Fundamentals of Boundary-Layer Heat Transfer With Streamwise Temperature Variations†

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Summary

Boundary-layer heat transfer is analyzed for the case of a sinusoidal distribution of temperature in the direction of flow. It is shown that for both laminar and turbulent flow the spatial distribution of heat transfer is generally out of phase with the wall temperature by an angle of 30° to 45°. This leads to the conclusion that in some areas the heat flow is opposite to the temperature difference as used in the definition of the heat-transfer coefficient, and points to the basic shortcomings of this concept. The physical explanation for this behavior is found to be the temperature-field distortion by the fluid motion. The distortion is measured by the Peclet number. Approximate equations representing a "conduction analogy" were used in this analysis and the validity of these equations for unsteady flow is examined with reference to limitations in frequency and wavelength. A solution of these equations is given for the case of a velocity profile which is not a straight line. The use of previously developed variational principles for the evaluation of convective heat transfer including cases of three-dimensional unsteady flow, turbulence, and nonparallel streamlines is also discussed.

(1) Introduction

It has been customary to describe the heat-transfer properties between a solid surface and a moving fluid by the so-called heat-transfer coefficient. While in many engineering problems dealing with simple configurations and average properties the use of such a coefficient is undoubtedly justified, many investigators have been aware of the vagueness and inadequacies attached to this concept. This is particularly true in problems requiring a detailed analysis of the temperature field in a solid whose boundary is in contact with a moving fluid.

Our main purpose is to show that the concept of a local-heat-transfer coefficient, which is generally used in the formulation of such problems, is in many cases grossly inadequate. This has been accomplished by analyzing the mechanisms of heat transfer to a moving fluid for the simplified case of a temperature distribution which varies sinusoidally along a streamwise coordinate.

The methods used in the course of this analysis also suggest some new and simplified procedures for the calculation of heat transfer for laminar and turbulent flow. In particular, simple and accurate variational methods may be used. Some new expressions are also derived which are applicable to problems of heat transfer in boundary layers associated with pressure gradients.

The present results justify the development of new procedures for the analysis of heat flow in systems combining solid conduction and fluid convection which do not require the use of heat-transfer coefficients. This has been presented in a separate companion paper.**

Section (2) deals with the case of uniform flow parallel to a plane boundary. The wall temperature is assumed to vary sinusoidally along the direction of flow and is also a sinusoidal function of time. It is possible to show that within the usual range of frequency and wavelength in subsonic air flow, a simplified quasi-steady equation for convective flow is valid. From this equation, the local-heat-transfer rate is easily evaluated. Its distribution along the space coordinate in the direction of flow is found to be out of phase by an angle of 45 degrees with the temperature. This means that over 25 percent of the surface the heat flow is in a direction opposite to the temperature difference between the solid and the fluid. The physical reason for this peculiarity is also analyzed. It is due to a distortion of the temperature field by the convective flow. The angle of distortion is measured by the Peclet number. The importance of this number as a measure of the convective properties is thereby emphasized.

The same problem is analyzed for a laminar boundary layer in Section (3) and for turbulent flow in Section (4). Inadequacies of the same order affecting the concept of a heat-transfer coefficient are found for these cases.

With reference to the case of laminar flow, attention is called to a remarkable memoir by Leveque published in 1928,[1] which has received very little attention until recently.†† In particular, he derived a solution for the heat transfer in a fluid moving with a velocity proportional to distance from the wall. The same solution was derived later by Lighthill,[3] in the context of boundary-layer theory, using a more elaborate operational procedure.

We have shown that Leveque's result is a particular case of a more general solution for nonlinear velocity profiles. This result leads immediately to simple expressions for the evaluation of heat transfer in boundary

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†† A short account of some of Leveque's results is given by Knudsen and Katz.[2]

layers under conditions of streamwise pressure gradient. The mathematical derivation is given in Appendix I.

In the last section, variational methods are applied to boundary-layer heat transfer by using a "conduction analogy." By this analogy, the Lagrangian equations developed earlier[4] are directly applicable to thermal convection, including the case of turbulent flow and nonparallel streamlines. By suitable use of the stream function the analogy extends to *three-dimensional* flow with axial symmetry and nonparallel streamlines. A numerical example indicates that the procedure is quite accurate. Earlier discussions in Section (2) lead to the conclusion that within certain practical limits *the analogy may be used for a time-dependent velocity* field. The analogy in conjunction with the writer's variational procedures has also been applied by Agrawal[5] in the context of laminar flow and parallel streamlines. The numerical results also show excellent accuracy.

A brief outline of the results obtained in this paper was given earlier.[11]

(2) Some Fundamental Aspects of Heat Transfer for Uniform Flow

Consider a two-dimensional incompressible flow field of velocity components u,v in the x,y plane. A two-dimensional temperature field θ in this fluid satisfies the equation

$$c\left(\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y}\right) = k\left(\frac{\partial\theta^2}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) \quad (2.1)$$

The thermal conductivity k and the heat capacity c per unit volume are assumed constant. For the purpose of deriving some of the essential qualitative aspects of the heat-transfer problem, along with significant orders of magnitude, let us simplify the analysis by putting

$$\left.\begin{aligned} u &= U_0 = \text{const.} \\ v &= 0 \end{aligned}\right\} \quad (2.2)$$

The fluid moves with a uniform velocity U_0 in the x direction. We assume that the fluid is located in the half-space $y > 0$ and that it flows along a solid plane boundary located at $y = 0$ (Fig. 1).

Eq. (2.1) becomes

$$\frac{\partial\theta}{\partial t} + U_0\frac{\partial\theta}{\partial x} = \kappa\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) \quad (2.3)$$

where

$$\kappa = k/c \quad (2.4)$$

is the thermal diffusivity of the fluid.

In particular, we want to find the influence of a streamwise temperature gradient on heat transfer for the case where the temperature is either time-dependent or not. For this purpose, it is natural to assume a temperature which is a sinusoidal function of both the coordinate x and the time t—viz.,

$$\theta = \theta_0 \exp(i\omega t + ilx - \beta y) \quad (2.5)$$

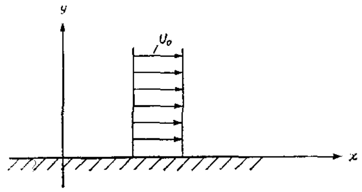

FIG. 1. Uniform flow near a wall.

Substituting in Eq. (2.3), the value of β is found to be given by

$$\beta^2/l^2 = 1 + [i/(\kappa l)][(\omega/l) + U_0] \quad (2.6)$$

Because we wish the solution to vanish for $y = \infty$, we must choose the root β which has a positive real part. If we call D the half-wavelength of the temperature along x, and t_1 the half-time-period, we may write

$$\omega t_1 = lD = \pi \quad (2.7)$$

Hence

$$(\omega/l) + U_0 = (D/t_1) + U_0 \quad (2.8)$$

The time-dependent term $\omega/l = D/t_1$ will be negligible if

$$D \ll U_0 t_1 \quad (2.9)$$

As a numerical example, let us consider an extreme case of unfavorable values,

$$\left.\begin{aligned} U_0 &= 1000 \text{ cm/sec} = 10 \text{ m/sec} \\ t_1 &= 10 \text{ sec} \\ D &= 100 \text{ cm} = 1\text{m} \end{aligned}\right\} \quad (2.10)$$

Even for such extreme values, the error in neglecting the time-dependent term will be only 1 percent. Hence, *in most aerodynamic applications for which* $U_0 t_1/D$ is larger than in the example (2.10), *it will be justifiable to write*

$$\beta^2 = l^2\{1 + i[U_0/(\kappa l)\} \quad (2.11)$$

and to replace Eq. (2.3) by the "quasi-steady" equation

$$(U_0/\kappa)(\partial\theta/\partial x) = (\partial^2\theta/\partial x^2) + (\partial^2\theta/\partial y^2) \quad (2.12)$$

The latter equation can be further simplified by considering numerical values in Eq. (2.11). For air, the value of the diffusivity is

$$\kappa = 0.187 \text{ cm}^2/\text{sec} \quad (2.13)$$

Hence, with the numerical values (2.10) for U_0 and D, we find

$$U_0/(\kappa l) = U_0 D/(\pi\kappa) = 168 \quad (2.14)$$

We may therefore neglect unity as compared with $U_0/(\kappa l)$ and write

$$\beta^2 = i[(U_0 l)/\kappa] \quad (2.15)$$

This amounts to neglecting the term $\partial^2\theta/\partial x^2$ which represents the streamwise conduction in Eq. (2.12). The final approximate equation for the temperature in the moving fluid is therefore

$$(U_0/\kappa)(\partial\theta/\partial x) = \partial^2\theta/\partial y^2 \quad (2.16)$$

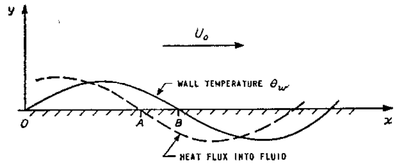

FIG. 2. Sinusoidal wall temperature θ_w and the corresponding distribution of heat flux.

The error involved becomes smaller as the streamwise temperature gradient and the fluid velocity become larger.

A solution of the simplified Eq. (2.16) is

$$\theta = \theta_0 e^{ilx} e^{-\beta y} \tag{2.17}$$

with

$$\beta = \sqrt{iU_0 l/\kappa} = e^{\pi i/4}\sqrt{\pi U_0/\kappa D} \tag{2.18}$$

The temperature at the boundary is

$$\theta_w = \theta_0{}^{ilx} \tag{2.19}$$

If we attempt to express the heat transfer at the wall by means of a quantity similar to the usual concept of a "local heat transfer coefficient," we are led to the expression

$$K = -(k/\theta_w)(\partial\theta/\partial y)_{y=0} \tag{2.20}$$

or

$$K = k\beta = e^{\pi i/4} k\sqrt{\pi U_0/(\kappa D)} \tag{2.21}$$

One of the important features of this results in the complex character of K. The heat flux at the wall is 45° out of phase with the streamwise temperature distribution. This is illustrated in Fig. 2, which shows a plot of wall temperature and heat flux. Between points A and B the heat flux is in a direction opposite to the difference between the wall temperature and the fluid initial "adiabatic" temperature. In this region, the "traditional" heat-transfer coefficient is negative and at point B it is negative and infinite.

Furthermore, this coefficient K is inversely proportional to $\sqrt{D}$. Hence, its magnitude also depends on the temperature gradient. Such behavior certainly does not fit the physical concept of a heat-transfer coefficient. The concept breaks down completely at points of negative or infinite value. In addition, the value of K depends on the distribution of the wall temperature—i.e., on the solution of the complete heat-flow problem for the combined systems of the boundary layer and the conductive solid adjacent to it. Since shorter wavelengths are associated with higher heat transfer, this also implies a "smoothing out" effect on the temperature distribution and a tendency for local hot spots to be evened out faster than would result from the application of the concept of a heat-transfer coefficient.

It is interesting to examine the physical reason why the heat flow reversal occurs in the region AB. We therefore consider the fluid temperature field given by Eq. (2.17). The value (2.18) of β may be written

$$\beta = (1 + i)\sqrt{\pi U_0/(2\kappa D)} \tag{2.22}$$

Hence, the temperature field is

$$\theta = \theta_0 \exp\left(ilx - iy\sqrt{\frac{\pi U_0}{2\kappa D}} - y\sqrt{\frac{\pi U_0}{2\kappa D}}\right) \tag{2.23}$$

The physical temperature is represented by the real part of this expression. The isothermal contours (θ = const.) in the moving fluid are shown schematically in Fig. 3. The reason for the flow reversal in the region AB is immediately apparent from this figure. Because of the convective entrainment by the fluid, the temperature field is distorted, producing a reversal of the temperature gradient near the wall in the region AB.

The significance of Eq. (2.23) is further clarified if we represent it in a skew coordinate system x',y' where the y' axis lies at an angle α with the x' axis (Fig. 3). In these coordinates, Eq. (2.23) is simplified to

$$\theta = \theta_0 \exp\,(ilx' - ly' \cos\alpha) \tag{2.24}$$

The angle of distortion α is given by

$$\tan\alpha = \sqrt{2\pi\kappa/(U_0 D)} \tag{2.25}$$

For the purpose of clarity, the distortion in Fig. 3 is minimized. Actually, with the numerical values (2.10) and (2.13), the angle α will be about 5°. Note that the quantity

$$\Delta = \sqrt{\kappa D/(\pi U_0)} \tag{2.26}$$

has the dimension of a length. For air with

$$\left.\begin{aligned} D &= 100 \text{ cm} \\ U_0 &= 10{,}000 \text{ cm/sec} = 100 \text{ m/sec} \end{aligned}\right\} \tag{2.27}$$

we find

$$\Delta = 0.024 \text{ cm} \tag{2.28}$$

We may write

$$\beta = e^{\pi i/4}/\Delta = [(1 + i)/\sqrt{2}](1/\Delta) \tag{2.29}$$

The solution (2.23) contains a real exponential factor $\exp[-y/(\sqrt{2}\Delta)]$, so that the temperature decreases by a factor $1/e$ in a distance from the boundary equal to

$$y_b = \sqrt{2}\Delta = 0.034 \text{ cm} \tag{2.30}$$

Note that the coefficient K may also be written

$$K = e^{\pi i/4}\, k/\Delta \tag{2.31}$$

The absolute value k/Δ represents the heat-transfer coefficient through a thin layer of fluid at rest and of thickness Δ. We may consider Δ to represent a "thermal-boundary-layer thickness."

Although we have been dealing here with a simplified velocity field, the analysis presented above leads to a clearer understanding of a basic mechanism of heat transfer between a solid wall and a moving fluid. It is,

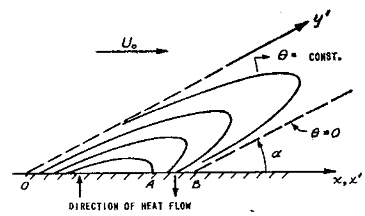

FIG. 3. Distortion of temperature field in the fluid due to its motion.

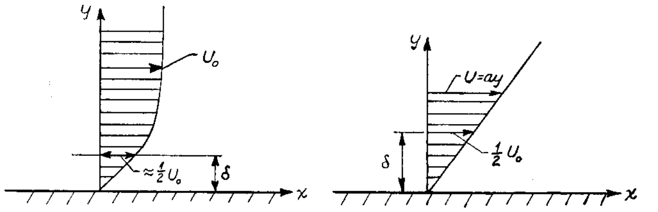

Fig. 4. (Left) Laminar boundary layer profile and displacement thickness δ. FIG. 5. (Right) Linear velocity distribution.

therefore, of interest to indicate which of the dimensionless parameters used in the traditional theories of heat transfer is most representative of the physics. In order to accomplish this, let us assume the fluid to be at rest. The temperature field would then satisfy Laplace's equation

$$(\partial^2\theta/\partial x^2) + (\partial^2\theta/\partial y^2) = 0 \tag{2.32}$$

and the fluid temperature corresponding to a sinusoidal distribution is

$$\theta = \theta_0 e^{ilx} e^{-ly} \tag{2.33}$$

This may be written

$$\theta = \theta_0 e^{ilx} e^{-y/\delta_p} \tag{2.34}$$

with

$$\delta_p = 1/l = D/\pi \tag{2.35}$$

The quantity δ_p is the thermal-boundary-layer thickness for the fluid at rest. We now compare this with the thermal-boundary-layer thickness Δ when the fluid is moving. As given by Eq. (2.26), we find

$$\delta_p{}^2/\Delta^2 = (1/\pi)(U_0 D/\kappa) \tag{2.36}$$

The dimensionless quantity

$$Pe = (U_0 D/\kappa) = (U_0 Dc/k) \tag{2.37}$$

is the Peclet number. (In the usual notation of physics, c should be replaced by ρc, where ρ is the specific mass and c the heat capacity per unit mass.) Hence,

$$\delta_p/\Delta = \sqrt{Pe/\pi} \tag{2.38}$$

In the case of uniform flow, the ratio of the thermal-boundary-layer thickness for a fluid at rest and in motion is, therefore, measured by the Peclet number.

The angle of distortion α considered above is, of course, closely related to the ratio δ_p/Δ and is also measured by the Peclet number. From Eq. (2.25), we derive

$$\tan \alpha = \sqrt{2\pi/Pe} \tag{2.39}$$

We must remember that the Peclet number is defined here in terms of the wavelength of the wall temperature distribution. Hence, it constitutes a dimensionless parameter for the influence of the streamwise temperature gradient. It will now be shown that the general conclusions obtained above by a simplified analysis are also valid for the case of heat transfer in laminar and turbulent boundary layers. This more detailed analysis is carried out in the next two sections.

(3) Heat Transfer in a Laminar Boundary Layer

Let us now examine the more general case when the flow is still laminar but the velocity U in the boundary layer is a function of the distance from the wall (Fig. 4)

$$U = U(y) \tag{3.1}$$

We shall retain the assumption that the velocity component normal to the boundary is negligible. With the same approximations introduced in the previous section for the case U = const., the temperature in the fluid will satisfy the equation

$$(1/\kappa)U(y)(\partial\theta/\partial x) = (\partial^2\theta/\partial y^2) \tag{3.2}$$

For the case of a sinusoidal temperature distribution, the solution will be qualitatively of the same general type as Eq. (2.17). For a flat plate lying parallel to a flow of undisturbed velocity U_0, the displacement thickness δ at a distance x from the leading edge is[6]

$$\delta = 1.73\sqrt{\nu x/U_0} \tag{3.3}$$

The velocity profile lying within this thickness is almost linear and the velocity at a distance $y = \delta$ from the wall is approximately

$$U \cong {}^1\!/_2 U_0 \tag{3.4}$$

Within this region, we may approximate the velocity distribution by the linear law

$$U = ay \tag{3.5}$$

with

$$a = {}^1\!/_2 U_0/\delta \tag{3.6}$$

It is, therefore, useful to analyze the problem of heat transfer for a boundary layer with the linear velocity distribution of Eq. (3.5) (Fig. 5). As already shown by Leveque,[1] it is possible to obtain an exact solution in this case. We have shown in Appendix I that it is a particular case of a more general solution, for which the velocity profile is given by the power law of Eq. (3.30). We shall, first, discuss the linear case ($\gamma = 1$).

With the velocity profile of Eq. (3.5), Eq. (3.2) becomes

$$(a/\kappa)y(\partial\theta/\partial x) = \partial^2\theta/\partial y^2 \tag{3.7}$$

An exact solution of this equation is found by putting

$$\theta = \phi(y/\sqrt[3]{x}) \tag{3.8}$$

By substitution in Eq. (3.7), we find an ordinary differential equation for the function ϕ which is readily integrable. The result is

$$\phi = C_1 + C_2 I(\eta) \tag{3.9}$$

with

$$\eta = \sqrt[3]{a/9\kappa}\cdot y/\sqrt[3]{x} \tag{3.10}$$

and

$$I(\eta) = \int_0^\eta e^{-\eta^3}\, d\eta \tag{3.11}$$

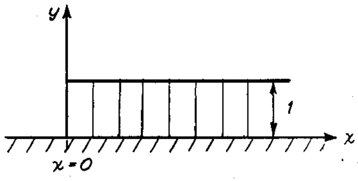

FIG. 6. Unit step temperature applied at the wall.

For convenience, this function is tabulated in Appendix II. It may also be expressed as*

$$I(\eta) = \Gamma(4/3)E_3(\eta) \tag{3.12}$$

The numerical value of the gamma function is

$$\Gamma(4/3) = 0.8929 \tag{3.13}$$

The asymptotic values are

$$E_3(\infty) = 1 \tag{3.14}$$

and

$$I(\infty) = \int_0^\infty e^{-\eta^3}\, d\eta = \Gamma(4/3) \tag{3.15}$$

We conclude that the particular solution

$$\theta = 1 - \{[I(\eta)]/[\Gamma(4/3)]\} \tag{3.16}$$

represents the case where a unit step temperature rise is applied at the wall on the positive side of the x axis (Fig. 6). The heat flux at any point in the fluid in a direction normal to the flow is

$$-k\frac{\partial\theta}{\partial y} = \frac{k}{\Gamma(4/3)}\sqrt[3]{\frac{a}{9\kappa x}}\, e^{-\eta^3} \tag{3.17}$$

At the wall, the rate of heat transfer per unit area into the fluid is found by putting $y = 0$ in Eq. (3.17). We find

$$F(x) = -k\left[\frac{\partial\theta}{\partial y}\right]_{y=0} = \frac{k}{\Gamma(4/3)}\sqrt[3]{\frac{a}{9\kappa x}} \tag{3.18}$$

The distribution of heat flux $f(x)$ due to an arbitrary temperature distribution $\theta_w(x)$ at the wall is given by Duhamel's integral

$$f(x) = \int_{x_0}^{x} \theta_w'(\xi)\, F(x-\xi)d\xi \tag{3.19}$$

with $$\theta'(x) = d\theta_w/dx$$

The wall temperature is assumed to be zero for $x < x_0$ and is arbitrary for $x > x_0$.

For comparison with the results of the previous section, we must investigate the heat flux due to a sinusoidal distribution of wall temperature

$$\theta_w = \theta_0 e^{ilx} \tag{3.20}$$

Applying Eq. (3.19), the heat flux is

$$f(x) = i\theta_0 l\int_{-\infty}^{x} e^{il\xi}F(x-\xi)d\xi \tag{3.21}$$

This integral is evaluated in Appendix III. We find

$$f(x) = \theta_0 e^{ilx}K \tag{3.22}$$

with

$$K = e^{i\pi/6}\frac{\Gamma(3/2)}{\Gamma(4/3)}k\sqrt[3]{\frac{al}{9\kappa}} \tag{3.23}$$

This complex coefficient is similar to Eq. (2.21) for the case of uniform flow.

Introducing the numerical values of the gamma functions

$$\Gamma(2/3) = 1.352 \tag{3.24}$$

along with the value (3.13) and substituting $l = \pi/D$, we derive

$$K = e^{i\pi/6}\,(k/\Delta_1) \tag{3.25}$$

with

$$\Delta_1 = 1.375\sqrt[3]{\kappa D/(\pi a)} = 1.73\sqrt[3]{\kappa D\delta/(\pi U_0)} \tag{3.26}$$

The factor $e^{i\pi/6}$ indicates that the heat flux is 30° out of phase with the wall temperature. Except for the phase factor, the heat transfer is the same as would occur through a stationary layer of fluid of thickness Δ_1. This quantity Δ_1 may be considered as a thermal-boundary-layer thickness in analogy with the quantity Δ defined by Eq. (2.26) for the case of uniform flow. Comparing Eqs. (2.26) and (3.26), we note that the difference between the cases of uniform and linear velocity profiles is twofold. The phase shift of the heat flux changes from an angle of 45° to one of 30°. The magnitude of the heat flux changes from an inverse square-root to an inverse cubic-root dependence on the wavelength.

We are now in a position to draw a general conclusion regarding the heat transfer in a laminar boundary of the more general type, as illustrated in Fig. 4. The boundary-layer displacement thickness was denoted by δ. For small enough values of the wavelength—i.e., small values of D—the thermal boundary layer lies within the displacement thickness

$$\Delta_1 < \delta \tag{3.27}$$

In this case, the results of the linear velocity distribution will apply. The heat transfer will then be measured by the complex coefficient K, as given by Eq. (3.25). Hence, for small enough wavelengths, the phase shift will be 30° and the heat flux will be proportional to $1/\sqrt[3]{D}$. For large wavelengths such that

$$\Delta_1 \gg \delta \tag{3.28}$$

the thermal boundary layer will penetrate deeply into the region of uniform velocity. We are then justified in applying Eq. (2.31) for the coefficient K, which

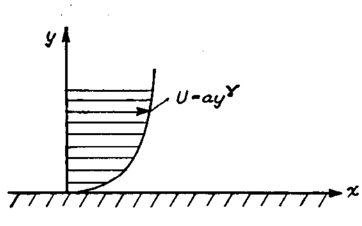

FIG. 7. Boundary layer with a power-law velocity profile.

* A plot of the function $E_3(\eta)$ is found in Ref. 7, and a table of $I(\eta)$ is also given in Ref. 2. The table in Appendix II was calculated independently.

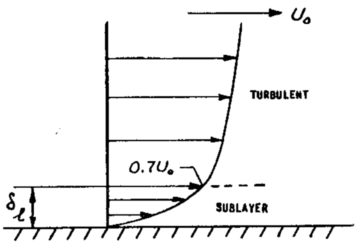

FIG. 8. Velocity profile of turbulent boundary layer on a flat plate. Thickness of the sublayer is δ_l.

corresponds to a constant velocity profile. The heat flux will tend to become proportional to $1/\sqrt{D}$ and the phase difference will approach 45°.

In the intermediate range where Δ_1 is somewhat larger than δ, the heat transfer will be represented by a value lying between the two existing cases just considered.

We conclude that the phase shift of the heat flux will always be between 30° and 45°, while its magnitude will vary as $1/D^n$, where n lies in the interval

$$1/3 \leq n \leq 1/2 \tag{3.29}$$

It is of interest to point out that Eq. (3.2) may be integrated for the case of the more general velocity profile

$$U(y) = ay^\gamma \tag{3.30}$$

For $0 < \gamma < 1$, this type of velocity profile is illustrated in Fig. 7. A solution of Eq. (3.2) in this case has been derived in Appendix I. This solution is

$$\theta = 1 - ([I_s(\eta)]/\{\Gamma[(s+1)/s]\}) \tag{3.31}$$

with

$$\eta = y[a/(s^2\kappa x)]^{1/s}, \quad s = \gamma + 2$$

$$I_s = \int_0^\eta \exp(-\eta^s)d\eta$$

Eq. (3.31) represents the fluid temperature for the case of a unit step temperature at the wall, as illustrated in Fig. 6.

The heat flux at the wall ($y = 0$) corresponding to this solution is

$$F(x) = -k\left(\frac{\partial\theta}{\partial y}\right)_{y=0} = \frac{k}{\Gamma[(s+1)/s]}\left(\frac{a}{s^2\kappa x}\right)^{1/s} \tag{3.32}$$

The heat transfer due to an arbitrary wall temperature distribution is given by Duhamel's integral [Eq. (3.19)], using Eq. (3.32). Eq. (3.18), obtained above for the linear velocity profile, is obtained by putting $s = \gamma + 2 = 3$ in the more general Eq. (3.32). By putting $a = U_0$ and $\gamma = 0$, the velocity profile becomes

$$U(y) = a = U_0 = \text{const.} \tag{3.33}$$

This corresponds to the case of uniform velocity considered in Section (2). The solution of Eq. (3.2) for this case is well-known. It may also be obtained here as a particular case of the more general solution by substituting $s = \gamma + 2 = 2$. Since

$$\Gamma(3/2) = \sqrt{\pi}/2 \tag{3.34}$$

Eq. (3.32) becomes

$$F(x) = k\sqrt{U_0/(\pi\kappa x)} \tag{3.35}$$

If we compare this expression with Eq. (3.18), obtained for the linear velocity profile ($\gamma = 1$), we see that the heat-flux distribution is now represented by the factor $1/\sqrt{x}$ instead of $1/\sqrt[3]{x}$. For velocity profiles with intermediate values of $\gamma (0 < \gamma < 1)$, the heat flux is proportional to the factor $x^{-1/s} (1/2 > 1/s > 1/3)$.

The velocity profile represented by Eq. (3.30) may be looked upon as an approximation for a boundary layer with a pressure gradient. The general solution [Eq. (3.31)] may therefore be used to evaluate the heat transfer under such conditions.

(4) Heat Transfer in a Turbulent Boundary Layer

We now turn our attention to the turbulent boundary. Let us consider the example of a flat plate lying parallel to the stream. Detailed measurements of the boundary-layer profile were made by Burgers and van der Hegge Zynen.[8]

The velocity profile of the boundary layer is shown schematically in Fig. 8. There is a sublayer of thickness δ_l. In this sublayer, the flow is mainly laminar. Outside the sublayer, the flow is fully turbulent. Actually, in the sublayer near the boundary of the turbulent region, there is a transition layer when the flow changes gradually from laminar to turbulent. For our purpose, since the analysis here is only for the purpose of obtaining orders of magnitude, we shall neglect this transition and assume that the flow is laminar in the sublayer. The stream velocity is U_0. At the boundary of the sublayer, the velocity is about 0.7 U_0. The dependence of the sublayer thickness δ_l on the distance x from the leading edge is plotted in nondimensional form in Fig. 9.

Let us now evaluate some orders of magnitude and consider a flow of air of 100-m/sec velocity at ordinary temperature. We put

$$U_0 = 10{,}000 \text{ cm/sec}, \quad \nu = 0.125 \text{ cm}^2/\text{sec} \tag{4.1}$$

where ν is the kinematic viscosity. For $U_0x/\nu = 3 \times 10^6$, the distance x from the leading edge is

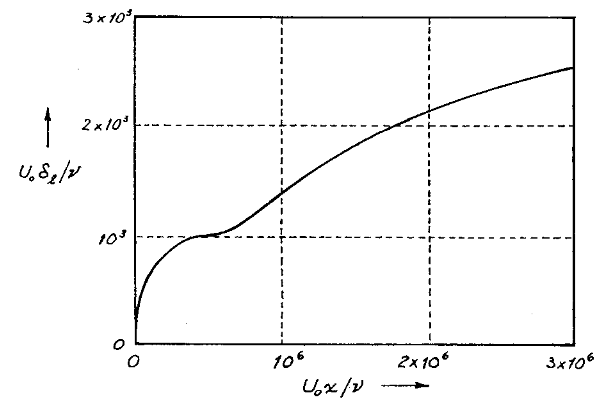

FIG. 9. Sublayer thickness δ_l as a function of the distance x to the leading edge of a plate (Ref. 8).

$$x = 37.5 \text{ cm} \quad (4.2)$$

From Fig. 9, the value of the laminar sublayer thickness at that point is

$$\delta_l = 0.031 \text{ cm} \quad (4.3)$$

This is of the same order as the thermal-boundary-layer thickness [Eq. (2.28)] calculated above for the case of uniform flow with the same stream velocity U_0 and a half-wavelength $D = 100$ cm. Actually, of course, the thermal-boundary-layer thickness should be computed on the basis of the average velocity within the layer. This amounts to replacing U_0 approximately by $^1/_2U_0$ in Eq. (2.26). We find the corrected value

$$\Delta = 0.034 \quad (4.4)$$

which is practically equal to the value [Eq. (3.3)] for δ_l.

As long as the thermal-boundary-layer thickness remains within the laminar sublayer, the heat transfer will obey the laws established for laminar boundary layers in the previous section. This means that the heat-flux phase difference will be between 30° and 45°, and the dependence of its magnitude on the wavelength will be the same as for the case of laminar flow.

The next step is to analyze the case when the thermal boundary layer penetrates into the turbulent region. Under the same conditions of flow, this will happen if the streamwise thermal gradient becomes smaller than assumed in the foregoing analysis. For example, let the thermal gradient correspond to a half wavelength.

$$D = 1{,}000 \text{ cm} \quad (4.5)$$

In that case, if we assume that Eq. (2.26) still applies, the thermal-boundary-layer thickness is

$$\Delta = 0.076 \text{ cm} \quad (4.6)$$

Comparing this result with the value [Eq. (4.3)] for δ_l indicates that the thermal boundary layer may extend appreciably into the turbulent region.

This analysis is, of course, only approximate, since Eq. (2.26) for Δ assumes the flow to be both laminar and uniform.

A more exact treatment requires the introduction of differential equations for the heat transfer in turbulent flow. As in the laminar case, we shall neglect the component of the average stream velocity which is normal to the wall. The differential equation for two-dimensional temperature field is now

$$\frac{\partial\theta}{\partial t} + U\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}\left(A\,\frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(A\,\frac{\partial\theta}{\partial y}\right) \quad (4.7)$$

The diffusion of heat in turbulent flow is represented by the total diffusivity A. It may be written

$$A = \kappa + \epsilon \quad (4.8)$$

where ϵ represents the diffusivity due to turbulent eddies and κ is the thermal conduction diffusivity already considered above. The quantity ϵ is generally a function of the coordinates and may also be a function of time. For convenience, we shall also define a total conductivity coefficient by

$$k_t = Ac \quad (4.9)$$

This may also be written

$$k_t = k + \epsilon c \quad (4.10)$$

Experimental data on the diffusion of heat in turbulent and laminar flow leads to the conclusion that A and k_t are about ten times larger than κ and k, respectively—i.e.,

$$A \cong 10\kappa, \quad k_t \cong 10k \quad (4.11)$$

A detailed evaluation of the magnitude and distribution of ϵ has been made by Rannie[9] on the basis of earlier theories by von Kármán. The latter were further discussed by Turcotte.[10] It is again possible to replace Eq. (4.7) by the approximate one

$$U(\partial\theta/\partial x) = (\partial/\partial y)[A(\partial\theta/\partial y)] \quad (4.12)$$

The argument leading to the neglection of the term $\partial\theta/\partial t$ in Section (2) does not involve the diffusivity and is, therefore, also valid for turbulent flow. Repeating the reasoning presented in Section (2) for the laminar flow and using the value of Eq. (4.11) for A leads to the conclusion that the term $\partial[A(\partial\theta/\partial x)]/\partial x$ will still remain negligible for air if $U_0 > 10{,}000$ cm/sec. This condition will generally be verified in practical problems with turbulent boundary layers.

In order to arrive at some estimate of the influence of turbulence on heat transfer without having to solve the differential Eq. (4.12), we shall introduce a simplified model for the boundary layer. Let us assume that the average velocity in the turbulent region is uniform and equal to the stream velocity U_0. Similarly, we also assume that the turbulent diffusivity is constant. If the sublayer is sufficiently thin, the heat transfer through it may be written as

$$K\theta_0 = k/\delta_l(\theta_0 - \theta_l) \quad (4.13)$$

In this expression, $\theta_0 e^{ilx}$ represents the streamwise temperature distribution at the wall, while $\theta_l e^{ilx}$ represents the temperature distribution at the boundary of the sublayer and turbulent flow. The factor K is a complex quantity representing the unknown heat transfer at the wall. The heat transfer from the sublayer to the turbulent flow may be expressed as in the laminar case considered previously, provided κ is replaced by A and k is replaced by k_t. With these substitutions in Eq. (2.31), this heat transfer may be written

$$K_t\theta_l = \theta_l e^{\pi i/4}\,(k_t/\Delta_t) \quad (4.14)$$

where

$$\Delta_t = \sqrt{AD/(\pi U_0)} \quad (4.15)$$

represents the thermal-boundary-layer thickness in the turbulent region. The expression

$$K_t = e^{\pi i/4}\,(k_t/\Delta_t) \quad (4.16)$$

is a complex coefficient representing the heat transfer into the turbulent region at the interface of the sublayer and the turbulent region.

Under the conditions assumed here that the thermal boundary layer penetrates appreciably into the turbulent region, we may write that the heat flow through the sublayer equals the heat entering the turbulent region. Hence, we may equate Eqs. (4.13) and (4.14) and write

$$(k/\delta_l)(\theta_0 - \theta_l) = \theta_l e^{\pi i/4} (k_t/\Delta_t) \qquad (4.17)$$

Solving this equation for θ_l and substituting in Eq. (4.13), we obtain the value of the coefficient K:

$$K = \frac{k}{\delta_l} \frac{1}{1 + (k/k_t)(\Delta_t/\delta_l)e^{-\pi i/4}} \qquad (4.18)$$

We see that, again, the concept of a local heat-transfer coefficient does not really apply. As before, we find that the heat transfer is out of phase with the wall temperature. Also, because Δ_t depends on the wavelength $(2D)$, the coefficient K also depends on the streamwise temperature gradient. The magnitude of these effects will depend on the numerical value of the factor $(k/k_t)(\Delta_t/\delta_l)$. *With the approximate values* of Eq. (4.11), we derive

$$(k/k_t)(\Delta_t/\delta_l) = (1/10)(\Delta_t/\delta_l) \qquad (4.19)$$

Hence,

$$K = \frac{k}{\delta_l} \frac{1}{1 + (1/10)(\Delta_t/\delta_l)e^{-\pi i/4}} \qquad (4.20)$$

This formula is significant only if the turbulent region contributes appreciably to the heat transfer. Assuming $\Delta_t/\delta_l > 10$, *we are led to the conclusion that the phase* angle of K lies between 30° and 45°. However in most cases the value of $\Delta\epsilon/\delta e$ will be smaller than assumed here leading to a smaller phase shift.

(5) A Conduction Analogy Leading to Variational Methods

The Lagrangian and variational methods developed earlier by this writer for heat-flow analysis[4] may be immediately extended to the evaluation of laminar and turbulent boundary-layer heat transfer. This can be shown as follows.

Let us first consider either a laminar or turbulent flow with a mean velocity independent of the streamwise coordinate. For two-dimensional flow, it was derived above that the temperature distribution in the boundary layer satisfies the equation

$$U(y)(\partial\theta/\partial x) = (\partial/\partial y)[A(y)(\partial\theta/\partial y)] \qquad (5.1)$$

For the laminar case, the equation is simplified by putting

$$A = \kappa = \text{const.} \qquad (5.2)$$

This type of equation is identical in form with the time-dependent heat conduction equation

$$c(\partial\theta/\partial t) = (\partial/\partial y)[k(\partial\theta/\partial y)] \qquad (5.3)$$

This equation represents the one-dimensional heat flow in a wall with a thermal conductivity k and a specific heat c per unit volume. This analogy provides an intuitive picture for the mechanism of boundary-layer heat transfer. Furthermore, and more important, it leads immediately to the use of variational techniques. This formulation acquires a special importance in conjunction with the intuitive understanding of boundary-layer heat transfer provided by the conduction analogy. As generally known, the practical value of variational procedures depends to a considerable degree on a physical understanding of the problem. Eq. (5.3) becomes identical with Eq. (5.1) for the boundary layer if we put

$$t = x, \quad c = U(y), \quad k = A(y) \qquad (5.4)$$

The analogy introduces the problem of heat conduction in a nonhomogeneous medium—i.e., for which both the heat capacity and thermal conductivity are functions of the location. By substituting the variables x, U, and A into the variational technique according to Eq. (5.4), we obtain an analog thermal potential

$$V = {}^1\!/_2 \int U\theta^2\, dy \qquad (5.5)$$

The temperature is related to the analogy heat flow by the equation

$$U\theta = -\partial H/\partial y \qquad (5.6)$$

We put

$$H = H(q_1 q_2 \ldots q_n, y) \qquad (5.7)$$

where the q's are generalized coordinate functions of x, and writing

$$\left.\begin{aligned} \dot{H} &= \frac{dH}{dx} = \sum^{i} \frac{\partial H}{\partial \dot{q}_i}\dot{q}_i \\ \dot{q}_i &= dq_i/dx \end{aligned}\right\} \qquad (5.8)$$

This defines an analog dissipation function

$$D = {}^1\!/_2 \int (1/A)\dot{H}^2 dy \qquad (5.9)$$

Finally, a thermal force at the boundary is defined by the variational equation

$$Q_i \delta q_i = \theta_w \delta H_w \qquad (5.10)$$

where θ_w is the temperature at the wall and δH_w a virtual heat flow at the wall.

With these definitions, the following Lagrangian equations are written for the unknowns q_i

$$(\partial V/\partial q_i) + (\partial D/\partial \dot{q}_i) = Q_i \qquad (5.11)$$

As an example, we shall consider the laminar flow with a uniform velocity

$$U = U_0 \qquad (5.12)$$

We assume the wall temperature to be a step function

$$\theta_w = \theta_0 \, 1(x) = \begin{Bmatrix} 0 & x < 0 \\ \theta_0 & x > 0 \end{Bmatrix} \tag{5.13}$$

A parabolic approximation is adopted for the temperature

$$\theta = \theta_0[1 - (y/q)]^2 \tag{5.14}$$

with q an unknown function of x. The quantity q represents the penetration of the temperature field into the boundary layer. In the present case of laminar flow, A becomes a constant $\kappa = k/c$. The problem has been solved in the context of conduction in a wall.[4] Using the correspondence of Eq. (5.4), the solution for q is found to be

$$q = 3.36 \sqrt{\kappa x/U_0} \tag{5.15}$$

The heat flux at the wall is obtained by integrating the total heat input in a slab of thickness q perpendicular to the flow. We write

$$\theta_0 F(x) = \frac{d}{dx}\int_0^q cU_0\theta dy = \frac{1}{3}\, cU_0\theta_0 \frac{dq}{dx} \tag{5.16}$$

For $\theta_0 = 1$, this equation becomes

$$F(x) = 0.560\, k\sqrt{U_0/(\kappa x)} \tag{5.17}$$

This result is very close to the exact solution [Eq. (3.35)] where the numerical factor is $1/\sqrt{\pi} = 0.564$, an error less than 1 percent. The parabolic approximation [Eq. (5.14)] may be used successfully to evaluate the heat transfer in the case of nonuniform and turbulent flow. For nonuniform Poiseuille flow between parallel walls, the calculation was carried out by Agrawal.[5] Excellent accuracy is obtained with very little calculation.

In the above, we have assumed the flow to be parallel to the wall. In some cases, it will be necessary to take into account the convective term in the direction perpendicular to the wall. In this case, the equation for the temperature field becomes

$$u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} = \frac{\partial}{\partial y}\left[A(x,y)\frac{\partial\theta}{\partial y}\right] \tag{5.18}$$

The total diffusivity A may be a function of the two coordinates x and y. The equation may be reduced to the same form [Eq. (5.1)] by using a transformation introduced by von Mises. This transformation replaces the independent variable x and y by x and the stream function ψ. This function satisfies the relations

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x \tag{5.19}$$

and is constant along the streamlines. Eq. (5.18) becomes

$$(\partial\theta/\partial x) = (\partial/\partial\psi)[uA(\partial\theta/\partial\psi)] \tag{5.20}$$

Note that the quantity uA is in general a function of both independent variables x and ψ. This introduces a difference with Eq. (5.1). However, the variational method is still applicable, as can be seen by considering the conduction analogy. In this analogy, the variable x represents the time and ψ is a space coordinate. Eq. (5.20), therefore, may be interpreted as a one-dimensional equation for transient heat conduction in a medium whose *thermal conductivity is a function of both time and location.* The variational methods which have been established in Ref. (4) are valid for this case. The dissipation function becomes an explicit function of time and the same Lagrangian Eq. (5.11) are applicable.

In three-dimensional flow with axial symmetry, a stream function also exists. It is easily shown that an equation of the type of Eq. (5.20) may be derived from the temperature field. In this case, the function uA is replaced by uAr^2, where r is the distance to the axis of symmetry. By this equation, it is possible to calculate the heat transfer in a circular duct, taking into account the streamwise variation of the velocity field due to the entrance conditions. A case where Eq. (5.20) is immediately reducible to the form of Eq. (5.1) appears when the quantity uA is the product of function of x and a function of ψ.

$$uA = f_1(x)f_2(\psi) \tag{5.21}$$

The factor $f_1(x)$ may then be brought to the left side and absorbed into a new independent variable x' related to x by a quadrature

$$x' = \int^x f_1(x)dx \tag{5.22}$$

Eq. (5.20) becomes

$$\partial\theta/\partial x' = (\partial/\partial\psi)[f_s(\psi)(\partial\theta/\partial\psi)] \tag{5.23}$$

The conduction analogy for this case corresponds to a medium of constant specific heat and thermal conductivity dependent on the one-dimensional coordinate ψ.

Appendix I

Consider the equation

$$(a/\kappa)y^\gamma(\partial\theta/\partial x) = \partial^2\theta/\partial y^2 \tag{I.1}$$

Our purpose is to find a solution of the type

$$\theta = \phi(\eta) \tag{I.2}$$

with a dimensionless variable

$$\eta = Byx^r \tag{I.3}$$

Coefficient B and exponent r must be determined by substitution in the differential Eq. (I.1). We find

$$(ar/B\kappa)y^{\gamma+1}x^{-r-1} = \frac{\phi''}{\phi'} \tag{I.4}$$

Let us make this equation an exact differential by writing the identity

$$(ar/B\kappa)y^{\gamma+1}\,x^{-r-1} \equiv -s\eta^{s-1} \tag{I.5}$$

or

$$(ar/B\kappa)y^{\gamma+1}x^{-r-1} \equiv -sB^{s-1}y^{s-1}x^{r(s-1)} \quad (I.6)$$

This will be an identity if

$$\left.\begin{aligned} \gamma + 1 = s - 1, \quad -r - 1 = r(s-1) \\ ar/B\kappa = -sB^{s-1} \end{aligned}\right\} \quad (I.7)$$

We derive

$$r = -1/s, \quad B = (a/\kappa s^2)^{1/s} \quad (I.8)$$

with

$$s = \gamma + 2 \quad (I.9)$$

Hence

$$\eta = y[a/(s^2\kappa x)]^{1/s} \quad (I.10)$$

The exact differential Eq. (I.4)

$$-s\eta^{s-1} = \phi''/\phi' \quad (I.11)$$

has the integral

$$\phi' = C_2 \exp(-\eta^s), \quad \phi = C_1 + C_2 I_s(\eta) \quad (I.12)$$

where

$$I_s(\eta) = \int_0^\eta \exp(-\eta^s)d\eta \quad (I.13)$$

If we choose

$$C_1 = 1 \quad C_2 = -\frac{1}{I_s(\infty)} = -\frac{1}{\Gamma[(s+1)/s]} \quad (I.14)$$

the temperature is given by

$$\theta = 1 - \Big([I_s(\eta)]/\{\Gamma[(s+1)/s]\}\Big) \quad (I.15)$$

and corresponds to a unit step boundary condition—i.e., $\theta = 1$ for $y = 0$ and $x > 0$.

Appendix II

Table of $I(\eta) = \int_0^\eta e^{-\eta^3}d\eta$

η	$I(\eta)$	η	$I(\eta)$
0	0		
0.1	0.0999	1.1	0.8389
0.2	0.1996	1.2	0.8609
0.3	0.2979	1.3	0.8751
0.4	0.3937	1.4	0.8838
0.5	0.4849	1.5	0.8886
0.6	0.5695	1.6	0.8910
0.7	0.6454	1.7	0.8922
0.8	0.7109	1.8	0.8927
0.9	0.7650	1.9	0.8928
1.0	0.8075	∞	0.8929

Appendix III

We must evaluate the integral [Eq. (3.21)]

$$f(x) = i\theta_0 l \int_{-\infty}^{x} e^{il\xi}F(x - \xi)d\xi \quad (III.1)$$

with

$$F(x) = \{k/[\Gamma(4/3)]\} \sqrt[3]{a/(9\kappa x)} \quad (III.2)$$

Introduce the change of variable

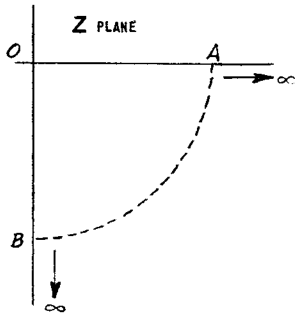

FIG. 10. Path of integration in complex z plane for Eq. III.5.

$$l(x - \xi) = z \quad (III.3)$$

Eq. (III.1) becomes

$$f(x) = K\theta_0 e^{ilx}$$

with

$$K = \frac{k}{\Gamma(4/3)}\sqrt[3]{\frac{al}{9\kappa}}\, i\int_0^\infty \frac{e^{-iz}}{\sqrt[3]{z}}dz \quad (III.4)$$

The integral in the last expression is evaluated by contour integration (Fig. 10). Consider the closed contour $OABO$ in the 4th quadrant of the complex plane. We may write

$$\oint_{OABO} \frac{ie^{-iz}}{\sqrt[3]{z}}dz = \int_{OA} + \int_{AB} + \int_{BO} \frac{ie^{-iz}}{\sqrt[3]{z}}dz = 0$$

(III.5)

On the infinite quarter circle AB the integral $\int_{AB}$ vanishes. Hence

$$\int_0^\infty \frac{ie^{-iz}}{\sqrt[3]{z}}dz = \int_0^{-i\infty} \frac{ie^{-iz}}{\sqrt[3]{z}}dz \quad (III.6)$$

Changing the variable z by

$$z = -iz' = e^{-i\pi/2}z'$$

the integral [Eq. (III.6)] becomes

$$\int_0^{-i\infty} \frac{ie^{-iz}}{\sqrt[3]{z}}dz = e^{i\pi/6}\int_0^\infty \frac{e^{-z'}}{\sqrt[3]{z'}}dz' = e^{i\pi/6}\Gamma\left(\frac{2}{3}\right) \quad (III.7)$$

Hence

$$K = e^{i\pi/6}\frac{\Gamma(2/3)}{\Gamma(4/3)}k\sqrt[3]{\frac{al}{9\kappa}} \quad (III.8)$$

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