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Variational Principles for Acoustic-Gravity Waves

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Variational principles are developed for the dynamics of a fluid under initial stress in an arbitrary potential field and disturbed from equilibrium. They are formulated in terms of the fluid displacement. Two distinct principles are obtained which are mathematically equivalent but differ fundamentally from the physical viewpoint. The difference results from expressing the potential energy in terms of buoyancy forces or strain energy. A very general stability criterion is obtained. An important new feature is the inclusion of surface integrals in the potential energy. The simplified principle for the case of a liquid is interpreted by means of an analog model. Lagrangian equations and methods of normal coordinates for the evaluation of transient propagation are applicable along with general theorems on the equivalence of group velocity and energy flux. As an illustration the case of a constant gravity field is discussed.

I. INTRODUCTION

IN a previous paper fluid displacement equations were derived for the dynamics of a fluid perturbed from equilibrium.[1] The body force acting on the fluid is represented by a completely general potential field.

The existence of a variational principle for these equations is inferred from the variational principle derived by the writer for the theory of elasticity under initial stress.[2] The derivation from this more general viewpoint is the object of the paper immediately following this one.

In the present paper variational principles are derived directly from the dynamical equations governing the fluid displacements. It was found[1] that these equations are of two fundamentally different types which we have called the modified and the unmodified equations. Correspondingly there are two different types of variational principles which we refer to as the modified and the unmodified principle.

The modified principle is derived in Sec. II. The potential energy is expressed in terms of the work done against the *bouyancy forces*. It is the sum of a *volume integral* extended to the fluid and a *surface integral* at the free surface. A condition of static stability is readily derived from this principle. It yields a generalization of the well-known stability criterion of a gas in a constant gravity field. A surface stability condition is found to be related to the "Taylor instability." A simplified principle for a liquid is also derived. It is found to be expressed in terms of the potential energy of the analog model which was discussed in the previous paper.[1]

The unmodified principle is derived in Sec. III. In this principle the potential energy is expressed in terms of the product of the stress by the volume change of the fluid. Because of the state of initial stress we must include the second-order terms in the volume change. The form of this variational principle is essentially the same as derived from the theory of elasticity in terms of the *strain energy*. An interesting feature of this principle is that in addition to a volume integral extended to the fluid the potential energy must include a *surface integral* extended to the rigid boundary of the fluid and depending essentially on the curvature of this boundary.

That the modified and unmodified principles are equivalent is demonstrated in Sec. IV.

The total potential energy may be expressed in alternate forms corresponding to these two principles. In any case, the general variational formulation leads to Hamilton's principle and opens the way to the use of generalized coordinates and Lagrangian equations as in the classical problem of oscillations of a conservative system with potential and kinetic energies.

As a consequence the method of normal coordinates[3] developed for the treatment of propagation of pulses and transients becomes readily applicable to acoustic-gravity waves.

It is also pointed out as another consequence of the same results that in waveguide propagation there is equivalence between group velocity and energy flux as already shown by the writer for the more general case of an elastic solid under initial stress.[4]

[1] M. A. Biot, Phys. Fluids **6**, 621 (1963). Multiple forms of the equations and the corresponding variational principles were originally derived by the writer in an Air Force Office of Scientific Research Report "Generalized Theory of Internal Gravity Waves" (1962).
[2] M. A. Biot, Phil. Mag. **27**, 468, 1939.
[3] M. A. Biot and I. Tolstoy, J. Acoust. Soc. Am. **29**, 381 (1957).
[4] M. A. Biot, Phys. Rev. **105**, 1129 (1957).

The variational principles for the particular case of a constant gravity field are discussed in Sec. 5.

II. MODIFIED VARIATIONAL PRINCIPLE

A "modified form" of the dynamical equations for a fluid under initial stress were derived as[1]

$$\frac{\partial s'}{\partial x_i} - \rho e X_i - u_j X_j \frac{\partial \rho}{\partial x_i} = \rho a_i, \tag{2.1}$$

with

$$s' = s + \rho u_j X_j, \tag{2.2}$$

where ρ is the initial density distribution; $X_i = -\partial U/\partial x_i$ is the body force field derived from a potential U; u_i is the fluid displacement; s is the incremental stress of a fluid particle; a_i is the acceleration of a fluid particle; and $e = \partial u_i/\partial x_i$ is the volume dilatation. It is easy to obtain directly a variational principle for these equations. The invariant to be considered is

$$\mathcal{W}_\tau = \iiint_\tau \left(\tfrac{1}{2}se + \rho e u_j X_j + \tfrac{1}{2} X_j \frac{\partial \rho}{\partial x_i} u_i u_j\right) d\tau, \tag{2.3}$$

where the integral is extended to a volume τ of the fluid. In accordance with the definition of s,[1] we must insert $s = \lambda e$.

That expression (2.3) leads to the correct variational principle may be guessed from inspection of the integrand. We notice that it contains a term $\frac{1}{2}\lambda e^2$ representing the elastic energy while the remaining terms represent the work done by the buoyancy forces.

This is readily verified by evaluating the variation of expression (2.3) due to virtual displacements δu_i. In this evaluation we take into account the following important property:

$$\frac{1}{2}\delta\left(X_j \frac{\partial \rho}{\partial x_i} u_i u_j\right) = \frac{1}{2}\left(X_j \frac{\partial \rho}{\partial x_i} + X_i \frac{\partial \rho}{\partial x_j}\right) u_j \delta u_i. \tag{2.4}$$

Because the fluid is initially in equilibrium in a body force field derived from a potential we may write[1]

$$X_j\, \partial\rho/\partial x_i = X_i\, \partial \rho/\partial x_j. \tag{2.5}$$

With this result and after integration by parts the variation of $\mathcal{W}_\tau$ becomes

$$\delta \mathcal{W}_\tau = \iint_A (s + \rho u_j X_j) n_i\, \delta u_i\, dA - \iiint_\tau \left(\frac{\partial s'}{\partial x_i} - \rho e X_i - u_j X_j \frac{\partial \rho}{\partial x_i}\right) \delta u_i\, d\tau. \tag{2.6}$$

The surface integral extends to the boundary A of the volume τ, and n_i denotes the components of the outward unit vector normal to the boundary. We recognize the left-hand side of Eqs. (2.1) to be the same as the integrand in the volume integral of expression (2.6). As a consequence if the dynamical equations (2.1) are verified throughout the volume τ, then the following variational equation is also verified.

$$\delta \mathcal{W}_\tau + \iiint_\tau \rho a_i\, \delta u_i\, d\tau = \iint_A (s + \rho u_j X_j) n_i\, \delta u_i\, dA. \tag{2.7}$$

This is a variational principle which must be verified for virtual displacements δu_i arbitrary in the volume and at the boundary. Because it is a direct consequence of Eqs. (2.1), we shall refer to it as the *modified variational principle*. The value of $\mathcal{W}_\tau$ may be written in a simpler form which brings out more clearly its physical significance. The vectors X_i and $\partial\rho/\partial x_i$ are perpendicular to the equipotential surfaces. Their algebraic projections on this normal direction are denoted by X and $\partial\rho/\partial n$. The normal component of u_i is denoted by u_n. With these definitions we write expression (2.3) as

$$\mathcal{W}_\tau = \iiint_\tau \left(\tfrac{1}{2}\lambda e^2 + \rho e u_n X + \tfrac{1}{2} x \frac{\partial \rho}{\partial n} u_n^2\right) d\tau. \tag{2.8}$$

In this form the physical significance of the terms in the integrand is self evident.

Potential Energy of the Free Surface

The variational principle (2.7) may be given a simpler form by transforming the surface integral on the right-hand side. We show that by introducing a constraint for the displacement at the solid boundary it reduces to a surface integral over the free surface which may then be incorporated in the total potential energy of the system.

We shall assume that on a solid boundary the displacements satisfy the following constraint:

$$n_i u_i = 0, \qquad n_i\, \delta u_i = 0. \tag{2.9}$$

These equations express the condition that the displacement u_i and its variation δu_i are tangent to the boundary surface *at the initial point*. This is because n_i is the normal direction of the boundary at the point x_i and not at the displaced point $\xi_i = x_i + u_i$. It is important to note this exact meaning since for a curved surface it violates the actual physical boundary condition with a *second-order* discrepancy. Of course, conditions (2.9) are also verified at boundaries when the displacement is required to vanish. We denote by B the solid boundary where conditions (2.9) are imposed and denote by F the free boundary where both the initial stress S and the incremental stress s must vanish, i.e.,

$$S = s = 0. \tag{2.10}$$

Because of conditions (2.9) and (2.10), the surface integral in the variational principle (2.7) becomes

$$\iint_A (s + \rho u_j X_j) n_i \, \delta u_i \, dA = \iint_F \rho u_j X_j n_i \, \delta u_i \, dA. \tag{2.11}$$

The domain of integration in this expression extends only to the free surface F. A further simplification arises from the condition of initial equilibrium of the fluid which implies that the free surface is an equipotential surface. Introducing X, the normal component of the body force, and u_n, the normal component of the surface displacement into Eq. (2.11) we obtain

$$\iint_A (s + \rho u_j X_j) n_i \, \delta u_i \, dA = -\delta \mathcal{W}_F, \tag{2.12}$$

with

$$\mathcal{W}_F = -\frac{1}{2} \iint_F \rho X u_n^2 \, dA. \tag{2.13}$$

Note that the value of X is positive along the outward normal.

By inserting expression (2.12) into the variational principle (2.7) it becomes

$$\delta(\mathcal{W}_\tau + \mathcal{W}_F) + \iiint_\tau \rho a_i \, \delta u_i \, d\tau = 0. \tag{2.14}$$

The total potential energy $\mathcal{P}$ in this case is represented as the sum of two terms

$$\mathcal{P} = \mathcal{W}_\tau + \mathcal{W}_F. \tag{2.15}$$

One term $\mathcal{W}_\tau$ corresponds to the potential energy stored in the volume τ. The other term $\mathcal{W}_F$ is contributed only by the deformation of the free surface.

Stability

Static stability requires that the potential energy (2.15) be positive definite. Hence $\mathcal{W}_F$ and $\mathcal{W}_\tau$ must be also positive definite. According to expression (2.13) $\mathcal{W}_F$ will be positive definite if

$$X < 0 \tag{2.16}$$

at the free surface. This inequality means that the body force must be directed inward at the free surface.

Referring to the value (2.8) for $\mathcal{W}_\tau$ we note that the integrand is a quadratic form in the two variables e and u_n. It will be positive definite if

$$\lambda X(\partial \rho / \partial n) - (\rho X)^2 > 0. \tag{2.17}$$

If we choose the normal to be oriented in the direction of the body force ($X > 0$) condition (2.17) becomes

$$(\partial/\partial n)(\log \rho) > \rho X/\lambda. \tag{2.18}$$

This criterion generalizes to the case of an arbitrary body force potential the well-known stability condition for a gas in a constant gravity field.

Analog Model and the Variational Principle for Internal Gravity Waves in a Liquid

For an incompressible fluid we put $e = 0$ in the value (2.8) of $\mathcal{W}_\tau$. It is simplified to

$$\mathcal{W}_\tau = \frac{1}{2} \iiint_\tau X \frac{\partial \rho}{\partial n} u_n^2 \, d\tau. \tag{2.19}$$

This result could also be derived immediately by considering the *analog model* introduced and discussion in the previous paper.[1] The value (2.19) of $\mathcal{W}_\tau$ is obviously the potential energy stored in the elastic restoring forces of the analog model inside the fluid. Similarly $\mathcal{W}_F$ is the potential energy of the surface restoring forces of the analog model. The stability of the equilibrium clearly depends on the sign of $X \, \partial\rho/\partial n$ in the fluid. If the body force and the density gradient are oriented everywhere in the same direction, $\mathcal{W}_\tau$ is positive definite. The same is true for $\mathcal{W}_F$ if the body force acts inward at the free surface. In this case the fluid is in stable equilibrium. It is possible to find examples where this condition is not fulfilled if the body force is produced by an *acceleration field*. In particular this is seen to be the explanation of the so-called "Taylor instability."

If the fluid is composed of a number of homogeneous layers with density discontinuities the potential energy is replaced by *a sum of surface integrals*. The normal gradient $\partial\rho/\partial n$ is replaced by the density discontinuity and the surface integral represents the potential energy of the restoring forces applied to the discontinuity surfaces in the analog model.

III. UNMODIFIED VARIATIONAL PRINCIPLE

Another form of dynamical equations for a fluid under initial stress derived in the previous paper[1] is written

$$\frac{\partial s}{\partial x_i} + e \frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i} \frac{\partial S}{\partial x_j} - \rho \frac{\partial^2 U}{\partial x_i \, \partial x_j} u_j = \rho a_i. \tag{3.1}$$

They were referred to as the unmodified equations. The initial stress field in the fluid is denoted by S. A variational principle for these equations is derived as follows. We put

$$\mathcal{P}_\tau = \iiint_\tau (\tfrac{1}{2}se + \mathcal{R} + \rho\Delta U)\, d\tau, \qquad (3.2)$$

with

$$\mathcal{R} = \tfrac{1}{2}S\left(e^2 - \frac{\partial u_i}{\partial x_i}\frac{\partial u_i}{\partial x_j}\right), \qquad \Delta U = \frac{1}{2}\frac{\partial^2 U}{\partial x_i\,\partial x_j}u_i u_j. \qquad (3.3)$$

It is seen that if Eqs. (3.1) are verified the following variational relation is also valid:

$$\delta\mathcal{P}_\tau + \iiint \rho a_i\, \delta u_i\, d\tau = \iint_A \left[(s + Se)n_i - S\frac{\partial u_i}{\partial x_i}n_j\right]\delta u_i\, dA. \qquad (3.4)$$

The surface A denotes the boundary of the volume τ of fluid and n_i is the outward unit normal to this boundary.

The variational equation (3.4) corresponds to the dynamical equations (3.1). We shall therefore refer to it as the *unmodified variational principle*. It will be shown[5] that in this form it is a particular case of a more general variational principle of the theory of elasticity of an initially stressed continuum.

We note an interesting physical interpretation of the quantity $\mathcal{R}$. It is the product of the initial stress S by a factor which represents the second-order volume increment as can be shown by expanding the Jacobian.

The surface integral on the right-hand side of Eq. (3.4) may be simplified by assuming displacements u_i tangent to the solid boundary at the initial point. Hence at the solid boundary we must satisfy the constraints (2.9). On the other hand, at the free surface $s = S = 0$. Under these conditions the surface integral in the variational principle (3.4) becomes

$$\iint_A \left[(s + Se)n_i - S\frac{\partial u_i}{\partial x_i}n_j\right]\delta u_i\, dA = -\iint_B S\frac{\partial u_j}{\partial x_i}n_j\, \delta u_i\, dA. \qquad (3.5)$$

The surface integral is now restricted to the rigid boundary B.

[5] M. A. Biot, Phys. Fluids **6**, 778 (1963).

Potential Energy of a Curved Rigid Boundary

As in the previous case, it is possible to incorporate the surface integral into an over-all potential energy of the fluid. We will show that the surface integral (3.5) may be expressed as an exact differential by taking into account the boundary constraints (2.9). Consider a function $F(x_1x_2x_3)$ such that the rigid boundary is defined by the equation

$$F(x_1x_2x_3) = 0. \qquad (3.6)$$

Putting

$$\phi = \pm\left[\left(\frac{\partial F}{\partial x_1}\right)^2 + \left(\frac{\partial F}{\partial x_2}\right)^2 + \left(\frac{\partial F}{\partial x_3}\right)^2\right]^{-\frac{1}{2}}, \qquad (3.7)$$

the unit normal vector is written

$$n_i = \phi\, \partial F/\partial x_i. \qquad (3.8)$$

The $\pm$ sign is chosen to correspond to the outward direction of n_i. Inserting this expression of n_i into the constraints (2.9) we find that the relation

$$\phi(\partial F/\partial x_j)u_j = 0 \qquad (3.9)$$

must be verified on the rigid boundary. Equation (3.9) also implies

$$(\partial F/\partial x_j)u_j = 0 \qquad (3.10)$$

on the same boundary.

If we consider $\phi(\partial F/\partial x_j)u_j$ as a function of the coordinates, Eq. (3.9) further implies that its gradient is normal to the rigid boundary. In view of the second condition (2.9), this is expressed by the relation

$$\delta u_i\frac{\partial}{\partial x_i}\left(\phi\frac{\partial F}{\partial x_j}u_j\right) = 0 \qquad (3.11)$$

or

$$\delta u_i\left(\frac{\partial\phi}{\partial x_i}\frac{\partial F}{\partial x_j}u_j + \phi\frac{\partial^2 F}{\partial x_i\,\partial x_j}u_j + \phi\frac{\partial F}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right) = 0. \qquad (3.12)$$

By Eqs. (3.8) and (3.10), this simplifies to

$$n_j\frac{\partial u_j}{\partial x_i}\delta u_i = -\phi\frac{\partial^2 F}{\partial x_i\,\partial x_j}u_j\,\delta u_i. \qquad (3.13)$$

The right-hand side of this equation is now an exact differential, i.e.,

$$n_j\frac{\partial u_j}{\partial x_i}\delta u_i = -\tfrac{1}{2}\delta\left(\phi\frac{\partial^2 F}{\partial x_i\,\partial x_j}u_i u_j\right), \qquad (3.14)$$

and the surface integral (3.5) becomes

$$-\iint_B S\frac{\partial u_i}{\partial x_i}n_j\ \delta u_i\ dA = \tfrac{1}{2}\delta\iint_B S\phi\frac{\partial^2 F}{\partial x_i\,\partial x_j}u_iu_j\ dA. \tag{3.15}$$

We may now introduce a surface potential energy of the rigid boundary.

$$\mathcal{P}_B = -\frac{1}{2}\iint_B S\phi\frac{\partial^2 F}{\partial x_i\,\partial x_j}u_iu_j\ dA. \tag{3.16}$$

By using relations (3.5) and (3.15) the variational principle (3.4) is written in the form

$$\delta(\mathcal{P}_\tau + \mathcal{P}_B) + \iiint_\tau \rho a_i\ \delta u_i\ d\tau = 0. \tag{3.17}$$

Hence we have again represented the total potential energy $\mathcal{P}$ as the sum of two terms

$$\mathcal{P} = \mathcal{P}_\tau + \mathcal{P}_B, \tag{3.18}$$

where $\mathcal{P}_\tau$ is the potential energy stored in the volume while $\mathcal{P}_B$ is the surface energy stored at the curved boundary. The reason for the contribution $\mathcal{P}_B$ to the total potential energy is due to the fact that the boundary displacement is assumed to be tangent to the boundary. As already mentioned above, this will violate the actual boundary constraint except if it is a plane surface. The discrepancy is of the second order and the initial normal stress contributes a second-order energy term represented by $\mathcal{P}_B$. It is seen from Eq. (3.16) that $\mathcal{P}_B = 0$ if F is a linear function of the coordinates, i.e., if the rigid boundary is a plane surface.

It can be verified that the surface integral in the variational equation (3.4) can also be made to vanish if the displaced point is forced to remain on the rigid boundary. However, this would introduce a nonlinear boundary condition.

IV. EQUIVALENCE OF THE TWO VARIATIONAL PRINCIPLES

In the preceding sections we have derived two essentially different forms of the variational principle. The modified form (2.7) corresponds to the dynamical equations (2.1). The unmodified form (3.4) corresponds to the dynamical equations (3.1).

That these two principles are rigorously equivalent may be shown as follows.

Consider the identity

$$\frac{\partial}{\partial x_i}(Se\ \delta u_i) - \frac{\partial}{\partial x_j}\left(S\frac{\partial u_j}{\partial x_i}\ \delta u_i\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial S}{\partial x_j}u_j\ \delta u_i\right) = \frac{\partial S}{\partial x_i}\ \delta(eu_i) + \tfrac{1}{2}S\ \delta\left(e^2 - \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}\right) + \frac{1}{2}\frac{\partial^2 S}{\partial x_i\,\partial x_j}\ \delta(u_iu_j). \tag{4.1}$$

In this identity S is an arbitrary function of the coordinates and δu_i an arbitrary variation. If we identify S with the initial stress in the fluid we may use the initial equilibrium conditions, i.e.,

$$\frac{\partial S}{\partial x_i} = -\rho X_i,\qquad \frac{\partial^2 S}{\partial x_i\,\partial x_j} = \rho\frac{\partial^2 U}{\partial x_i\,\partial x_j} - X_i\frac{\partial \rho}{\partial x_j}. \tag{4.2}$$

By introducing these expressions into the identity (4.1) it becomes

$$\frac{\partial}{\partial x_i}(\rho X_ju_j\ \delta u_i) - \delta Y = \frac{\partial}{\partial x_i}(Se\ \delta u_i) - \frac{\partial}{\partial x_j}\left(S\frac{\partial u_j}{\partial x_i}\ \delta u_i\right) - \delta(\mathcal{R} + \rho\ \Delta U). \tag{4.3}$$

We have put

$$Y = \rho X_ju_je + \tfrac{1}{2}X_j(\partial\rho/\partial x_i)u_iu_j, \tag{4.4}$$

while $\mathcal{R}$ and ΔU are expressions (3.3). If we add

$$(\partial/\partial x_i)(s\ \delta u_i) - \tfrac{1}{2}\ \delta(se) - \rho a_i\ \delta u_i \tag{4.5}$$

to both sides of Eq. (4.3) and integrate throughout the volume τ of the fluid, it becomes

$$\iint_A (s + \rho X_ju_j)n_i\ \delta u_i\ dA - \delta\mathcal{W}_\tau - \iiint_\tau \rho a_i\ \delta u_i = \iint_A\left[(s + Se)n_i - S\frac{\partial u_j}{\partial x_i}n_j\right]\delta u_i\ dA - \delta\mathcal{P}_\tau - \iiint_\tau \rho a_i\ \delta u_i. \tag{4.6}$$

Putting one side or the other of this equation equal to zero yields one of the two variational principles (2.7) or (3.4). Hence they are equivalent.

Note a fundamental difference in the potential energy for these two principles. In the modified form as shown by the expression of $\mathcal{W}_\tau$ the state of initial stress is taken into account through the body force X_i alone. On the other hand, in the unmodified form the potential energy contains the initial stress S itself and a term ΔU which depends on the variation of the body force field. The latter term does not appear in the modified equations.

V. LAGRANGIAN EQUATIONS AND HAMILTON'S PRINCIPLE

The variational principles are written as a single equation

$$\delta\mathcal{P} + \iiint_\tau \rho a_i \, \delta u_i \, d\tau = 0. \tag{5.1}$$

When the potential energy $\mathcal{P}$ is expressed by Eq. (2.15) this yields the modified variational principle (2.14). When the value (3.18) is used for $\mathcal{P}$ we obtain the unmodified variational principle (3.17).

In any case, $\mathcal{P}$ is a homogeneous quadratic function of the displacements expressed in various ways. When there is no Coriolis term we may introduce the invariant

$$T = \frac{1}{2} \iiint_\tau \rho u_i u_i \, d\tau. \tag{5.2}$$

The variational principle (5.1) then becomes

$$\delta(\mathcal{P} + p^2 T) = 0, \tag{5.3}$$

where

$$p = \partial/\partial t \tag{5.4}$$

is a time differential operator treated as an algebraic quantity. By introducing the kinetic energy

$$\mathfrak{J} = \frac{1}{2} \iiint_\tau \rho \dot{u}_i \dot{u}_i \, d\tau \tag{5.5}$$

and writing

$$\delta \int_0^t dt(\mathfrak{J} - \mathcal{P}) = 0, \tag{5.6}$$

the variational equation becomes a particular case of Hamilton's principle.

Lagrange's equations with generalized coordinates q_i are obtained from these principles by expressing the displacements as linear combinations of fixed configuration fields $U_{ij}(x_1 x_2 x_3)$, i.e.,

$$u_i = U_{ij} q_j. \tag{5.7}$$

Substituting in the expression for $\mathcal{P}$ and T we find

$$\mathcal{P} = \tfrac{1}{2} a_{ij} q_i q_j, \qquad T = \tfrac{1}{2} m_{ij} q_i q_j. \tag{5.8}$$

The variational principle (5.3) yields the Lagrangian equations

$$\partial\mathcal{P}/\partial q_i + p^2 \, \partial T/\partial q_i = 0. \tag{5.9}$$

They are the same as the classical equations for the mechanics of a conservative system. The natural oscillations are normal coordinates.

Formulation of Wave Propagation by Normal Coordinates

The natural oscillations derived from the Lagrangian equations (5.9) are normal coordinates of the fluid medium. As shown earlier,[3] these normal coordinates may be used in a new method of evaluating the transient propagation in the medium under pulse excitation. The method becomes readily applicable for acoustic-gravity and internal gravity waves by inserting in the Lagrangian equations the value of the potential energy $\mathcal{P}$ derived above. The important point here is the fact that this potential energy must include the surface integral represented by the terms $\mathcal{W}_F$ or $\mathcal{P}_B$.

Equivalence of Group Velocity and Energy Transport

The variational principle for the theory of elasticity under initial stress was used by the writer to derive general theorems on the equivalence of group velocity and energy flux for any type of waveguide system.[4] The theorem was also derived for electromagnetic waveguides. The theorem is valid of course for a fluid under initial stress by considering that it is the limiting case of a solid when we assume that all elastic moduli vanish except the bulk modulus. It may also be derived directly in the present case. The proof depends essentially on the possibility of expressing the dynamical equations in the variational-Lagrangian form (5.3) and follows exactly the procedure of reference 4.

VI. CONSTANT GRAVITY FIELD

We shall consider the case of a uniform gravity field of acceleration g. With a vertical z axis directed positively upward, the body force field is represented by the components

$$X_i = (0, 0, -g). \tag{6.1}$$

The modified variational principle (2.14) and (5.3) is

$$\delta(\mathcal{W}_\tau + \mathcal{W}_F) + p^2 \, \delta T = 0, \tag{6.2}$$

where

$$\mathcal{W}_\tau = \iiint_\tau \left(\tfrac{1}{2}\lambda e^2 - \rho g e w - \tfrac{1}{2} g \frac{d\rho}{dz} w^2 \right) d\tau,$$

$$\mathcal{W}_F = \tfrac{1}{2} g \iint_F \rho w^2 \, dA, \tag{6.3}$$

$$T = \frac{1}{2} \iiint_\tau \rho(u^2 + v^2 + w^2) \, d\tau$$

(u, v, w, are displacements). An extensive discussion of the properties of acoustic-gravity waves based on the writers fluid-displacement equations is given by Tolstoy in a simultaneous paper.[6] He also shows for the particular case of a constant gravity field how the dynamical equations may be derived from a Lagrangian density.

[6] I. Tolstoy, Rev. Mod. Phys. 35, 207 (1963).

The variational principle (6.2) yields directly the differential equations (5.2) of reference 1 for the case of a constant gravity.

In particular consider a liquid of constant depth h. Putting $e = 0$ the value of $\mathcal{W}_\tau$ becomes

$$\mathcal{W}_\tau = -\tfrac{1}{2}g \iiint_\tau \frac{d\rho}{dz} w^2 \, d\tau. \tag{6.4}$$

Consider a wave sinusoidal along x with plane motion in the xy plane. The displacements are expressed as

$$\begin{aligned} u &= \frac{df}{dz} \cos kx, \\ w &= kf \sin kx, \\ v &= 0. \end{aligned} \tag{6.5}$$

The time factor exp $(i\omega t)$ is omitted and we put $p = i\omega$. These displacements satisfy the constraint of incompressibility with an arbitrary function $f(z)$.

The variational principle (6.2) becomes

$$\delta\left[\int_0^h \frac{\rho}{V^2}(\omega^2 - \omega_c^2)f^2 \, dz + \int_0^h \rho\left(\frac{df}{dz}\right)^2 dz\right] - \delta\left(\frac{g\rho f^2}{V^2}\right)_{z=h} = 0 \tag{6.6}$$

with

$$\omega_c^2 = -g\frac{1}{\rho}\frac{d\rho}{dz} \qquad V = \frac{\omega}{k}. \tag{6.7}$$

The variational principle (6.6) yields for the unknown f the Sturm–Liouville equation and the boundary condition already derived for this problem as shown by Eqs. (5.8) and (5.10) of reference 1.

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