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Internal buckling under initial stress in finite elasticity

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It is shown that elastic instability may occur in the interior of a medium which is of infinite extent or confined by rigid boundaries. This type of buckling is the mathematical analogue of body waves in dynamics. The analysis is based on the writer's theory of elasticity under initial stress. Significant incremental coefficients are derived for a medium in an initial state of finite strain. The existence of internal buckling is shown to be a consequence of the mixed hyperbolic-elliptic nature of the equations. Additional insight is provided by a variational analysis. The phenomenon may also be derived from earlier results of the writer for acoustic propagation under initial stress. In general, internal buckling requires the material to be anisotropic. However, it may occur in a medium of finite isotropy under exceptional conditions which recall the appearance of slip line in plasticity.

1. INTRODUCTION

A general theory of elasticity under initial stress was developed by the writer in a series of earlier publications (Biot 1939, 1940). The theory was applied to problems of instability and wave propagation. More recently it was applied to an exact analysis of the buckling of a plate embedded in an infinite medium (Biot 1959).

The problems of elastic instability or buckling which have been considered thus far require the presence of either free surfaces, or discontinuities, or inhomogeneities. In the present paper the theory is applied to a stability problem of an entirely different type. We shall see that elastic instability may occur under initial stress in a homogeneous medium of infinite extent or confined by rigid boundaries. In contrast with the usual case of buckling, the dimensions and the geometry of the system are not essential in the determination of the critical stress. Hence, we have referred to this phenomenon as an *internal buckling*.

The distinction is entirely similar to two types of waves which may propagate in elastic media. One type, which is exemplified by bending waves in a plate, is analogous to the usual type of buckling instability. The other type—body waves is analogous to what we call here internal buckling.

Sections 2 and 3 constitute a brief introduction to the general theory in the context of plane strain. Certain elastic coefficients which play an important role in internal buckling are introduced, and their physical significance is discussed in detail. A coefficient of particular importance which is introduced here is a '*slide* modulus'.

In §4, some important results are derived with respect to the incremental coefficients of a medium which is isotropic in finite strain. It is shown that for plane strain such a medium may retain its isotropy for incremental deformations. The particular finite stress-strain law which is required for this to be the case is also

derived.* Elastic coefficients for an anisotropic laminated medium are derived in **§5**.

It is shown that internal buckling is associated with hyperbolic equations and the existence of characteristics. In §6, the existence of such solutions is discussed, and the value of the critical stress is derived.

Section 7 discusses the occurrence of internal buckling in a medium of finite extent confined by rigid boundaries.

Results obtained previously for the theory of elastic waves under initial stress (Biot 1940) are recalled in §8. Attention is called to an important relation verified by transversal waves which is independent of the elastic properties and involves only the initial stress and the density. The appearance of internal buckling is associated with the vanishing of the velocity of propagation of certain types of elastic waves.

In some of the discussion, it was assumed that a certain inequality is verified by the incremental elastic coefficients. The inequality is usually verified for ordinary materials, and if this is the case, internal buckling is not possible for a material which is isotropic in finite strain. For exceptional materials, however, the inequality may not be verified and in that case internal instability is possible for a material which is isotropic in finite strain. This exceptional behaviour is discussed in the appendix and is illustrated by an example. The medium considered, although elastic, is assumed to become softer as the deformation increases in a way analogous to plastic yielding. The internal buckling in this case is a phenomenon closely related to the appearance of slip lines in plasticity.

2. INCREMENTAL STRESSES IN PLANE STRAIN

A brief outline will first be presented of certain basic equations and properties of the incremental stress field for a continuum under initial stress. The problem was the object of detailed treatment some twenty years ago by this writer (1939, 1940). The derivation was also briefly reproduced in a more recent publication (1959) for the case of plane strain.

We shall discuss here the particular case of an elastic continuum under an initial state of stress represented by three constant principal stresses S_{11} , S_{22} , S_{33} parallel with the co-ordinate axes x, y, z (figure 1).

The incremental strain is restricted to a two-dimensional field represented in the x, y plane by the components

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{xy} = \frac{1}{2} \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right].$$
 (2.1)

The co-ordinates of a material point in the initial state of stress are denoted by x and y. The incremental deformation produces a displacement field such that the co-ordinates become x + u and y + v. Since we are dealing with a two-dimensional

^{*} A more elaborate discussion of this result and of incremental elastic coefficients is presented in another paper (1961).

[†] These two types of internal buckling are also analyzed in a forthcoming book by the writer where they are referred to as internal instability of the first and second kind.

state of incremental strain, the z component of the displacement is put equal to zero.

Consider now the incremental stress field generated by the strain components (2.1). The incremental stress components in the x, y plane are denoted by s_{11}, s_{22} , s_{12} . They are referred to axes 1, 2, which rotate locally with the continuum. The local angle of rotation is given by

$$\omega = \frac{1}{2} \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right], \qquad (2.2)$$

These stresses are defined as the incremental forces acting per unit area after deformation on the faces of a unit cube whose sides are parallel with the locally rotated axes 1, 2.



FIGURE 1. State of initial stress in the x, y plane.

The author has shown (1939, 1940, 1959) that for the case of initial stress considered here, the incremental stress field satisfies the equilibrium equations

$$\begin{aligned} & \left. \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + (S_{11} - S_{22}) \frac{\partial \omega}{\partial y} = 0, \right| \\ & \left. \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + (S_{11} - S_{22}) \frac{\partial \omega}{\partial x} = 0. \right| \end{aligned}$$

$$(2.3)$$

As should be, these equations are independent of the initial stress component S_{33} perpendicular to the x, y plane. This is, of course, a consequence of the fact that we are considering the incremental deformation to be two-dimensional in the x, yplane.

In the present analysis, the physical significance of the equations appear more clearly if the initial stress is represented as

$$S_{11} = -p_f - P, \\S_{22} = -p_f.$$
(2.4)

The initial stress field is then considered as the superposition of a hydrostatic pressure p_f and a compressive stress P acting horizontally in the medium (figure 2). We may also write

$$P = S_{22} - S_{11}. \tag{2.5}$$

With this definition, equations $(2\cdot 3)$ are written

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P \frac{\partial \omega}{\partial y} = 0,
\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P \frac{\partial \omega}{\partial x} = 0.$$
(2.6)



FIGURE 2. State of initial stress represented as a superposition of hydrostatic pressure p_t and an additional horizontal compressive stress P.



FIGURE 3. Forces acting on a deformed boundary C'.

In the following discussion, we shall also make use of expressions for boundary forces. Consider the material enclosed initially in a contour C. After deformation, the contour becomes C' (figure 3). Before deformation, an element of arc ds is acted upon by forces due to the initial stresses. After deformation, the element of arc becomes ds'. The x and y components of the forces acting on this element ds' after deformation are denoted by

$$\begin{aligned}
df_x &= f_x ds, \\
df_y &= f_y ds.
\end{aligned}$$
(2.7)

The components f_x , f_y therefore represent the boundary force *per unit initial area*. These force components were found (1939) to be

$$\begin{aligned} f_x &= (S_{11} + s_{11} + S_{11}e_{yy})\cos{(n, x)} + (s_{12} - S_{22}\omega - S_{11}e_{xy})\cos{(n, y)}, \\ f_y &= (s_{12} + S_{11}\omega - S_{22}e_{xy})\cos{(n, x)} + (S_{22} + s_{22} + S_{22}e_{xx})\cos{(n, y)}. \end{aligned}$$
 (2.8)

The directional cosines of the outward normal direction to the *initial contour* C are designated by $\cos(n, x)$ and $\cos(n, y)$. The quantities $S_{11} \cos(n, x)$ and $S_{22} \cos(n, y)$ represents the forces acting initially on the contour C in the prestressed state. Hence, the incremental forces acting on the deformed boundary per unit initial area are

$$\Delta f_x = \Delta_{xx} \cos(n, x) + \Delta_{xy} \cos(n, y), \Delta f_y = \Delta_{yx} \cos(n, x) + \Delta_{yy} \cos(n, y),$$
(2.9)

with

$$\begin{aligned} \Delta_{xx} &= s_{11} + S_{11} e_{yy}, \\ \Delta_{yx} &= s_{12} + S_{11} \omega - S_{22} e_{xy}, \\ \Delta_{xy} &= s_{12} - S_{22} \omega - S_{11} e_{xy}, \\ \Delta_{yy} &= s_{22} + S_{22} e_{xx}. \end{aligned}$$

$$(2.10)$$

These last expressions have a definite physical significance. For instance, Δ_{xx} and Δ_{yx} are the x and y components of the incremental force acting *per unit initial* area on a surface initially perpendicular to the x direction.

We shall find that expressions such as (2.9) for the boundary forces play an important role in the physical interpretation of certain coefficients.

3. ANISOTROPIC STRESS-STRAIN RELATIONS FOR ELASTIC STRESS INCREMENTS

The problem is the choice of the relations between the incremental stresses and the strain. Since we are dealing with a linear theory of prestressed continua, the strain components $(2\cdot 1)$ are the same as in the classical theory of infinitesimal strain. Consider the case of an elastic body such that the co-ordinate axes are in planes of symmetry for the elastic properties of the material. Such a material is called *orthotropic*. We shall assume that the state of initial stress is one in which the principal stresses S_{11} , S_{22} , S_{33} are oriented along the three co-ordinate axes. Such an initial state of stress does not disturb the symmetry of the medium. Therefore, the incremental stress-strain relations will also retain the same orthotropic symmetry.

We are interested in the plane strain increments in the x, y plane. Since orthotropic symmetry is retained, the incremental stress-strain relations in two dimensions are

$$s_{11} = B_{11}e_{xx} + B_{12}e_{yy},$$

$$s_{22} = B_{21}e_{xx} + B_{22}e_{yy},$$

$$s_{12} = 2Qe_{xy}.$$
(3.1)

In the classical theory of elasticity, where there are no initial stresses, the existence of an elastic potential energy requires that the coefficients satisfy the relation

$$B_{12} = B_{21}.$$
 (3.2)

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As shown in an earlier publication (1939), this is not so if there is an initial stress. Under the presently assumed state of initial stress, condition $(3\cdot 2)$ is replaced by

$$B_{12} + S_{11} = B_{21} + S_{22}, \tag{3.3}$$

$$B_{12} = B_{21} + P, (3.4)$$

or

where P is defined by equation (2.5).

When the theory is applied to specific problems, it is advantageous to assume the material to be *incompressible*. This introduces considerable simplifications in the algebra without restricting the general character of the physical results. For this purpose, we shall now investigate how the stress-strain relations $(3\cdot1)$ are to be modified for an orthotropic material which is incompressible. In this case, as we shall see, conditions $(3\cdot4)$ required for the existence of an elastic strain-energy does not enter into consideration. We may write

$$s_{11} - s_{22} = 4Ne_{xx}, s_{22} - s_{11} = 4Ne_{yy}.$$
 (3.5)

If we introduce the average two-dimensional stress

$$s = \frac{1}{2}(s_{11} + s_{22}),\tag{3.6}$$

relations (3.5) are equivalent to

$$s_{11} - s = 2Ne_{xx}, \\ s_{22} - s = 2Ne_{yy}.$$
(3.7)

It should be noted here that s is defined by equation $(3\cdot 6)$. For an anisotropic medium, this two-dimensional definition of s is not equal to the average of the three normal stress components.

We can also use these relations without specifying that s is defined by equation (3.6), provided we impose the condition of incompressibility

$$e_{xx} + e_{yy} = 0. \tag{3.8}$$

The value (3.6) of s is then a consequence of the three equations (3.7) and (3.8).

The complete stress-strain relations in two dimensions for the orthotropic incompressible medium are

$$\begin{array}{l}
s_{11} - s = 2Ne_{xx}, \\
s_{22} - s = 2Ne_{yy}, \\
s_{12} = 2Qe_{xy}.
\end{array}$$
(3.9)

These relations contain two elastic coefficients N and Q. For an isotropic medium, if the initial stress is zero or if $S_{11} = S_{22}$, the incremental properties remain isotropic in the plane of deformation. In this case the coefficients N and Q become identical, and the incremental properties in two dimensions are characterized by a single shear modulus

$$\mu = N = Q. \tag{3.10}$$

We should remember that when an isotropic medium is under an initial stress for which $S_{11} \neq S_{22}$, the incremental properties do not generally remain isotropic. They may, however, remain isotropic in some special cases discussed in §4. Furthermore, the incremental coefficients N and Q themselves become functions of the initial stress.

A point of considerable importance in the present theory lies in the possibility of expressing the elastic coefficients in terms of measurable quantities.

Let us first turn our attention to the coefficient Q, which appears in the stressstrain relations (3.1) and (3.9) for either the compressible or the incompressible material. We shall assume that the state of initial stress is reduced to a horizontal compressive stress P; hence,

$$S_{11} = -P, \\S_{22} = 0.$$
 (3.11)

Consider a strip of material cut in a direction parallel with the x direction (figure 4a). The strip is maintained in its initial state of stress. This condition of pre-stress is



FIGURE 4. Measurements and physical significance of the elastic coefficients L and M.

maintained by applying a compressive stress P at both ends. In order to prevent buckling, the strip is sandwiched between two rigid blocks. We now produce a horizontal shear displacement

$$u = \gamma (y - y_0) \tag{3.12}$$

by applying a tangential force Δ_{xy} per unit area to the upper face of the strip. The lower face, located at $y = y_0$, remains fixed. This can be accomplished by providing adherence between the strip and the blocks and applying horizontal forces to the blocks. The relation between the force Δ_{xy} and the shear deformation γ introduces a measurable elastic coefficient L defined as

$$\Delta_{xy} = L\gamma. \tag{3.13}$$

The tangential force Δ_{xy} is also given by the third equation of (2.10), in which we introduce the values (3.11) of the initial stress and put

$$e_{xy} = \frac{1}{2}\gamma,$$

$$s_{12} = 2Qe_{xy} = Q\gamma.$$

$$(3.14)$$

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Hence,

$$\Delta_{xy} = (Q + \frac{1}{2}P)\gamma. \tag{3.15}$$

Comparing with equation (3.13), we derive

$$L = Q + \frac{1}{2}P.$$
 (3.16)

This relation defines the elastic coefficient Q in terms of a measurable L which will be referred to as the *slide modulus*. Note that equation (3.15) is *valid for either the incompressible or the compressible medium*.

According to equations (2.10), there is also a vertical force Δ_{yx} which must be applied at both ends of the strip. This force is

or

$$\begin{array}{l} \Delta_{yx} = s_{12} + \frac{1}{2} P \gamma, \\ \Delta_{yx} = \Delta_{xy}. \end{array} \right\}$$

$$(3.17)$$

This turns out to be the force required to balance the couple due to the tangential force Δ_{xy} .

We shall now consider the coefficient N, which appears in the stress-strain relations (3.9) for the incompressible material. Again let us assume that the initial stress is a compression P acting in the horizontal direction. As in the previous case, a strip of the material under this compressive stress is sandwiched between two blocks (figure 4b). Lubrication is applied between the blocks and the strip. The total compressive force in the x direction can then be increased. This increase per unit initial area is $-\Delta_{xx}$. A compressive force of value $-\Delta_{yy}$ per unit area may be applied across the thickness. These incremental forces are given by equations (2·10).

Substituting in these equations the initial stresses (3.11) and using the condition of incompressibility (3.8) and the stress-strain relations (3.5), we derive

$$\begin{array}{l} \Delta_{xx} - \Delta_{yy} = 4Me_{xx}, \\ \Delta_{yy} - \Delta_{xx} = 4Me_{yy}, \end{array}$$

$$(3.18)$$

$$M = N + \frac{1}{4}P.$$
 (3.19)

We can evaluate the coefficient M by applying an increment of compressive force $-\Delta_{xx}$ with no forces applied normally to the strip $(\Delta_{yy} = 0)$ and measuring the extensional strain, or we can apply a compressive force $-\Delta_{yy}$ across the thickness while maintaining the horizontal stress constant $(\Delta_{xx} = 0)$. We remember that the strain considered here is two-dimensional. Therefore, when measuring the coefficient N, we must apply a normal stress in the direction perpendicular to the x, y plane in order to maintain zero strain in that direction. An initial stress S_{33} can also be applied simultaneously in that direction.

The same measurements can be carried out in a closed vessel containing a fluid at the pressure p_f . The initial stresses S_{11} and S_{22} are then given by expression (2.4); i.e.

$$S_{11} = -p_f - P, \\S_{22} = -p_f,$$
(3.20)

$$P = S_{22} - S_{11}. \tag{3.21}$$

with

and

Measurements with superimposed fluid pressures yield elastic coefficients in a state of triaxial stress. In this case we consider the forces

$$\begin{aligned} \delta_{xx} &= \Delta_{xx} - p_f e_{xx}, \\ \delta_{yy} &= \Delta_{yy} - p_f e_{yy}. \end{aligned}$$
 (3.22)

These quantities represent the forces in excess of those due to hydrostatic pressure. With the definition (3.19) for M, where P is now given by expression (3.21), the following relations are derived

$$\delta_{xx} - \delta_{yy} = 4Me_{xx}, \\ \delta_{yy} - \delta_{xx} = 4Me_{yy}.$$
(3.23)

Measurement of the coefficient M under the initial stress (3.20) can then be carried out as described above for the case of zero fluid pressure.

Equation (3.16), defining the slide modulus L, is also valid for this case with the value (3.21) for P. Measurement of the slide modulus is not modified by the presence of the fluid pressure.

4. Incremental coefficients for an isotropic elastic medium in finite initial strain

In the preceding analysis, we have considered the stress-strain relations in an elastic medium under initial stress and have assumed the elastic symmetry to be orthotropic. In particular, the elastic medium may be isotropic in the original unstressed state. In a state of finite initial stress, the incremental elastic properties of such a material will generally not remain isotropic but will acquire an orthotropic symmetry defined by the principal directions of stress. In the case of original isotropy, however, it is possible to evaluate the elastic coefficient Q directly in terms of the initial stress alone and the corresponding finite strain.

In order to show this, let us call (a) the original unstressed state, (b) the initial stressed state, and (c) the state obtained after incremental strains. In order to simplify the writing, we shall consider a deformation which is two-dimensional in both the finite and the incremental strain. We shall see that this does not restrict the generality of the results.* The original co-ordinates in the representative plane are denoted by X, Y in state (a). The co-ordinates in state (b) are

$$\begin{array}{l} x = a_{11}X, \\ y = a_{22}Y. \end{array}$$
 (4.1)

The principal directions are along the co-ordinate axes. In this initial state, there are principal stresses S_{11} , S_{22} along x and y and a normal stress S_{33} perpendicular to the plane of deformation.

We now superpose upon state (b) a pure shear strain e_{xy} . This shear strain is assumed to be small of the first order. The new co-ordinates become

$$\begin{cases} \xi = x + e_{xy}y, \\ \eta = e_{xy}x + y. \end{cases}$$
(4.2)

* A more elaborate analysis of the incremental coefficients was developed in another paper (1961).

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The total transformation from (a) to (c) is

$$\xi = a_{11} X + e_{xy} a_{22} Y,$$

$$\eta = e_{xy} a_{11} X + a_{22} Y.$$

$$(4.3)$$

This may be identified with the following transformation

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}.$$
 (4.4)

This transformation is obtained by applying to the material a solid rotation through an angle θ followed by a pure deformation corresponding to the symmetric matrix b_{ij} . By identifying the coefficients in the two transformations (4.3) and (4.4), we find, by neglecting second- and higher-order terms,

$$\begin{array}{l} b_{11} = a_{11}, \\ b_{22} = a_{22}, \\ b_{12} = e_{xy} \frac{a_{11}^2 + a_{22}^2}{a_{11} + a_{22}}. \end{array}$$

$$(4.5)$$

This pure deformation is equivalent to a principal elongation b_I in a direction I which makes an angle α with the x axis. This angle is given by

$$\tan 2\alpha = \frac{2b_{12}}{b_{11} - b_{22}}.$$
(4.6)

Neglecting second-order quantities, we can write

$$\alpha = e_{xy} \frac{a_{11}^2 + a_{12}^2}{a_{11}^2 - a_{22}^2}.$$
(4.7)

There is also an elongation b_{II} in the other principal direction II perpendicular to I. Calculating the values of b_I and b_{II} from the matrix b_{ij} by a standard procedure, we find that if we again neglect second-order quantities, we can write

$$b_{I} = b_{11} = a_{11}, b_{II} = b_{22} = a_{22}.$$
(4.8)

This shows that, except for second order in the magnitude of the principal strains, states (b) and (c) differ only by a rotation of the directions of principal strains through an angle α .

Here the property of *isotropy* enters into play. Because of this isotropy, the principal stresses in states (b) and (c) must be of the same magnitude, except for second-order errors. In other words, the magnitude of the principal stresses in state (c) to this approximation is the same as that of the initial stresses S_{11} and S_{22} . The stress S_{11} now acts in a direction which makes an angle α with the x axis.

If we resolve these principal stresses along the x and y directions, we obtain a shear stress component which to the first order is

$$s_{12} = (S_{11} - S_{22}) \alpha. \tag{4.9}$$

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If we substitute the value (4.7) for α ,

$$s_{12} = (S_{11} - S_{22}) \frac{a_{11}^2 + a_{22}^2}{a_{11}^2 - a_{22}^2} e_{xy}.$$
 (4.10)

Hence, the coefficient Q in the stress-strain relations (3.1) is equal to

$$Q = \frac{1}{2} (S_{11} - S_{22}) \frac{a_{11}^2 + a_{22}^2}{a_{11}^2 - a_{22}^2}.$$
 (4.11)

By going through the derivation, we see that this result is also valid if the initial strain is three-dimensional. Expression $(4\cdot11)$ is therefore completely general and is valid in each of the three planes of symmetry of the initial strain. The three corresponding values of Q are obtained by cyclic permutation of indices in expression $(4\cdot11)$. The result was also derived in another paper (Biot 1961), in which it is the object of a more elaborate discussion.

In the same paper (Biot 1961) an additional and alternate derivation of expression $(4\cdot11)$ is also given by the method of tensor invariants, showing that the latter procedure is by comparison obscure and considerably more involved.

Let us now examine the particular case of an incompressible medium. In this case the medium is characterized by only one other elastic parameter, i.e. the coefficient N appearing in the stress-strain relations (3.9). We shall consider the case of twodimensional finite strain. Because of incompressibility, this finite strain is characterized by a single strain variable λ . We can write

$$\begin{array}{l} a_{11} = \lambda, \\ a_{22} = 1/\lambda. \end{array}$$

$$(4.12)$$

If this strain is produced by a stress difference $S_{11} - S_{22}$, the finite stress-strain law can be written $S_{11} - S_{22} = \kappa(\lambda) = -P.$ (4.13)

$$s_{11} - s_{22} = (\mathrm{d}\kappa/\mathrm{d}\lambda)\,\mathrm{d}\lambda.\tag{4.14}$$

Putting $e_{xx} = d\lambda/\lambda$ and comparing with the stress-strain relation (3.5), we derive the value of the incremental elastic coefficient

$$N = \frac{1}{4}\lambda(\mathrm{d}\kappa/\mathrm{d}\lambda). \tag{4.15}$$

This coefficient depends on the slope of the stress-strain curve. The value of Q given by expression (4.11) can be written

$$Q = \frac{1}{2}\kappa(\lambda)\frac{\lambda^4 + 1}{\lambda^4 - 1}.$$
(4.16)

For a vanishing initial stress, i.e. for $\lambda = 1$, the limiting value of Q is

$$Q = \frac{1}{4} (\mathrm{d}\kappa/\mathrm{d}\lambda). \tag{4.17}$$

Hence, in this case
$$N = Q = \mu_0$$
, (4.18)

where μ_0 represents the shear modulus in the unstressed state. The condition (4.18) is, of course, an expression of the isotropy of the medium in that state.

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It is most interesting to determine whether there exists a particular stressstrain relation $\kappa(\lambda)$ such that the material remains isotropic in the *x*, *y* plane for incremental stresses in the vicinity of a state of finite initial strain. This will be the case for all values of λ if we have

 $N = Q, \qquad (4.19)$ $\frac{1}{\kappa} \frac{\mathrm{d}\kappa}{\mathrm{d}\lambda} = \frac{2}{\lambda} \frac{\lambda^4 + 1}{\lambda^4 - 1}.$

This is a differential equation for $\kappa(\lambda)$. Its solution is

$$\kappa = \mu_0 \{\lambda^2 - (1/\lambda^2)\}. \tag{4.20}$$

It is remarkable that this finite deformation law is identical with that derived from the statistical thermodynamic model of polymer chains (Treloar 1955). It may be considered typical of rubber-like materials. The incremental properties in the x, yplane are characterized by a single shear modulus

$$\mu(\lambda) = N = Q = \frac{1}{2}\mu_0 \{\lambda^2 + (1/\lambda^2)\}.$$
(4.21)

It is a function of the strain and increases with the initial deformation.

Let us finally evaluate the slide modulus L defined by equation (3.16). For the case of a material which is isotropic in the unstrained state, this coefficient is

$$L = \frac{\kappa(\lambda)}{\lambda^4 - 1} = \frac{P}{1 - \lambda^4}.$$
 (4.22)

For rubber obeying the stress-strain relation (4.20), it becomes

$$L = \mu_0 / \lambda^2. \tag{4.23}$$

This coefficient increases with the magnitude of the compression and decreases when the material is stretched. This behaviour is in contrast with that of the shear modulus μ as given by expression (4.21).

The *M* coefficient is obtained by substituting expressions (4.20) and (4.21) into equation (3.19). We find

$$M = \frac{1}{4}\mu_0 \{\lambda^2 + (3/\lambda^2)\}.$$
 (4.24)

This is the coefficient which appears in equations (3.23).

5. A THINLY LAMINATED MEDIUM AS AN EXAMPLE OF ANISOTROPY

Anisotropy can be created or increased artificially by stacking alternately soft and hard layers of materials. The layers are assumed to be thin. Our problem is to evaluate the relation between average stresses and deformation in this medium. This is equivalent to replacing the inhomogeneous solid by an equivalent homogeneous anisotropic continuum.

It is obvious that there is a limit to this equivalence. This limitation is related to the exact meaning of the requirement that the layers be sufficiently thin. For instance, in the analysis of internal buckling presented below, sinusoidal deformations will be considered. In such deformations, however, the laminated medium will behave as a continuum only if the wavelength is 'sufficiently' large relative to the

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lamination thickness. This point will be examined in more detail in a forthcoming publication.

For simplicity, we shall assume the material to be incompressible. The stressstrain relations for the composite laminated medium in the form (3.9) are most easily obtained by first considering the coefficients M and L as defined in §3. The properties of the hard layer are characterized by the coefficients M_1 and L_1 , and those of the soft layer are characterized by M_2 and L_2 . The hard and soft layers occupy, respectively, fractions α_1 and α_2 of the unit thickness. Hence,

$$\alpha_1 + \alpha_2 = 1. \tag{5.1}$$

The initial stress differences in the layers are

$$P_{1} = S_{22} - S_{11}^{(1)}, P_{2} = S_{22} - S_{11}^{(2)}.$$
(5.2)

The x direction is chosen parallel with the lamination.

The total stress in the x direction is

$$S_{11} = \alpha_1 S_{11}^{(1)} + \alpha_2 S_{11}^{(2)}. \tag{5.3}$$

The total stress difference is

$$P = S_{22} - S_{11} = \alpha_1 P_1 + \alpha_2 P_2. \tag{5.4}$$

We now apply equations (3.23) to each type of material separately and write

$$\begin{cases} \delta_{xx}^{(1)} - \delta_{yy} = 4M_1 e_{xx}, \\ \delta_{xx}^{(2)} - \delta_{yy} = 4M_2 e_{xx}. \end{cases}$$

$$(5.5)$$

The quantities $\delta_{xx}^{(1)}$ and $\delta_{xx}^{(2)}$ are the incremental forces acting in the x direction in each material per unit initial area as defined in §3. Multipling these equations by α_1 and α_2 and adding, we obtain

$$\delta_{xx} - \delta_{yy} = 4Me_{xx},\tag{5.6}$$

with

$$\delta_{xx} = \alpha_1 \delta_{xx}^{(1)} + \alpha_2 \delta_{xx}^{(2)}. \tag{5.7}$$

Hence,
$$M = M_1 \alpha_1 + M_2 \alpha_2 \tag{5.8}$$

is the coefficient for the composite material.

Similarly, the slide modulus can be evaluated by applying a tangential force Δ_{xy} as described above and as illustrated in figure 4. This force will produce a shear displacement parallel with the layers. The slide moduli of each layer have been denoted by L_1 and L_2 . Hence, the total shear angle γ is

$$\gamma = \left[\frac{\alpha_1}{L_1} + \frac{\alpha_2}{L_2}\right] \Delta_{xy}.$$
(5.9)

Comparing this equation with relation (3.13), we derive the slide modulus of the composite medium

$$L = 1 / \left(\frac{\alpha_1}{\overline{L}_1} + \frac{\alpha_2}{\overline{L}_2} \right). \tag{5.10}$$

Expressions (5.8) and (5.9) for the composite coefficients are valid when the materials of the individual layers are either isotropic or not isotropic. If the individual layers are isotropic in finite strain, their incremental properties may still be anisotropic, but in that case the slide coefficients are derived from equation (4.22); i.e.

$$L = \frac{P_1}{1 - \lambda^4}, \quad L_2 = \frac{P_2}{1 - \lambda^4},$$
 (5.11)

where λ denotes the extension ratio in the direction of the layers in the state of initial stress. When equation (5.10) is applied, the slide modulus of the composite material becomes

$$L = \frac{1}{1 - \lambda^4} \frac{P_1 P_2}{\alpha_1 P_2 + \alpha_2 P_2}.$$
 (5.12)

As shown in §4, it is possible for a material to retain plane isotropy under initial stress, provided it obeys the finite stress-strain law $(4\cdot20)$ characteristic of rubberlike materials. If the laminations are made of such material, their individual finite stress-strain relations are

$$P_{1} = \mu_{01}\{(1/\lambda^{2}) - \lambda^{2}\}, \quad P_{2} = \mu_{02}\{(1/\lambda^{2}) - \lambda^{2}\}, \quad (5.13)$$

and expressions (4.23) for the corresponding slide moduli yield

$$L_1 = \frac{\mu_{01}}{\lambda^2}, \quad L_2 = \frac{\mu_{02}}{\lambda^2}.$$
 (5.14)

In this case, the slide modulus (5.10) for the composite material becomes

$$L = \frac{1}{\lambda^2} \frac{1}{\frac{\alpha_1}{\mu_{01}} + \frac{\alpha_2}{\mu_{02}}}.$$
 (5.15)

Applying relation (4.24) to each layer, we write

$$M_1 = \frac{1}{4}\mu_{01}\{\lambda^2 + (3/\lambda^2)\}, \quad M_2 = \frac{1}{4}\mu_{02}\{\lambda^2 + (3/\lambda^2)\}.$$
 (5.16)

Substituting these values in equation (5.8) yields

$$M = \frac{1}{4} (\alpha_1 \mu_{01} + \alpha_2 \mu_{02}) \{ \lambda^2 + (3/\lambda^2) \}$$
 (5.17)

for the other coefficient of the composite material.

The coefficients Q and N are obtained immediately, once L and M are known, by applying equations (3.16) and (3.19).

6. INTERNAL BUCKLING AND ITS RELATION TO HYPERBOLIC SOLUTIONS

We shall now consider the displacement and incremental stress fields and shall investigate the general solutions of the equations which govern these fields.

Assuming incompressibility, we write the displacement components as

$$u = -\partial \phi / \partial y, \quad v = \partial \phi / \partial x.$$
 (6.1)

Introducing this expression in the stress-strain relations (3.9) and using the equilibrium equations (2.6) of the stress field, we obtain, after the required eliminations,

$$[Q + \frac{1}{2}P]\frac{\partial^4\phi}{\partial y^4} + 2(2N - Q)\frac{\partial^4\phi}{\partial x^2 \delta y^2} + [Q - \frac{1}{2}P]\frac{\partial^4\phi}{\partial x^4} = 0.$$
 (6.2)

For the case of an isotropic medium initially unstressed, i.e. for

$$N = Q, \quad P = 0, \tag{6.3}$$

the equation reduces to the well-known biharmonic equation

$$\nabla^4 \phi = 0, \tag{6.4}$$

whose solutions are of the elliptic type. This same elliptic character is retained in the absence of initial stress for the anisotropic case.

A radical change occurs in the nature of the solution in the presence of initial stress beyond a critical value. To show this, let us assume a solution of the form

$$\phi = \varphi(x - \sigma y). \tag{6.5}$$

Substitution of this solution into the general equation (6.2) yields a characteristic equation σ

$$r^4 + 2m\sigma^2 + k^2 = 0, (6.6)$$

with

$$m = \frac{2N - Q}{Q + \frac{1}{2}P} = \frac{2M - L}{L},$$

$$k^{2} = \frac{Q - \frac{1}{2}P}{Q + \frac{1}{2}P} = \frac{L - P}{P}.$$
(6.7)

The roots of equation (6.6) are

$$\sigma_1^2 = -m + \sqrt{(m^2 - k^2)}, \qquad (6.8)$$

$$\sigma_2^2 = -m - \sqrt{(m^2 - k^2)}.$$

For the purpose of restricting the complexity of the present discussion, we shall introduce the assumption

$$m > 0. \tag{6.9}$$

Since, for physical reason, the coefficient L is positive, this is equivalent to the condition 2N > 0

2M > L.

$$2N > Q, \tag{6.10}$$

(6.11)

or,

Obviously, a material for which this condition is not fulfilled must be of a special nature, since it is considerably softer for elongation than it is for shear, whereas on the other hand it must be capable of sustaining an initial stress P. For the sake of generality, the case in which the inequality (6.11) is not verified is discussed in the appendix.

Under the assumption of m > 0, there can be real roots only if

$$k^2 < 0.$$
 (6.12)

Let us denote the real roots as

Choosing the positive determination of the radical, we can write

$$\xi = \sqrt{\{-m + \sqrt{(m^2 - k^2)}\}}.$$
 (6.14)

There exists, therefore, a real solution of equation (6.2) which can be written

$$\phi = \varphi_1(x - \xi y) + \varphi_2(x + \xi y), \tag{6.15}$$

where φ_1 and φ_2 are arbitrary functions of the argument. This solution is of the *hyperbolic type*, and the roots $\pm \xi$ represent the slopes of the characteristics relative to the y axis. The inequality (6.12) can be written

$$P > 2Q, \tag{6.16}$$

or, equivalently

$$P > L. \tag{6.17}$$

This inequality represents the condition for the existence of real characteristics. The appearance of such hyperbolic type solutions corresponds to the phenomenon of *internal buckling*. The nature of this phenomenon will be discussed in more detail in the next section.

Note that a more general solution is obtained by including the imaginary roots $\pm \sigma_2$. This is written

$$\phi = \varphi_1(x - \xi y) + \varphi_2(x + \xi y) + \mathscr{R}\varphi_3(x - \sigma_2 y) + \mathscr{R}\varphi_4(x + \sigma_2 y), \qquad (6.18)$$

where $\Re \varphi_3$ and $\Re \varphi_4$ are the real parts of arbitrary analytic functions of the argument. The general solution is of a *mixed elliptic-hyperbolic type*.

An obvious question immediately arises here; that is, whether internal buckling is possible for a material which is originally isotropic under finite deformations. The answer is that internal buckling *cannot occur in this case if* 2M > L. This is readily shown by substituting expression (4.22) for the slide molulus into the inequality (6.17). The inequality becomes

$$P > P/(1-\lambda^4).$$
 (6.19)

Since $\lambda < 1$ and P > 0 this inequality cannot be verified. In the exceptional case in which 2M < L, however, finite isotropy does not exclude internal buckling. This is briefly discussed in the appendix.

7. INTERNAL BUCKLING OF A MEDIUM CONFINED BY RIGID BOUNDARIES

It is possible to find a particular case of the general hyperbolic solution (6.15) which corresponds to a rectangular region under initial stress but confined between rigid boundaries. Such a solution is written

$$\phi = -(1/2l)\cos l(x-\xi y) - (1/2l)\cos l(x+\xi y), \tag{7.1}$$

$$\phi = -(1/l)\cos lx\cos l\xi y. \tag{7.2}$$

The displacements are

or

$$\begin{array}{l} u = -\partial \phi / \partial y = -\xi \cos lx \sin l\xi y, \\ v = \partial \phi / \partial x = \sin lx \cos l\xi y. \end{array} \right\}$$
(7.3)

This solution may, of course, be multiplied by a constant of arbitrary magnitude.

The displacement field $(7\cdot3)$ results from an interference pattern of sinusoidal solutions along the two characteristic directions. It is formally analogous to a pattern of standing acoustic waves in a rectangular domain.

The physical significance of this field is further clarified by considering the pattern in the x and y directions. There is a wavelength associated with each of these directions. They are

$$\mathscr{L}_{x} = 2\pi/l, \quad \mathscr{L}_{y} = 2\pi/\xi l. \tag{7.4}$$

This yields an additional physical interpretation of the variable ξ as the ratio of these two wavelengths; i.e. $\xi = \mathscr{L}_r / \mathscr{L}_n.$ (7.5)



buckling (case L < 2M).

or

FIGURE 6. Metastable configuration corresponding to $n_x = 1$, $n_y = 1$.

On the other hand, this parameter is a root of equation (6.6) and is therefore related to the initial stress P and the elastic coefficients L and M by the relation

$$P = L\xi^4 + 2(2M - L)\xi^2 + L, \tag{7.6}$$

 $P = L(1 - \xi^2)^2 + 4M\xi^2. \tag{7.7}$

This equation provides the condition under which a solution of the type $(7\cdot3)$ is possible. Since, in the general case, the coefficients L and M are functions of the initial stress P, equation $(7\cdot7)$ is really an implicit equation for P. In order to discuss this equation, let us assume for the time being that L and M vary only insignificantly with the initial stress so that they may be considered as constant. Let us also assume, as above, that 2M > L. In that case the functional relation between P and ξ is represented schematically by the curve in figure 5. Consider now a medium confined by a rigid rectangular boundary of sides h_x and h_y (figure 6). The solution $(7\cdot3)$ fits these boundary conditions, provided

$$\mathscr{L}_x = 2h_x/n_x, \quad \mathscr{L}_y = 2h_y/n_y, \tag{7.8}$$

where n_x and n_y are integers. In this solution the normal displacements vanish at the boundaries. It is readily verified that the tangential stresses also vanish at these boundaries, but the tangential displacements do not. The boundaries are

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therefore *rigid and perfectly lubricated*. The value of the parameter ξ for any of these solutions is $\xi = \pi + \frac{1}{2} - \frac$

$$\xi = n_y h_x / n_x h_y. \tag{7.9}$$

Let us assume that the initial compression is

$$P = P_0 > L. \tag{7.10}$$

Figure 5 shows that a value ξ_0 of ξ is associated with P_0 .

If it is possible to find two integers n_x and n_y such that

$$\xi_0 = n_y h_x / n_x h_y, \tag{7.11}$$

the particular solution corresponding to these integers represents an *internal buckling* of a rigidly confined medium. Although it is an equilibrium configuration under the compression P_0 , it is a metastable configuration. This can be shown by considering other solutions such that

$$n_y h_x / n_x h_y \leqslant \xi_0. \tag{7.12}$$

There are an infinite number of solutions satisfying this inequality. All these solutions correspond to buckling loads which are *smaller than* P_0 . Hence, the whole range of values

$$0 < \xi \leqslant \xi_0 \tag{7.13}$$

corresponds to an infinite set of buckling modes which are all hypercritically unstable under the compression P_0 . Three of the possible buckling configurations are illustrated in figures 6, 7, and 8. They correspond to

$$n_{x} = 1, \quad n_{y} = 1; n_{x} = 5, \quad n_{y} = 1; n_{x} = 5, \quad n_{y} = 2.$$
(7.14)

We may well ask which one in this infinite set of internal buckling modes is the most unstable. The answer is immediately furnished by the diagram in figure 5. It is the buckling mode for which ξ becomes vanishingly small, since this is the one for which the difference between the critical load P = L and the actual compression P_0 is maximized. This mode is represented by the values.

$$n_x = \infty, \quad n_y = 1. \tag{7.15}$$

In other words, it exhibits a vanishing wavelength in the direction of the initial compression.

Hence, theoretically, this buckling with vanishing wavelength appears as soon as the compression P becomes greater than L.

We are dealing here with a state of microscopic internal collapse of the medium. This result may seem paradoxical, but it leads to important conclusions regarding the actual behaviour of the medium if we consider that in an actual case other factors not taken into account in the present theory will enter into play. In a laminated medium, for example, the wavelength of internal buckling will be restricted by the lamination thickness. This aspect of the problem will be discussed in a forthcoming publication. The peculiar behaviour exhibited by internal buckling also points to the physical impossibility of measuring the elastic coefficients beyond the critical point.

The physical significance of the present result is further brought into light by a consideration of the stability problem from the energy viewpoint.

It was shown by this writer in an early paper (1939) that the theory of elasticity under initial stress can be formulated completely by introducing an 'incremental strain energy'. In the particular case of two-dimensional strain and with the initial



FIGURE 7. Metastable configuration corresponding to $n_x = 5$, $n_y = 1$.



FIGURE 8. Metastable configuration corresponding to $n_x = 5$, $n_y = 2$.

compression P in the x direction, the incremental strain energy per unit initial volume becomes

$$\Delta V = \frac{1}{2} t_{11} e_{xx} + \frac{1}{2} t_{22} e_{yy} + t_{12} e_{xy} - P(e_{xy} \omega + \frac{1}{2} \omega^2).$$
(7.16)

In this expression, an alternate stress system t_{ij} is used. It represents the incremental forces per unit initial area. The components of these forces are projected on locally rotated axes. It was shown (Biot 1939) that these stress components are related to s_{ij} by relations which, in the present case, take the form

$$t_{11} = s_{11} - Pe_{yy}, t_{22} = s_{22}, t_{12} = s_{12} + \frac{1}{2}Pe_{xy}.$$
 (7.17)

We substitute these expressions in ΔV , taking into account the stress-strain relations (3.9) and the condition of incompressibility (3.8). This yields

$$\Delta V = (2N + \frac{1}{2}P) e_{xx}^2 + (2Q + P) e_{xy}^2 - \frac{1}{2}P (\partial v/\partial x)^2.$$
(7.18)

With the coefficients M and L, as defined by equations (3.16) and (3.19), it is written

$$\Delta V = 2Me_{xx}^{2} + 2Le_{xy}^{2} - \frac{1}{2}P(\partial v/\partial x)^{2}.$$
(7.19)

The incremental strain energy over the rectangular region confining the material under the unit stress is derived by integrating ΔV over this area. If we substitute the solution (7.3) for u and v, the integration amounts to averaging the squares of sines and cosines. The average energy per unit area is found to be

$$\Delta V_{\rm av.} = \frac{1}{4} l^2 [L(1-\xi^2)^2 + 4M\xi^2 - P]. \tag{7.20}$$

We see that the condition $\Delta V_{av.} = 0$ coincides with equation (7.7). The potential energy is negative for $P > L(1-\xi^2)^2 + 4M\xi^2$. (7.21)

This illustrates the physical significance of the instability.

When the inequality (7.21) is satisfied, more energy is available in the initial compression than is required to initiate a mode of buckling characterized by the parameter ξ . As the material buckles, the elastic strain energy associated with the uniform compression is released and is transferred into the buckling mode.

We have assumed in the discussion that L and M are constant. We can easily verify that the qualitative features of the results are not affected if we assume Land M to depend on P, provided 2M > L. The case 2M < L is briefly discussed in the appendix.

8. The influence of internal buckling on acoustic propagation

In a previous paper (1940), the writer analyzed the influence of initial stress on the propagation of elastic waves. The general theory was applied to the twodimensional problem of an anisotropic medium satisfying the stress-strain relations (3.1) and under initial stresses S_{11} and S_{22} . In this case we must add the acceleration terms to equations (2.3). They are replaced by

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P \frac{\partial \omega}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2}, \\
\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P \frac{\partial \omega}{\partial x} = \rho \frac{\partial^2 v}{\partial t^2},$$
(8.1)

where $P = S_{22} - S_{11}$, ρ is the mass density, and u and v are the displacements in the x, y plane. The propagation equations were solved for the case of transversal waves propagating in the x and y directions with velocities V_x and V_y . These velocities were found to be

$$V_x = \sqrt{\left(\frac{Q-\frac{1}{2}P}{\rho}\right)}, \quad V_y = \sqrt{\left(\frac{Q+\frac{1}{2}P}{\rho}\right)}.$$
(8.2)

An immediate consequence of these equations is the result

$$V_y^2 - V_x^2 = P/\rho;$$
 (8.3)

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i.e. the difference of the squares of these two velocities is *independent of the elastic properties*. As pointed out in the earlier paper (1940), this result shows that the effect of initial stress on acoustic propagation cannot be accounted for by simply modifying the elastic coefficients.

As P increases, the velocity V_x will usually decrease and will become zero for

$$Q - \frac{1}{2}P = L - P = 0. \tag{8.4}$$

Comparing with equation (7.7), we see that the vanishing of the velocity V_x corresponds to internal buckling. Physically, this means that when the initial compression in the x direction is high enough to satisfy equation (8.4), the transversal rigidity disappears, and the wave velocity tends to zero.

For a medium which is initially isotropic in the unstressed state, we replace the value of Q by the expression (4.11). We find the velocities

$$V_x = \sqrt{\left(\frac{P}{\rho} \frac{a_{11}^2}{(a_{22}^2 - a_{11}^2)}\right)}, \quad V_y = \sqrt{\left(\frac{P}{\rho} \frac{a_{22}^2}{(a_{22}^2 - a_{11}^2)}\right)}.$$
(8.5)

In these expressions a_{11} and a_{22} are the principal extension ratios in the state of initial stress. In this case the velocities V_x and V_y cannot be made to vanish.

Note the property

$$V_x/a_{11} = V_y/a_{22} \tag{8.6}$$

which means that the transit time remains the same between two pairs of points attached to the medium equidistant in the unstressed state and located on the x, y axes.

Appendix. The case 2M < L

The case m < 0 will be briefly discussed. This is done most conveniently by considering equation (7.6); i.e.

$$P = L + 2(2M - L)\xi^{2} + L\xi^{4}.$$
 (A1)

In this case,

$$2M - L < 0. \tag{A2}$$

Let us assume the coefficients L and M to be constant. The plot of P against ξ exhibits a minimum at the abscissa (figure A 1):

$$\xi = \sqrt{\{(L-2M)/L\}}.$$
 (A3)

The minimum value of P is $P_{\min} = 4M(L-M)/L.$ (A4)

If

$$P_{\min} < P < L, \tag{A5}$$

there are two real roots ξ_1 and ξ_2 of equation (A1), and instability occurs when ξ is located in the range

$$\xi_1 < \xi < \xi_2, \tag{A6}$$

as shown in figure A1.

For the present case, internal buckling occurs when $P > P_{\min}$, and the slope of the characteristics at which this instability appears is given by expression (A3). In contrast with the case analyzed in §7, the slope at incipient buckling is not zero.

A discussion of the case in which L and M are functions of P must take into account the specific finite deformation properties of the material under consideration, as illustrated by the example below.

Another fundamental difference between this case and that considered in the text is that internal buckling may occur for a material which is isotropic under



FIGURE A1. Instability diagram for the case L > 2M.



FIGURE A2. Finite stress-strain law represented by equation (A7).

finite deformation. To show this, let us consider an isotropic medium whose finite deformation for plane strain obeys the equation

$$S_{11} - S_{22} = \kappa = 2\mu_0(\lambda^4 - 1)/(\lambda^4 + 1).$$
 (A7)

The type of stress-strain law represented by this equation is illustrated in figure A 2. Applying equations (4.15) and (4.16), we find

$$N = \frac{4\lambda^4}{(\lambda^4 + 1)^2} \mu_0,$$

$$Q = \mu_0.$$
(A8)

This material becomes 'softer' as the deformation increases. Evaluating the coefficients L and M, we can write the condition (7.7) for internal buckling in the form

$$\xi^4 - (z - 8/z)\,\lambda^2 \xi^2 + \lambda^4 = 0,\tag{A9}$$

where

$$z = \lambda^2 + 1/\lambda^2. \tag{A10}$$

Equation (A9) has a real positive double root for ξ^2 when z = 4. One root,

$$\lambda = \sqrt{(2 + \sqrt{3})} = 1.93, \tag{A11}$$

corresponds to an extension. There is another root corresponding to a compression:

$$\lambda = \frac{1}{\sqrt{(2+\sqrt{3})}} = 0.517.$$
 (A12)

The material becomes internally unstable for deformations corresponding to this extension. At this critical point the slope of the characteristics is given by $\xi = \pm \lambda$. Hence, they make an angle of about 27 degrees with the direction of the extension.

The behaviour of such material which is considered here in the context of elasticity is obviously related to problems of stability and the generation of slip lines in *plasticity*.

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