specifically with the local heat-transfer coefficient, shows that this concept is physically unsound and is grossly inadequate for problems in which the streamwise temperature gradient is appreciable. Our purpose was to furnish a practical justification for the new methods presented in the second paper.[2] These new methods are based on the extension to convective heat transfer of the same variational principles and Lagrangian methods developed earlier.[3-5] They do not involve the concept of heat-transfer coefficient and provide a unified analysis of convection and conduction for composite systems comprising a solid structure and moving fluids in turbulent or laminar flow. This unified theory[2] proceeds along two distinct lines. In the first the fluid transfer properties are embodied in a new concept which leads directly to Lagrangian equations for conductive-convective thermal systems of a very general nature. In the second method the Lagrangian equations are formulated by using a dissipation function applicable to convective flow and based on the thermodynamic concept of an entropy production which includes the effect of turbulent diffusion. Along this line an important result was recently obtained by Nigam and Agrawal in the context of laminar flow.[6] They proposed a formal expression for the dissipation function which is different from Eq. (7) below but which coincides with it in some particular cases.

The new developments presented here indicate the possibility of extending the thermodynamics of irreversible processes to systems which are not in the vicinity of an equilibrium state and for which Onsager's relations are not verified. This involves generalizations beyond the narrow field of heat transfer and to principles of wider range than those of nonequilibrium thermodynamics as they stand at this time.

We shall briefly summarize the contents of these two papers. The basic inadequacy of the local heat-transfer coefficient[1] may be demonstrated very simply by considering the particular case of a fluid moving with a uniform velocity V_0 along a straight wall. With the x coordinate along the wall and the y coordinate in the normal direction a well known approximate equation for the fluid temperature θ is

$$V_0(\partial\theta/\partial x) = \kappa(\partial^2\theta/\partial y^2) \tag{1}$$

where κ is the thermal diffusivity of the fluid. A solution of this equation is

$$\theta = \theta_0 \exp\,(ilx)\exp(-y\sqrt{iV_0 l/\kappa}) \tag{2}$$

It corresponds to a sinusoidal distribution of temperature along x at the wall. The local rate of heat-transfer is also sinusoidal, and the ratio of this heat transfer to the wall temperature is a complex quantity

$$K = k\sqrt{V_0 l/\kappa}\exp\,(i\pi/4) \tag{3}$$

where k is the fluid thermal conductivity. This shows that the local heat flow is 45° out of phase with the temperature. Hence over 25 percent of the wall *the heat flows in a direction opposite to the so-called adiabatic temperature difference.* Moreover the value of K depends on the wavelength, hence on the temperature gradient.

The analysis was carried out for the more general case of a laminar or turbulent boundary layer[1] and inadequacies of the same order were obtained.

As a sideline to its main theme the first paper[1] also calls attention to the possibility of evaluating the convective heat flow from existing variational principles by extending the conduction analogy to boundary-layer heat transfer. It is well known that Eq. (1) represents a conduction analogy if the coordinate x is considered to represent the time variable. For a boundary layer with nonparallel streamlines the temperature distribution θ satisfies the equation

$$\partial\theta/\partial x = (\partial/\partial\psi)[uA(\partial\theta/\partial\psi)] \tag{4}$$

where ψ is the stream function and $u(x,\psi)$ the x component of the velocity. The quantity $A(x,\psi)$ is the total diffusivity—i.e.,

$$A = \kappa + \epsilon \tag{5}$$

where κ and ϵ are, respectively, the thermal and the turbulent diffusivities. Eq. (4) represents a *conduction analogy* with a time variable x and a space coordinate ψ in a medium of variable conductivity uA. The variational methods developed earlier for conduction[3] are directly applicable to this case. An example shows that they are quite accurate. In practice the method is also applicable when the mean flow is unsteady.

The generalized variational and Lagrangian methods developed in the second paper[2] follow a twofold approach.

In the first the Lagrangian equations for transient heat flow in a solid in contact with a moving fluid are formulated by introducing a *trailing function.* This function represents the influence of a heat injection into the fluid at a certain point at the surface. This new concept is well suited to the Lagrangian and variational formulation. It is found that the evaluation of the trailing function by variational methods is remarkably accurate. The method of "associated fields" which was developed earlier[4] in order to simplify the formulation of two- and three-dimensional problems is also extended to convective heat transfer. In addition a "reverse-flow" theorem has been derived which reflects the non-self-adjoint properties of the system.

The second approach is more fundamental from the thermodynamic viewpoint. In it the variational principle and the Lagrangian equations

$$(\partial v/\partial q_i) + (\partial D/\partial \dot{q}_i) = Q_i \tag{6}$$

are extended to include convective heat transfer in laminar and turbulent flow. The Lagrangian equations in this case govern the transient heat flow in a composite system comprising solid structures and moving fluids considered as a single system. Therefore no explicit use need be made of any surface transfer properties. The extension results from the introduction of a dissipation function proportional to the rate of entropy production, namely

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$$D = \frac{1}{2} \int_\tau \frac{1}{cA} (\dot{\mathbf{H}} - h\mathbf{u})^2 d\tau \tag{7}$$

This is a volume integral extended to the composite fluid-solid system, where $\dot{\mathbf{H}}$ and $\mathbf{u}$ are respectively the local rate of heat flow and the velocity. The total diffusivity A may be a function of location, time, and temperature. The heat content h is defined as in a previous paper[3]

$$h = \int_0^\theta c d\theta \tag{8}$$

where c, the heat capacity per unit volume, may be a function of the temperature and in the solid may also be function of the location. The dissipation function defined by Eq. (7) implies the concept of entropy production associated with turbulent diffusivity.

The coefficients in Eqs. (6) for convective systems do not constitute a symmetric matrix and the usual reciprocity properties of the thermodynamics of irreversible processes do not apply. The effect of heat generation by viscous dissipation in the fluid is easily introduced by adding a source in Eq. (8) as indicated in an earlier paper.[4]

References

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