Reprinted without change of pagination from the
Proceedings of the Royal Society, A, *volume* 273, pp. 329–339, 1963

Surface instability in finite anisotropic elasticity under initial stress

By M. A. Biot

Shell Development Company (A Division of Shell Oil Company), Exploration and Production Research Division, Houston, Texas

(*Communicated by Sir Edward Bullard, F.R.S.—Received* 12 *September* 1962)

The writer's theory of surface instability of an elastic body under initial stress in finite strain is extended to anisotropic elasticity. The general characteristic equation for surface instability is derived. In addition, the surface deflexion under a normal load is evaluated for the subcritical case of initial stress. The analysis includes the case in which gravity is taken into account. It is found that for a certain value of the initial stress the surface behaves as if the solid were a fluid, and surface loads sink to a depth at which they are supported by buoyancy forces only.

1. Introduction

The instability of an elastic half-space under initial stress has been derived and discussed in some recent publications (Biot 1958, 1959, 1961 *a*). The analysis is based on the general theory of elasticity of a continuum under initial stress developed in another series of papers (Biot 1938, 1939, 1940 *a*, *b*). The stability analysis of the elastic half-space referred to above was restricted to the case of a medium which is isotropic for incremental plane strain. In particular it was shown that this property applies to rubber-type elasticity, and it was used to derive an exact analysis for the surface instability of rubber in finite strain (Biot 1961 *a*).

The more elaborate problem of the instability of a continuously non-homogeneous half-space with exponential distribution of rigidity was treated in a detailed analysis including the effect of gravity (Biot 1960).

The existence of surface instability may also be derived from the exact equation for the buckling of a thick slab obtained earlier (Biot 1938) and from the theory of surface waves under pre-stress (Buckens 1958).

The problem of surface indentation of a homogeneous isotropic half-space restricted to a particular type of elasticity and uniaxial initial strain has also been analyzed by using an entirely different approach based on the classical theory of tensor invariants (Green, Rivlin & Shield 1952). The method is handicapped by an intricate formalism which obscures the interpretation. The authors conclude that the result 'seems' to indicate the possibility of instability.

The next step in the development of the theory is, of course, to extend the analysis to the case of a non-isotropic medium. However, this extension is not immediate because a new phenomenon appears which we have referred to as *internal buckling* and which is the object of a separate and detailed analysis presented in an accompanying paper (1963).

Internal buckling is a type of instability which may occur in a homogeneous medium of infinite extent under initial stress. Its existence is not conditioned by the presence of discontinuities or free surfaces. Surface instability on the other hand cannot exist without the presence of the free surface. The distinction is entirely the same as between body waves and Rayleigh waves in acoustic propagation.

In the present paper the surface instability of the elastic half-space is analyzed for the case of a non-isotropic medium. The surface instability is characterized by a buckling mode whose amplitude decreases exponentially with the distance from the surface. This distinguishes it from internal buckling. The latter may be influenced by the free surface but occurs throughout the medium in analogy with the reflexion of acoustic body waves at a boundary. Within the range of parameters assumed in the present analysis, internal buckling is excluded.

The incremental elastic properties are assumed to be of orthotropic symmetry, one plane of symmetry being parallel with the free surface. The anisotropy considered here includes two physically distinct cases. In one case the medium is orthotropic in the original unstressed state. It is then deformed into a state of finite initial strain by applying principal stresses whose directions coincide with the planes of elastic symmetry. The property of orthotropic symmetry in this case is retained for the incremental deformations.

In the other case the medium is originally isotropic in finite strain, and the anisotropy is due to the state of initial deformation. The incremental deformations in this case will exhibit orthotropic symmetry along the directions of the initial principal stresses.

In either case the theory deals with the stability of incremental deformation about a state of finite homogeneous strain. The principal initial stresses are parallel with the free surface, and if the medium is originally orthotropic, it is assumed that they also are parallel with the planes of elastic symmetry.

In the present analysis the medium is assumed incompressible. While this assumption provides drastic simplifications of the algebra, it does not restrict the general character of the results.

In § 2 we have briefly recalled the basic equations of the stability problem as formulated in more detail in the earlier work.

The analytical solution of the stability problem for the anisotropic half-space is carried out in § 3, and a numerical discussion of the result is presented in § 4. The discussion is carried out in the context of a more general problem than mere instability by evaluating the deflexion of the surface under a normal load.

The influence of gravity on the surface instability is discussed in § 5. The surface deflexion under a given surface load is examined. It is found that for a compression which would produce buckling in the absence of gravity, the surface is still stable but the load sinks into the surface down to a certain depth as if it were a buoyant fluid. In other words, the apparent elastic rigidity of the surface vanishes and its carrying capacity is the same as if the load were floating on a fluid of the same density as the half space.

For a compression somewhat higher than this value the surface becomes unstable, but only in a range of wavelengths below a certain cut-off value, depending on the compression. The smallest wavelengths in this unstable range exhibit the highest instability.

2. General formulation

A solid half-space is subject to a uniform compressive stress P parallel with the surface. We shall consider an incompressible elastic medium of orthotropic incremental properties. The x axis coincides with the surface, and the y axis is directed positively outward (figure 1). They are also axes of symmetry for the mechanical properties of the medium. The incremental deformation analyzed here is a state of

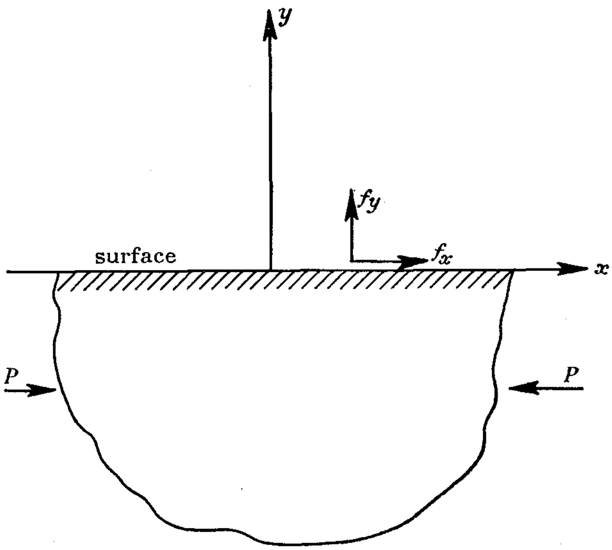

Figure 1. Half-space and co-ordinate system.

plane strain where all variables are functions of x and y. The two-dimensional equations of equilibrium for the stress field are

$$\left.\begin{aligned} \frac{\partial s_{11}}{\partial x}+\frac{\partial s_{12}}{\partial y}-P\frac{\partial \omega}{\partial y} &= 0, \\ \frac{\partial s_{12}}{\partial x}+\frac{\partial s_{22}}{\partial y}-P\frac{\partial \omega}{\partial x} &= 0. \end{aligned}\right\} \tag{2·1}$$

These equations were derived in earlier publications (Biot 1938, 1940*a*,*b*, 1959). The rotation ω is defined by

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right), \tag{2·2}$$

where u and v are the displacement components. The stress components s_{11}, s_{22}, s_{12} are the incremental stresses referred to rectangular axes rotated locally through the angle ω.

The strain components are

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right).$$

They are related to the incremental stress by the relations

$$\left.\begin{aligned} s_{11}-s &= 2Ne_{xx}, \\ s_{22}-s &= 2Ne_{yy}, \\ s_{12} &= 2Qe_{xy}. \end{aligned}\right\} \tag{2·4}$$

These relations introduce two elastic coefficients N and Q and represent a material of orthotropic symmetry. It reduces to the familiar isotropic stress-strain relation for an incompressible material if

$$N = Q. \tag{2·5}$$

The medium being incompressible, we must add the condition

$$e_{xx}+e_{yy} = 0. \tag{2·6}$$

Solutions of the above equations will be sought which are sinusoidal in the x coordinate. The boundary conditions at the surface will be introduced later.

In previous work (1961 *c*, 1963) we have also introduced the elastic coefficients

$$\left.\begin{aligned} M &= N+\tfrac{1}{4}P, \\ L &= Q+\tfrac{1}{2}P, \end{aligned}\right\} \tag{2·7}$$

which are directly measurable and have a simple physical interpretation. The coefficient L was referred to as the slide modulus.

3. Solution of the stability problem

The condition (2·6) of incompressibility is satisfied by putting

$$u = -\frac{\partial\phi}{\partial y}, \quad v = \frac{\partial\phi}{\partial x}. \tag{3·1}$$

The field equations of the previous section then reduce to two equations with two unknowns

$$\left.\begin{aligned} \frac{\partial s}{\partial x}-\frac{\partial}{\partial y}\left[(2N-Q+\tfrac{1}{2}P)\frac{\partial^2\phi}{\partial x^2}+(Q+\tfrac{1}{2}P)\frac{\partial^2\phi}{\partial y^2}\right] &= 0, \\ \frac{\partial s}{\partial y}+\frac{\partial}{\partial x}\left[(2N-Q-\tfrac{1}{2}P)\frac{\partial^2\phi}{\partial y^2}+(Q-\tfrac{1}{2}P)\frac{\partial^2\phi}{\partial x^2}\right] &= 0. \end{aligned}\right\} \tag{3·2}$$

Elimination of s yields

$$(Q+\tfrac{1}{2}P)\frac{\partial^4\phi}{\partial y^4}+2(2N-Q)\frac{\partial^4\phi}{\partial x^2\partial y^2}+(Q-\tfrac{1}{2}P)\frac{\partial^4\phi}{\partial x^4} = 0. \tag{3·3}$$

Solutions of these equations are of the form

$$\left.\begin{aligned} \phi l^2 &= f(ly)\sin lx, \\ s &= F(ly)\cos lx, \end{aligned}\right\} \tag{3·4}$$

Substituting this expression for ϕ into equation (3·3), we derive

$$(Q+\tfrac{1}{2}P)f'''' - 2(2N-Q)f''+(Q-\tfrac{1}{2}P)f = 0. \tag{3·5}$$

The primes denote differentiation with respect to the argument ly. Substitution of s and ϕ in the first equation (3·2) yields F in terms of f

$$F(ly) = (2N - Q + \tfrac{1}{2}P)f' - (Q + \tfrac{1}{2}P)f'''. \tag{3·6}$$

The function of f is the general solution of equation (3·5), i.e.

$$f = \sum^{i} C_i e^{\beta_i ly}, \tag{3·7}$$

where β_i are any of the four roots of the equation

$$\beta^4 - 2m\beta^2 + k^2 = 0. \tag{3·8}$$

We have put

$$k^2 = \frac{1-\zeta}{1+\zeta}, \quad \zeta = \frac{P}{2Q}, \quad m = \frac{2(N/Q)-1}{1+\zeta}. \tag{3·9}$$

In terms of the coefficients M and L these expressions are written

$$k^2 = \frac{L-P}{L}, \quad \zeta = \frac{P}{2L-P}, \quad m = \frac{2M-L}{L}. \tag{3·10}$$

Of considerable importance in this problem is the behaviour of the roots of equation (3·8).

We are interested here in surface instability. Hence we must restrict ourselves to solutions which vanish at $y = -\infty$. Such solutions will decay exponentially with depth. In order to satisfy the boundary condition at the surface ($y = 0$), there must be two independent solutions of this type. Therefore, we must exclude all cases where one of the roots is a pure imaginary.

There are two such cases,

$$\left.\begin{array}{lll}(1) & m > 0 \quad \text{with} & k^2 < 0,\\ (2) & m < 0 \quad \text{with} & m^2 - k^2 > 0.\end{array}\right\} \tag{3·11}$$

The first case is equivalent to

$$2M > L, \quad P > L \tag{3·12}$$

and the second to

$$2M < L, \quad P > (4M/L)(L-M). \tag{3·13}$$

The physical significance of these inequalities is evident from the analysis of a previous paper (1963). We have shown that they correspond to what we have called internal buckling. Hence, exclusion of the two cases (3·11) amounts to the assumption that the parameters lie outside the range of internal buckling, i.e. we must assume that

$$\left.\begin{array}{lll}(1) & m > 0 \quad \text{with} & k^2 > 0,\\ (2) & m < 0 \quad \text{with} & m^2 - k^2 < 0.\end{array}\right\} \tag{3·14}$$

In either case this implies

$$k^2 > 0 \quad \text{or} \quad P < L, \tag{3·15}$$

and if $\quad 2M < L \quad \text{then} \quad P < (4M/L)(L-M).$

Under these conditions the roots β of equation (3·8) are either real or complex conjugate. Their real part is different from zero, and it is always possible to choose two of them such that their real parts are positive. We shall designate these two roots by

$$\left.\begin{aligned}\beta_1 &= \sqrt{\{m+\sqrt{(m^2-k^2)}\}},\\ \beta_2 &= \sqrt{\{m-\sqrt{(m^2-k^2)}\}}.\end{aligned}\right\} \tag{3·16}$$

The solution adopted is then

$$f = C_1 e^{\beta_1 ly} + C_2 e^{\beta_2 ly}. \tag{3·17}$$

It vanishes at $y = -\infty$.

We shall now introduce the boundary forces. These forces were evaluated in the earlier papers (1938, 1939). In the present case the force components at the boundary are

$$\left.\begin{aligned}f_x &= s_{12}+Pe_{xy},\\ f_y &= s_{22}.\end{aligned}\right\} \tag{3·18}$$

These forces act at the deformed surface, represented by $y = 0$, per unit initial area. Substituting the solutions derived above, we find

$$\left.\begin{aligned}f_x &= \tau \sin lx,\\ f_y &= q \cos lx,\end{aligned}\right\} \tag{3·19}$$

with

$$\left.\begin{aligned}\tau/L &= -f(0)-f''(0),\\ q/L &= (2m+1)f'(0)-f'''(0).\end{aligned}\right\} \tag{3·20}$$

On the other hand, the surface deflexion may be written

$$\left.\begin{aligned}u &= U \sin lx,\\ v &= V \cos lx,\end{aligned}\right\} \tag{3·21}$$

and again using the above solution

$$\left.\begin{aligned}lU &= -f'(0),\\ lV &= f(0).\end{aligned}\right\} \tag{3·22}$$

By substitution of the solution (3·17) for f, this is written

$$\left.\begin{aligned}lU &= -C_1\beta_1-C_2\beta_2,\\ lV &= C_1+C_2.\end{aligned}\right\} \tag{3·23}$$

Solving these two equations for C_1 and C_2, and substituting in expression (3·17) and (3·20), we obtain

$$\left.\begin{aligned}\tau/lL &= U(\beta_1+\beta_2)+V(\beta_1\beta_2-1),\\ q/lL &= U(\beta_1\beta_2-1)+V\beta_1\beta_2(\beta_1+\beta_2).\end{aligned}\right\} \tag{3·24}$$

The product $\beta_1\beta_2$ and the sum $\beta_1+\beta_2$ can be expressed very simply in terms of k and m. Equation (3·8) is quadratic in β^2, and its roots satisfy the relations

$$\left.\begin{aligned}\beta_1^2\beta_2^2 &= k^2,\\ \beta_1^2+\beta_2^2 &= 2m.\end{aligned}\right\} \tag{3·25}$$

Because of assumptions stated above, k^2 is positive. Its positive square root is denoted by

$$k = \sqrt{\frac{1-\zeta}{1+\zeta}} = \sqrt{\frac{L-P}{P}}. \tag{3·26}$$

Furthermore, we have shown that under the same assumptions the roots β_1 and β_2 are either real and positive or complex conjugate with a positive real part. It follows that without ambiguity in sign we may write

$$\beta_1 \beta_2 = k. \tag{3·27}$$

Combining this result with the second relation (3·25) yields

$$(\beta_1+\beta_2)^2 = 2(m+k). \tag{3·28}$$

Because of the assumptions (3·14), $k > |m|$ hence $m+k$ is positive. Since $\beta_1+\beta_2$ is also positive, we derive

$$\beta_1+\beta_2 = \sqrt{\{2(m+k)\}} \tag{3·29}$$

where the right-hand side represents the positive square roots.

With these results, equations (3·24) become

$$\begin{aligned} \tau/lL &= U\sqrt{\{2(m+k)\}}+V(k-1), \\ q/lL &= U(k-1)+Vk\sqrt{\{2(m+k)\}}. \end{aligned} \tag{3·30}$$

The physical implications of this result will now be discussed.

4. Discussion of the instability for the elastic half-space

The case of vanishing initial stress is found by putting $P = 0$ in equations (3·30). In this case we find†,

$$\left.\begin{aligned} \tau &= 2Ul\sqrt{(NQ)}, \\ q &= 2Vl\sqrt{(NQ)}. \end{aligned}\right\} \tag{4·1}$$

For isotropic material $N = Q$, and equations (4·1) become

$$\left.\begin{aligned} \tau &= 2UlQ, \\ q &= 2VlQ. \end{aligned}\right\} \tag{4·2}$$

This leads to the conclusion that in the absence of initial stress the surface deflexion of the anisotropic medium is derived by the same expression as for the isotropic medium, except that we must replace the shear modulus by the geometric mean of the two moduli N and Q. The tangential and normal deflexions also remain uncoupled for the anisotropic case.

Let us now investigate how these conclusions are modified if an initial compression P is present. In that case k is different from unity and the normal and tangential deflexions are coupled. Of particular interest is the case where $\tau = 0$. The surface is then subject only to a normal load q proportional to the normal deflexion V. We may write this relation in the form

$$V = \frac{q}{2l\varphi\sqrt{(NQ)}}, \tag{4·3}$$

with

$$\varphi = \frac{1+\zeta}{2\sqrt{(N/Q)}}\,\frac{k^2-1+2k(m+1)}{\sqrt{\{2(m+k)\}}}. \tag{4·4}$$

† A point of interest here is the application of the correspondence principle to these equations. For a viscous solid we substitute the operators $N^* = N'p$ and $Q^* = Q'p$. It can be seen that this medium behaves as an isotropic solid of viscosity $\eta = \sqrt{(N'Q')}$. (The operator p represents the time derivative.)

It is obtained by putting $\tau = 0$ in equations (3·30) and eliminating U between the two equations. Comparing with expressions (4·1) we conclude that $1/\varphi$ represents an amplification factor due to the presence of the initial compression P. It is easily verified that $\varphi = 1$ for $P = 0$. When $\varphi = 0$ the surface deflexion becomes infinite. This condition corresponds to surface buckling. It may be written

$$k^2 - 1 + 2k(m+1) = 0. \tag{4·5}$$

It is an equation for ζ and contains N/Q as a parameter. This parameter may be considered as a measure of the anisotropy of the medium. For $N = Q$ and after rationalization, it degenerates into the equation

$$\zeta^3 + 2\zeta^2 - 2 = 0, \tag{4·6}$$

where the real root is
$$\zeta_{cr.} = 0·84. \tag{4·7}$$

This checks with the result already established previously for this particular case (Biot 1958, 1959, 1960, 1961 *a*). The more general equation (4·5) may be written

$$\frac{N}{Q} = \tfrac{1}{2}\zeta\left[\sqrt{\left(\frac{1+\zeta}{1-\zeta}\right)} - 1\right]. \tag{4·8}$$

The variable ζ in the equation represents the critical value $\zeta_{cr.}$ at instability. The ratio N/Q corresponding to various critical values is given in table 1.

TABLE 1. CRITICAL VALUE, $\zeta_{cr.} = P/2Q$, FOR SURFACE INSTABILITY AS A FUNCTION OF N/Q

N/Q	$\zeta_{cr.}$	N/Q	$\zeta_{cr.}$
0	0	1·00	0·84
0·20	0·02	1·51	0·90
0·40	0·10	2·45	0·95
0·60	0·30	4·37	0·98
0·70	0·48	6·50	0·99
0·80	0·80	∞	1·00

There is a range for which the solution is oscillatory with an amplitude decreasing with the distance from the surface. This is the region for which the roots β_1 and β_2 are complex conjugate. The condition that this be the case is

$$m^2 - k^2 < 0. \tag{4·9}$$

Since $m + k > 0$ this is equivalent to

$$m < k$$

or
$$N/Q < \tfrac{1}{2}\{1 + \sqrt{(1-\zeta^2)}\}. \tag{4·10}$$

The cross-over point occurs when this relation is replaced by an equality. Equating this value of N/Q with expression (4·8) at the critical point yields the cross-over value $\zeta_{cr.} = N/Q = 0·8$. Hence, if

$$0 < N/Q < 0·8, \tag{4·11}$$

the solution is of the oscillatory type.

The amplification factor $1/\varphi$ was computed earlier (Biot 1959, 1961 *b*) as a function of ζ in the particular case of incremental isotropy, i.e. for $N/Q = 1$. A similar dependence of the amplification on ζ is found for the case of anisotropy.

Evaluation of the finite strain at which surface instability occurs is illustrated as follows. Assume a material for which the compressive stress P in two-dimensional strain is given as a function of the extension ratio λ, by

$$P = -\kappa(\lambda), \tag{4·12}$$

for a compression $\lambda < 1$ and $\kappa < 0$. If the material is isotropic in finite strain, we have shown (1961*b*, 1963) that the coefficients N and Q are given by

$$N = \tfrac{1}{4}\lambda\frac{d\kappa}{d\lambda}, \quad Q = \tfrac{1}{2}\kappa\frac{\lambda^4+1}{\lambda^4-1}. \tag{4·13}$$

Hence,
$$\frac{N}{Q} = \frac{1}{2}\frac{\lambda}{\kappa}\frac{d\kappa}{d\lambda}\frac{\lambda^4-1}{\lambda^4+1}, \quad \zeta = \frac{P}{2Q} = \frac{1-\lambda^4}{1+\lambda^4}. \tag{4·14}$$

We now plot a curve with the abscissa ζ and the ordinate N/Q as a parametric function of λ. The point at which this curve crosses the plot represented by table 1 yields the critical finite strain at which surface instability occurs.

If the material is originally anisotropic, we need an additional measurement besides the stress-strain curve in order to determine Q. We have discussed earlier (1963) how Q may be determined by measuring the slide modulus L.

5. The influence of gravity on surface instability

The author has shown (1960) that for an incompressible material the influence of gravity is equivalent to the addition of a surface load proportional to the deflexion. The boundary conditions (3·19) must, therefore, be replaced by

$$\left.\begin{aligned} f_x &= \tau \sin lx, \\ f_y &= q\cos lx - \rho g V \cos lx, \end{aligned}\right\} \tag{5·1}$$

where ρ is the mass density of the medium and g the acceleration of gravity acting in the negative y direction. Results obtained above are all applicable to the case where gravity is taken into account. All we have to do is to replace q by $q-\rho g V$. This substitution may be performed in equation (4·3). It becomes

$$V = \frac{q-\rho g V}{2l\varphi\sqrt{(NQ)}}. \tag{5·2}$$

Hence
$$V = \frac{q}{2l\varphi\sqrt{(NQ)}+\rho g}. \tag{5·3}$$

This gives the deflexion $V\cos lx$ of the surface under a disturbed normal load $q\cos lx$. It is assumed that no horizontal force is acting at the surface, i.e. $\tau = 0$. Instability occurs for

$$2l\varphi\sqrt{(NQ)}+\rho g = 0. \tag{5·4}$$

We note that in this case the instability depends on the wavelength. This is in contrast with the weightless case analyzed in the previous section.

The buckling condition (5·4) can be verified only if φ is negative, i.e. if the value of ζ is larger than those of table 1. As expected, the presence of gravity has a stabilizing effect. Another point of interest is the existence of a cut-off wavelength. For example, in the case $N = Q$, the value of φ varies from $\varphi = 0$ to $\varphi = -1$ when ζ varies from $\zeta = 0{\cdot}84$ to $\zeta = 1$. Hence, for a value $\zeta_{cr.}$ such that

$$0{\cdot}84 < \zeta_{cr.} < 1, \tag{5·5}$$

the value of φ is

$$\varphi = -\varphi' \quad (0 < \varphi' < 1). \tag{5·6}$$

Substitution in equation (5·4) yields a critical wave number

$$l_{cr.} = \rho g / 2\varphi' \sqrt{(NQ)}. \tag{5·7}$$

This defines a cut-off wavelength such that all deformations of smaller wavelengths are unstable and the instability increases as the wavelength tends to zero.

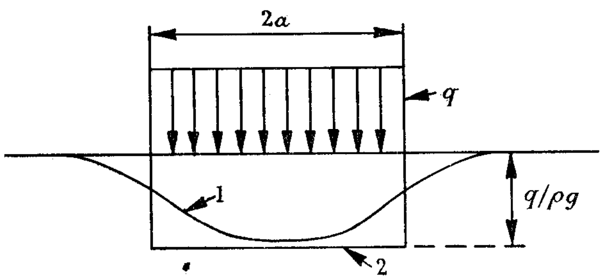

FIGURE 2. Deflexion of the surface under a uniform load: curve 1 for P smaller than the critical value of table 1 and curve 2 for P given by table 1.

Another interesting aspect of this problem is brought out by evaluating the deflexion under a given surface load. For a compression P corresponding to $\zeta_{cr.}$ of table 1 the value of φ is zero. Hence, the surface deflexion is

$$V = q/\rho g. \tag{5·8}$$

This value is independent of the wavelength. The load sinks to a depth where it is balanced by the hydrostatic pressure. Hence, in such a case the *solid behaves like a fluid* and supports the load only through a buoyancy effect.

For example, if a uniform load of magnitude q is applied over a distance $2a$ (figure 2), we may calculate the deflexion by the use of equation (5·3) and simple Fourier transforms. The reader will easily verify that this involves the well-known tabulated functions Si and Ci.

When the lateral compression in the medium is smaller than the value corresponding to table 1, the sinking of the surface is represented by a smooth curve represented by line 1 in figure 2. When the compression reaches the critical value of table 1, the deflexion is represented by curve 2 in the same figure. The surface load sinks in by a uniform amount $q/\rho g$ as if it were floating on a fluid of the same density. Of course, we must remember that the vertical slopes in this last case cannot occur within the limits of the linearized theory.

References

Biot, M. A. 1938 *Proc. Fifth Int. Congr. Appl. Mech.* **11**, 117–122.
Biot, M. A. 1939 *Phil. Mag.* (7), **27**, 468–489.
Biot, M. A. 1940*a* *Z. Angew. Math. Mech.* **20**, 89–99.
Biot, M. A. 1940*b* *J. Appl. Phys.* **11**, 522–530.
Biot, M. A. 1958 *Proc. IUTAM Colloquium on Inhomogeneities in Elasticity and Plasticity, Warsaw*, 1958, pp. 311–321. Pergamon Press.
Biot, M. A. 1959 *Quart. Appl. Math.* **17**, 185–204.
Biot, M. A. 1960 *J. Franklin Inst.* **270**, 190–201.
Biot, M. A. 1961*a* Incremental elastic coefficients of an isotropic medium in finite strain. Air Force Office of Scientific Research, Report T.N. 1772 (to be published in *Applied Scientific Research*, Ser. A).
Biot, M. A. 1961*b* Surface instability of rubber in compression. Air Force Office of Scientific Research, Report T.N. 1771 (to be published in *Applied Scientific Research*, Ser. A).
Biot, M. A. 1961*c* Exact theory of buckling of a thick slab. Air Force Office of Scientific Research, Report T.N. 1770 (to be published in *Applied Scientific Research*, Ser. A).
Biot, M. A. 1963 *Proc. Roy. Soc.* A, **273**, 306–328.
Buckens, F. 1958 *Annali di Geofisica* (2), **11**, 99–112.
Green, A. E., Rivlin, R. S. & Shield, R. T. 1952 *Proc. Roy. Soc.* A, **211**, 128–154.

PRINTED IN GREAT BRITAIN AT THE UNIVERSITY PRESS, CAMBRIDGE
(BROOKE CRUTCHLEY, UNIVERSITY PRINTER)