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Acoustic-Gravity Waves as a Particular Case of the Theory of Elasticity under Initial Stress

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General equations for acoustic-gravity waves in a fluid are derived as a particular case of the theory of elasticity of initially stressed continua. The differential equations for the fluid dynamics and the corresponding variational principles are obtained from the more general results for the elastic solid established earlier by the writer. The transition from solid to fluid is illustrated for the special case of a constant gravity field. The dynamics of a fluid under initial stress is thus brought within the scope of the theory of elasticity providing a unified treatment of wave propagation in composite fluid–solid systems.

I. INTRODUCTION

DYNAMICAL equations and corresponding variational principles for acoustic-gravity waves have been derived and discussed in two preceding papers[1,2] in terms of the fluid displacement field. It is now shown that these results constitute a particular case of the theory of elasticity of an initially stressed continuum.[3,4]

There are many advantages in considering the problem of acoustic gravity waves in a fluid from this viewpoint. As we have seen, use of the displacement field leads directly to the expression of the potential energy and to the corresponding variational principles. In addition, a unified theory renders possible the treatment of propagation in coupled fluids and solids as a single system.

In Sec. II the general equations for the dynamics of an elastic solid under initial stress are briefly recalled.[3] By inserting isotropic components of stress in these equations we obtain the unmodified equations for acoustic-gravity waves which were previously derived.[1]

[1] M. A. Biot, Phys. Fluids **6**, 621 (1963).
[2] M. A. Biot, Phys. Fluids **6**, 772 (1963).
[3] M. A. Biot, J. Appl. Phys. 11, 522 (1940).
[4] M. A. Biot, Phil. Mag. **27**, 468 (1939).

The variational principle for an elastic solid under initial stress was derived by the writer in 1939.[4] In Sec. III it is applied to the case of a fluid by inserting isotropic components for the stress. This yields the unmodified variational principle for acoustic-gravity waves which was derived by a direct method in the previous paper.[2] The modified principle also follows as a rigorous consequence since it was shown in reference 2 that the two principles are mathematically equivalent.

In Sec. IV equations are written for elastic waves in an isotropic solid in a constant gravity field with initial hydrostatic stress. With zero value of the shear modulus these equations become those of acoustic-gravity waves in a fluid.

II. DYNAMICAL EQUATIONS

In a paper[3] dealing with the propagation of elastic waves in a solid under initial stress the following results were derived. Initially the solid is in equilibrium in a state of initial stress S_{ij}. The external forces acting on the medium are the body force field X_i and certain boundary forces f_i. The coordinates are x_i. In a small perturbation of this state the particle coordinates become

$$\xi_i = x_i + u_i \tag{2.1}$$

and u_i represents the displacement field. The strain is

$$e_{ij} = \tfrac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i). \tag{2.2}$$

The local rotation of the medium is

$$\omega_{ij} = \tfrac{1}{2}(\partial u_i/\partial x_j - \partial u_j/\partial x_i). \tag{2.3}$$

The incremental stresses due to the perturbation are denoted by s_{ij} and are referred to axes which have been subject to a solid rotation ω_{ij}. They are linearly related to the strain components e_{ij} through suitable elastic coefficients. The dynamical equations for this medium as derived by the writer[3] are

$$\frac{\partial s_{ij}}{\partial x_j} + \rho\,\Delta X_i - \rho\omega_{ik}X_k - \rho e X_i + S_{jk}\frac{\partial \omega_{ik}}{\partial x_j} + S_{ik}\frac{\partial \omega_{jk}}{\partial x_j} - e_{jk}\frac{\partial S_{ik}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2}. \tag{2.4}$$

The initial density distribution is denoted by ρ and

$$\Delta X_i = X_i(\xi) - X_i(x) \tag{2.5}$$

represents the increment of body force on a particle due to its displacement. The volume dilatation is

$$e = \partial u_i/\partial x_i. \tag{2.6}$$

For a fluid the stresses are isotropic. We write

$$s_{ij} = s\delta_{ij}, \qquad S_{ij} = S\delta_{ij}, \tag{2.7}$$

where

$$\delta_{ij} = \begin{cases} 0 & (i \neq j), \\ 1 & (i = j). \end{cases} \tag{2.8}$$

By inserting these values into Eqs. (2.4) we derive

$$\frac{\partial s}{\partial x_i} - \rho e X_i - \rho\omega_{ij}X_j - e_{ij}\frac{\partial S}{\partial x_j} + \rho\Delta X_i = \rho\frac{\partial^2 u_i}{\partial t^2}. \tag{2.9}$$

The condition of initial equilibrium of the fluid is

$$\partial S/\partial x_i + \rho X_i = 0. \tag{2.10}$$

By taking this relation into account, Eq. (2.9) becomes

$$\frac{\partial s}{\partial x_i} + e\frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i}\frac{\partial S}{\partial x_j} + \rho\Delta X_i = \rho\frac{\partial^2 u_i}{\partial t^2}. \tag{2.11}$$

Introducing a body force derived from a potential U and linearizing ΔX_i we write

$$X_i = -\partial U/\partial x_i$$
$$\Delta X_i = -\left(\frac{\partial^2 U}{\partial x_i\,\partial x_j}\right)u_j. \tag{2.12}$$

With this value of ΔX_i, Eqs. (2.11) become

$$\frac{\partial s}{\partial x_i} + e\frac{\partial S}{\partial x_i} - \frac{\partial u_j}{\partial x_i}\frac{\partial S}{\partial x_j} - \rho\frac{\partial^2 U}{\partial x_i\,\partial x_j}u_j = \rho\frac{\partial^2 u_i}{\partial t^2}. \tag{2.13}$$

Equations (2.11) and (2.13) are identical with Eqs. (2.24) and (3.4) derived in a previous paper[1] by a more direct method for the dynamics of a fluid under initial stress.

III. VARIATIONAL PRINCIPLE

In the theory of elasticity under initial stress developed by the writer,[3] the following variational principle was established for the static case.

$$\delta\iiint_\tau \Delta V\,d\tau = \iiint_\tau \Delta X_i\rho\delta u_i d\tau + \iint_A \Delta f_i\,\delta u_i\,dA, \tag{3.1}$$

where A is the boundary of the volume τ and

$$\Delta V = \tfrac{1}{2}t_{ij}e_{ij} + \tfrac{1}{2}S_{ij}(e_{ik}\omega_{kj} + e_{jk}\omega_{ki} + \omega_{ik}\omega_{jk}),$$
$$t_{ij} = s_{ij} + S_{ij}e - \tfrac{1}{2}(S_{ik}e_{jk} + S_{jk}e_{ik}). \tag{3.2}$$

The incremental boundary force is

$$\Delta f_i = (s_{ij} + S_{kj}\omega_{ik} + S_{ij}e - S_{ik}e_{jk})n_j, \tag{3.3}$$

where n_j is the unit outward normal to the boundary.

Validity of the principle (3.1) for dynamics is implicit from d'Alemberts principle by including the inertia force into the body force. This is obtained by replacing ΔX_i by $\Delta X_i - a_i$, where a_i is the particle acceleration. The variational principle (3.1) becomes

$$\delta \iiint_\tau \Delta V \, d\tau = \iiint_\tau (\Delta X_i - a_i)\rho \, \delta u_i \, d\tau + \iint_A \Delta f_i \, \delta u_i \, dA. \tag{3.4}$$

Assuming a body force potential U according to Eq. (2.12) we may write

$$\Delta X_i \, \delta u_i = -\delta \Delta U, \tag{3.5}$$

with

$$\Delta U = \frac{1}{2}\left(\frac{\partial^2 U}{\partial x_i \, \partial x_j}\right) u_i u_j.$$

Hence Eq. (3.4) takes the form

$$\delta \iiint_\tau (\Delta V + \rho \Delta U) \, d\tau + \iiint_\tau \rho a_i \, \delta u_i \, d\tau = \iint_A \Delta f_i \, \delta u_i \, dA. \tag{3.6}$$

Let us now introduce the isotropic fluid stresses (2.7) into the values of ΔV and Δf_i. We find

$$\Delta V = \tfrac{1}{2} se + \mathcal{R},$$

$$\mathcal{R} = \tfrac{1}{2} S\left(e^2 - \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right), \tag{3.7}$$

$$\Delta f_i = (s + Se)n_i - S \frac{\partial u_j}{\partial x_i} n_j.$$

The variational principle (3.6) becomes

$$\delta \iiint_\tau (\tfrac{1}{2} se + \mathcal{R} + \rho \Delta U) \, d\tau + \iiint_\tau \rho a_i \, \delta u_i \, d\tau = \iint_A \left[(s + Se)n_i + S \frac{\partial u_j}{\partial x_i} n_j\right] \delta u_i \, dA.$$

This result is identical with the unmodified variational principle expressed by Eq. (3.4) of the previous paper.[2]

IV. CONSTANT GRAVITY FIELD

Consider an elastic solid initially in equilibrium in a constant gravity field

$$X_i = (0, 0, -g). \tag{4.1}$$

The z axis is directed vertically upward.

Let us assume that the state of initial stress is hydrostatic. Hence

$$\partial S/\partial x = \partial S/\partial y = 0, \qquad \partial S/\partial z = \rho g. \tag{4.2}$$

The density ρ is a function of z.

We also assume isotropic stress–strain relations for the incremental stresses

$$s_{ij} = 2\mu e_{ij} + \delta_{ij}\lambda e. \tag{4.3}$$

The dynamical equations (2.4) become

$$\begin{aligned}
\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} + \frac{\partial s_{13}}{\partial z} - \rho g \frac{\partial w}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2}, \\
\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + \frac{\partial s_{23}}{\partial z} - \rho g \frac{\partial w}{\partial y} &= \rho \frac{\partial^2 v}{\partial t^2}, \\
\frac{\partial s_{31}}{\partial x} + \frac{\partial s_{32}}{\partial y} + \frac{\partial s_{33}}{\partial z} - \rho g \frac{\partial w}{\partial z} + \rho g e &= \rho \frac{\partial^2 w}{\partial t^2}.
\end{aligned} \tag{4.4}$$

(displacements are u, v, w). These equations were derived by the writer in 1940.[3] By putting the rigidity equal to zero ($\mu = 0$) we obtain the dynamical equations for a fluid. They coincide with Eqs. (5.2) discussed in a previous paper.[1]

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