# *Theory of Stability and Consolidation of a Porous Medium Under Initial Stress*

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# *Theory of Stability and Consolidation of a Porous Medium Under Initial Stress*[1]

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**Abstract.** Fundamental equations are derived for the mechanics of a fluid-filled porous medium under initial stress. The theory takes into account elastic and viscoelastic properties, including the most general case of anisotropy. It includes the theory of stability, and of acoustic propagation under initial stress, and by thermodynamic analogy the dynamics of a thermoelastic continuum under initial stress. Equations are also developed for a medium which is isotropic in finite strain. General variational principles are derived by which problems are easily formulated in curvilinear coordinates or in Lagrangian form by using generalized coordinates. It is shown that the variational principles are a direct consequence of the general equations of the thermodynamics of irreversible processes, and lead to real characteristic roots for instability.

**1. Introduction.** In the mechanics of porous media which has been developed to date the effect of the initial stress has not been introduced into the basic equations. The generalization of the theory to include this effect is of considerable interest in many applications. In civil engineering and geophysics the problems of consolidation and tectonics involve earth masses which are initially under high initial stress. In problems of foundation engineering the influence of the initial stress appears in a buoyancy effect which is actually used in design procedures and amounts to "floating" a building on its foundation. Since earth masses are generally porous and fluid-saturated, a consolidation theory taking into account this initial stress is obviously needed. On the other hand, problems of tectonic folding in geology are to a large extent problems of stability of porous media under initial stress. This requires an extension of the stability theory of continua to porous media. The theory presented in this paper is essentially the mechanics of fluid saturated porous media initially in equilibrium in a stressed condition and subjected to small perturbations. However, by a trivial limiting process where the variables become infinitesimals the theory leads also to instantaneous time rate equations valid for finite deformations.

The consolidation theory [1] [10] with suitable adaptations is combined with

[1] This work was sponsored by the Shell Development Co.

the mechanics of continua under initial stress [2], [3], [4], [5], [6]. That such a generalization is readily possible was already pointed out many years ago by the writer in the original paper on consolidation [1].

The implications of the theory extend beyond the scope of stability and the mechanics of porous media. This is outlined briefly in the Appendix. The results are immediately applicable to acoustic propagation in a porous medium under initial stress including problems of seismic propagation in porous rock and geological structures. By thermodynamic analogy the same equations also govern the dynamics and stability of a thermoelastic continuum under initial stress.

**2. Equilibrium equations for the incremental stress field.** A porous medium is assumed to be in equilibrium in a state of initial stress. The total initial stress in the bulk material is denoted by $S_{ij}$. With a body force $X_i$ per unit mass and a mass density $\rho$ for the bulk medium the equilibrium condition for the state of initial stress is

$$\frac{\partial S_{ij}}{\partial x_j} + \rho X_i = 0 \tag{2.1}$$

The pore contains a fluid of mass density $\rho_f$. In the initial equilibrium the fluid pressure $P$ in the pores must satisfy the condition

$$-\frac{\partial P}{\partial x_i} + \rho_f X_i = 0 \tag{2.2}$$

In the present section the properties of this fluid will not appear explicitly in the equations.

A first order perturbation of the equilibrium is now introduced. The coordinates $x_i$ of a point attached to the medium become

$$\xi_i = x_i + u_i \tag{2.3}$$

after deformation. The stress initially $S_{ij}$ becomes $\bar{\sigma}_{ij}$ at the displaced point. The strain is

$$e_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.4}$$

and the local rotation is

$$\omega_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{2.5}$$

A volume $V$ in the initial state becomes $V'$ after deformation and its surface $S$ becomes $S'$. The total force acting on $S'$ after deformation is

$$F_i = \iint_{S'} \bar{\sigma}_{ij} n'_j \, dS'. \tag{2.6}$$

The stress at the displaced point referred to fixed $x_i$ coordinates is $\bar{\sigma}_{ij}$ and $n'_i$ is the unit normal to $S'$. The surface integral may be transformed to the original coordinate system by the transformation rules of surface integrals. We put

$$A_{ij} = \bar{\sigma}_{ik} M_{kj}, \tag{2.7}$$

where

$$M_{ij} = (1 + e)\delta_{ij} - e_{ij} + \omega_{ij} \tag{2.8}$$

are the partial Jacobians of the coordinate transformation written by retaining only zero and first order terms. The dilatation is denoted by

$$e = \delta_{ij} e_{ij}. \tag{2.9}$$

The transformed integral is

$$F_i = \iint_S A_{ij} n_j \, dS, \tag{2.10}$$

where $n_j$ is the unit normal on the initial boundary.

Consider now the resultant of the body force acting on $V'$. Denote by $X'_i$ the body force field at the displaced point and by $\rho'$ the mass density of the bulk material at that point. The body force acting on $V'$ is

$$B_i = \iiint_{V'} X'_i \rho' \, dV'. \tag{2.11}$$

*This is transformed to an integral over the original volume $V$ by introducing the* Jacobian

$$J = \frac{d(\xi_i)}{d(x_i)} \tag{2.12}$$

we find

$$B_i = \iiint_V X'_i \rho' J \, dV. \tag{2.13}$$

The quantity $\rho' J$ is the mass of an element of bulk material originally of unit volume with a mass $\rho$. Hence we may write

$$\rho' J = \rho + \Delta\rho \tag{2.14}$$

where $\Delta\rho$ is the mass of fluid which has entered the initial unit volume through the pores.

Putting equal to zero the sum of the forces (2.10) and (2.13) acting on the volume $V'$ we derive

$$\iint_S A_{ij} n_j \, dS + \iiint_V (\rho + \Delta\rho) X'_i \, dV = 0. \tag{2.15}$$

Transforming the first term to a volume integral yields

$$\iiint_V \left[ \frac{\partial A_{ij}}{\partial x_j} + (\rho + \Delta\rho)X' \right] dV = 0. \tag{2.16}$$

Since $V$ is arbitrary we must have the equilibrium condition

$$\frac{\partial A_{ij}}{\partial x_j} + (\rho + \Delta\rho)X_i' = 0. \tag{2.17}$$

Following the procedure used in the writer's earlier work [2], [3], [4], [5], [6] we now introduce incremental stress components $s_{ij}$ referred to local axes obtained by giving the original coordinate axes a solid rotation defined by $\omega_{ij}$ . These stress components are related to $\bar{\sigma}_{ij}$ by the equations

$$\bar{\sigma}_{ij} = S_{ij} + s_{ij} + S_{kj}\omega_{ik} + S_{ik}\omega_{jk}. \tag{2.18}$$

Terms of order higher than the first have been dropped. Again with a first order approximation

$$A_{ij} = S_{ij} + s_{ij} + S_{ij}e + S_{kj}\omega_{ik} - S_{ik}e_{kj}. \tag{2.19}$$

We substitute this expression into the equilibrium equations (2, 17) taking into account the equilibrium condition (2.1) for the initial stress. This yields

$$\frac{\partial}{\partial x_j}(s_{ij} + S_{ij}e + S_{kj}\omega_{ik} - S_{ik}e_{kj}) + \rho\Delta X_i + X_i'\Delta\rho = 0, \tag{2.20}$$

where

$$\Delta X_i = X_i' - X_i. \tag{2.21}$$

This is the difference between the body force field at the displaced and initial points. To the first order we may express it as

$$\Delta X_i = \frac{\partial X_i}{\partial x_j} u_j. \tag{2.22}$$

For a uniform gravity field $\Delta X_i$ vanishes. To the first order we may also replace $X'$ by $X$ and write equation (2.20) as

$$\frac{\partial}{\partial x_j}(s_{ij} + S_{ij}e + S_{kj}\omega_{ik} - S_{ik}e_{kj}) + \rho\Delta X_i + X_i\Delta\rho = 0. \tag{2.23}$$

Except for the term $X_i\Delta\rho$ these equations are identical with those obtained previously for the continuum under initial stress [2], [3], [4], [5], [6]. We have shown that they may be written in an alternate form by using the equilibrium conditions (2.1) for the initial stress and well known differential relations between $e_{ij}$ and $\omega_{ij}$ . Equations (2.23) then take the form

$$\frac{\partial s_{ij}}{\partial x_j} + \rho\Delta X_i - \rho\omega_{ik}X_k - \rho e X_i + X_i\Delta\rho + S_{jk}\frac{\partial \omega_{ik}}{\partial x_j} + S_{ik}\frac{\partial \omega_{jk}}{\partial x_j} - e_{jk}\frac{\partial S_{ik}}{\partial x_j} = 0. \tag{2.24}$$

The boundary forces are obtained immediately from expression (2.10). The force *per unit initial area* acting on the deformed boundary is

$$f_i = A_{ij}n_j. \tag{2.25}$$

This may be written

$$f_i = S_{ij}n_j + \Delta f_i, \tag{2.26}$$

where $\Delta t_i$ is the incremental boundary force per unit initial area. Introducing the value (2.19) for $A_{ij}$ we derive

$$\Delta f_i = (s_{ij} + S_{ij}e + S_{kj}\omega_{ik} - S_{ik}e_{kj})n_j. \tag{2.27}$$

This equation is used to express the boundary conditions.

It should be noted that if the boundary is in contact with a fluid where the hydrostatic stress field is $S(x_i)$ the boundary condition is obtained by inserting into expression (2.27) the values

$$\begin{aligned} s_{ij} &= [S(\xi_i) - S(x_i)]\delta_{ij} \\ S_{ij} &= S\delta_{ij} \end{aligned} \tag{2.28}$$

The incremental boundary force is then

$$\Delta f_i = [S(\xi_i) - S(x_i)]n_i + \left(en_i - \frac{\partial u_j}{\partial x_i} n_j\right)S \tag{2.29}$$

The boundary condition is found by equating expressions (2.27) and (2.29). This equation is useful in problems where the boundaries of the porous medium are submerged.

**3. Strain and incremental stresses.** Consider a cube of unit size of the bulk material oriented along the original fixed axes and under initial stresses. Let us give the material an homogeneous deformation without rotation. After the deformation the cube becomes a parallelepiped. From equation (2.10) it is seen that the forces acting on the deformed faces of the element are determined by $A_{ij}$ after putting equal to zero the rotation $\omega_{ij}$. We denote this value of $A_{ij}$ by

$$A'_{ij} = S_{ij} + s_{ij} + S_{ij}e - S_{ik}e_{kj}. \tag{3.1}$$

As done in earlier papers [2], [3], [4] it is convenient to introduce a non-symmetric tensor

$$t'_{ij} = s_{ij} + S_{ij}e - S_{ik}e_{kj} \tag{3.2}$$

which represents incremental forces per unit initial area. Hence

$$A'_{ij} = S_{ij} + t'_{ij}. \tag{3.3}$$

During the deformation the fluid content in the pores also changes. We denote by $w_i$ a vector representing the volume of fluid which has flowed into the element through the face perpendicular to the $i^{th}$ coordinate axis. We denote by $p_f$ the increment of fluid pressure in the pores. We shall first assume that the initial fluid pressure and fluid density are uniform.

For isothermal deformations there is a strain energy represented by the isothermal free energy. This increment of free energy of the element is

$$dW = A'_{ij}\, de_{ij} + p_f\, d\zeta. \tag{3.4}$$

We have defined a fluid content variable as

$$\zeta = -\frac{\partial w_i}{\partial x_i}. \tag{3.5}$$

We also introduce the symmetric part of $t'_{ij}$ as

$$t_{ij} = \tfrac{1}{2}(t'_{ij} + t'_{ji}). \tag{3.6}$$

Hence

$$dW = S_{ij}\, de_{ij} + t_{ij}\, de_{ij} + p_f\, d\zeta. \tag{3.7}$$

This must be an exact differential. The incremental stresses $t_{ij}$ and $p_f$ are linear functions of $e_{ij}$ and $\zeta$. Hence

$$\begin{aligned} t_{ij} &= C_{ij}^{\mu\nu} e_{\mu\nu} + M_{ij}\zeta, \\ p_f &= M'_{ij} e_{ij} + M\zeta. \end{aligned} \tag{3.8}$$

Because of the symmetry of $t_{ij}$ and $e_{ij}$ we put

$$\begin{aligned} C_{ij}^{\mu\nu} &= C_{ji}^{\mu\nu} = C_{ij}^{\nu\mu}, \\ M_{ij} &= M_{ji}, \\ M'_{ij} &= M'_{ji}. \end{aligned} \tag{3.9}$$

Since $dW$ is an exact differential we must also satisfy the conditions

$$\begin{aligned} C_{ij}^{\mu\nu} &= C_{\mu\nu}^{ij}, \\ M_{ij} &= M'_{ij}. \end{aligned} \tag{3.10}$$

Hence equations (3.8) become

$$\begin{aligned} t_{ij} &= C_{ij}^{\mu\nu} e_{\mu\nu} + M_{ij}\zeta, \\ p_f &= M_{ij} e_{ij} + M\zeta. \end{aligned} \tag{3.11}$$

We may express these equations in terms of the stresses $s_{ij}$ by using relation

$$t_{ij} = s_{ij} + S_{ij}e - \tfrac{1}{2}(S_{ik}e_{kj} + S_{jk}e_{ki}) \tag{3.12}$$

obtained from equations (3.2) and (3.6). It may be written in the form

$$t_{ij} = s_{ij} + S_{ij}\delta_{\mu\nu}e_{\mu\nu} - \tfrac{1}{2}(S_{i\mu}\delta_{\nu j} + S_{j\mu}\delta_{i\nu})e_{\mu\nu}. \tag{3.13}$$

Since $e_{\mu\nu} = e_{\nu\mu}$ this may also be written

$$t_{ij} = s_{ij} + S_{ij}\delta_{\mu\nu}e_{\mu\nu} - D_{ij}^{\mu\nu}e_{\mu\nu} \tag{3.14}$$

with

$$D_{ij}^{\mu\nu} = \tfrac{1}{4}(S_{i\mu}\delta_{\nu j} + S_{j\mu}\delta_{i\nu} + S_{i\nu}\delta_{\mu j} + S_{j\nu}\delta_{i\mu}), \quad D_{ij}^{\mu\nu} = D_{\mu\nu}^{ij}. \tag{3.15}$$

We now substitute $t_{ij}$ into equations (3.8). This yields

$$s_{ij} = B_{ij}^{\mu\nu} e_{\mu\nu} + M_{ij}\zeta, \quad p_f = M_{ij}e_{ij} + M\zeta, \tag{3.16}$$

with

$$B_{ij}^{\mu\nu} = C_{ij}^{\mu\nu} - S_{ij}\delta_{\mu\nu} + D_{ij}^{\mu\nu}. \tag{3.17}$$

The matrix $B_{ij}^{\mu\nu}$ is not symmetric, and obeys the following relations,

$$B_{ij}^{\mu\nu} + S_{ij}\delta_{\mu\nu} = B_{\mu\nu}^{ij} + S_{\mu\nu}\delta_{ij}. \tag{3.18}$$

Hence in general

$$B_{ij}^{\mu\nu} \neq B_{\mu\nu}^{ij}. \tag{3.19}$$

This is the same property as already derived for the coefficients in the earlier theory for the continuum under initial stress [3], [4], [5].

In the derivation we have assumed the initial fluid pressure and density to be uniform throughout the element considered. When dealing with a medium under the action of a body force this is not the case and we must take into account the fact that the fluid density is not uniform. This becomes clear if we assume a steady state of flow for the fluid through the pores. The stresses remain constant in this case however, this is in contradiction with the stress-strain relations (3.16) because $\zeta$ in this case is not zero. However, this difficulty is easily taken care of by redefining $\zeta$ as

$$\zeta = -\frac{1}{\rho_f}\frac{\partial}{\partial x_i}(\rho_f w_i). \tag{3.20}$$

It coincides with (3.5) when $\rho_f$ is uniform.

As shown by the results obtained from the thermodynamic and variational principles discussed in section 6 below, it is also necessary to substitute another variable for the incremental pressure $p_f$. Assuming isothermal transformations we must consider the function

$$\psi = \int \frac{dP}{\rho_f}. \tag{3.21}$$

For a given fluid at a given temperature it depends only on the pressure $P$. The incremental value $\Delta\psi$ of $\psi$ at a given location must be considered and the expression $\rho_f\,\Delta\psi$ must be substituted for $p_f$. With the definitions (3.20) for $\zeta$ the stress-strain relations (3.11) and (3.16) become

$$t_{ij} = C_{ij}^{\mu\nu} e_{\mu\nu} + M_{ij}\zeta, \quad \rho_f\Delta\psi = M_{ij}e_{ij} + M\zeta \tag{3.22}$$

and

$$s_{ij} = B_{ij}^{\mu\nu} e_{\mu\nu} + M_{ij}\zeta, \qquad (3.23)$$
$$\rho_f \Delta\psi = M_{ij} e_{ij} + M\zeta.$$

As an example we shall consider the case of an orthotropic medium whose axes of elastic symmetry coincide with the coordinate axes. We assume that the principal directions of the initial stress also coincide with the same directions. The initial stresses are denoted by $S_{11}S_{22}S_{33}$. The stress-strain relations (3.16) take the particular form

$$\begin{aligned}
s_{11} &= B_{11}e_{xx} + B_{12}e_{yy} + B_{13}e_{zz} + M_1\zeta,\\
s_{22} &= B_{21}e_{xx} + B_{22}e_{yy} + B_{23}e_{zz} + M_2\zeta,\\
s_{33} &= B_{31}e_{xx} + B_{32}e_{yy} + B_{33}e_{zz} + M_3\zeta,\\
s_{23} &= 2Q_1 e_{yz},\\
s_{31} &= 2Q_2 e_{zx},\\
s_{12} &= 2Q_3 e_{xy},\\
\rho_f\Delta\psi &= M_1 e_{xx} + M_2 e_{yy} + M_3 e_{zz} + M\zeta.
\end{aligned} \qquad (3.24)$$

The matrix $B_{ij}$ is non-symmetric and relations (3.18) become

$$\begin{aligned}
B_{12} + S_{11} &= B_{21} + S_{22} = C_{12},\\
B_{23} + S_{22} &= B_{32} + S_{33} = C_{23},\\
B_{31} + S_{33} &= B_{13} + S_{11} = C_{31}.
\end{aligned} \qquad (3.25)$$

The stress system $t_{ij}$ is

$$\begin{aligned}
t_{11} &= s_{11} + S_{11}(e_{yy} + e_{zz}),\\
t_{22} &= s_{22} + S_{22}(e_{zz} + e_{xx}),\\
t_{33} &= s_{33} + S_{33}(e_{xx} + e_{yy}),\\
t_{23} &= s_{23},\\
t_{31} &= s_{31},\\
t_{12} &= s_{12}.
\end{aligned} \qquad (3.26)$$

With these stresses the first three equations (3.24) are replaced by

$$\begin{aligned}
t_{11} &= B_{11}e_{xx} + C_{12}e_{yy} + C_{31}e_{zz} + M_1\zeta,\\
t_{22} &= C_{12}e_{xx} + B_{22}e_{yy} + C_{23}e_{zz} + M_2\zeta,\\
t_{33} &= C_{31}e_{xx} + C_{23}e_{yy} + B_{33}e_{zz} + M_3\zeta.
\end{aligned} \qquad (3.27)$$

The matrix in this case is symmetric.

The medium may be isotropic in finite strain. This means that starting from the unstressed condition the finite stress-strain relations are the same for any orientation of the medium. In that case the anisotropy of the incremental properties is due to the initial stress. The medium become orthotropic with directions of symmetry along the principal directions of the initial stress. If these directions coincide with the coordinate axes the stress-strain relations have the same form as equations (3.24). However in this case it is possible to write explicitly the shear coefficients $Q_1$, $Q_2$, $Q_3$. If $\lambda_1$, $\lambda_2$, $\lambda_3$ are the extension ratios in the principal directions of the finite initial strain we may write

$$
\begin{aligned}
Q_1 &= \tfrac{1}{2}(S_{22} - S_{33}) \frac{\lambda_2^2 + \lambda_3^2}{\lambda_2^2 - \lambda_3^2}, \\
Q_2 &= \tfrac{1}{2}(S_{33} - S_{11}) \frac{\lambda_3^2 + \lambda_1^2}{\lambda_3^2 - \lambda_1^2}, \\
Q_3 &= \tfrac{1}{2}(S_{11} - S_{22}) \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2}.
\end{aligned}
\tag{3.28}
$$

These expressions where derived by the writer in earlier papers [7], [8] for the case of the elastic continuum isotropic in finite strain. The derivation is obviously valid also for the porous isotropic medium.

**4. Thermodynamics of Darcy's Law under initial stress.** In the previous sections we have considered the equilibrium equations of the stress field and the stress-strain properties under reversible quasi-static conditions. In order to complete the necessary set of equations we must now consider the irreversible process of fluid flow through the pores. For isotropic media the relation between the relative rate of flow of the pore fluid, the fluid pressure and the body forces is known as Darcy's law. A generalization of Darcy's law based on the thermodynamic of irreversible processes was given earlier by the writer [9], [10]. This result will now be extended to a medium under initial stress. Before analyzing the effect of the viscosity it is important to understand the simpler problems of a porous medium containing a frictionless fluid. The same fluid is assumed to occupy the pores at all points. However, the pressures and densities of this fluid may vary from point to point. On the other hand the system is assumed to be in thermal equilibrium, the temperature is uniform throughout, and remains constant during the deformation process. Hence we are dealing with isothermal transformations such that the fluid density $\rho_f(\xi)$ is a function only of the fluid pressure $P(\xi)$ at the same point. We shall use the function

$$\psi = \int \frac{dP}{\rho_f}, \tag{4.1}$$

which depends only on the fluid pressure.

As shown by Hubbert [11] the following potential function is fundamental in expressing the mechanics of flow in porous media,

$$\phi = \psi + U. \tag{4.2}$$

The body force potential $v$ is a function of the coordinates, and the body force field per unit mass is given by

$$X_i = -\frac{\partial U}{\partial x_i}. \tag{4.3}$$

The condition of equilibrium of the fluid is expressed by the equation

$$\phi = \text{Constant}. \tag{4.4}$$

It is readily verified, by taking the gradient of $\phi$, that

$$\frac{\partial \phi}{\partial \xi_i} = \frac{1}{\rho_f} \frac{\partial P_f}{\partial \xi_i} - X_i = 0. \tag{4.5}$$

This is the equation for static equilibrium of the fluid. Let us assume now that the fluid is not in equilibrium. The departure from equilibrium may be measured by a "disequilibrium force" per unit volume defined as

$$\rho_f X_i' = -\rho_f \frac{\partial \phi}{\partial \xi_i}. \tag{4.6}$$

If instead of a perfect fluid we are dealing with a viscous fluid of viscosity $\eta$ the disequilibrium force produces a fluid motion through the pores and brings into the picture a dissipation function which may be written,

$$D = \tfrac{1}{2}\eta r_{ij} \dot{w}_i \dot{w}_j. \tag{4.7}$$

Time derivatives of $w_i$ at the point $\xi_i$ are denoted by $\dot{w}_i$ . Principles of non-equilibrium thermodynamics for perturbations around an equilibrium state lead to the equation

$$\rho_f X_i' = \frac{\partial D}{\partial \dot{w}_i} \tag{4.8}$$

or

$$-\rho_f \frac{\partial \phi}{\partial \xi_i} = \eta r_{ij} \dot{w}_j, \tag{4.9}$$

The reciprocity property of the coefficient,

$$r_{ij} = r_{ji}, \tag{4.10}$$

is a consequence of Onsager's principle [12], [13]. Introducing the inverse matrix of elements $k_{ij}$

$$[k_{ij}] = [r_{ij}]^{-1} \tag{4.11}$$

equation (4.9) becomes

$$\dot{w}_i = -\frac{\rho_f}{\eta} k_{ij} \frac{\partial \phi}{\partial \xi_j}. \tag{4.12}$$

This equation is formally the same as that obtained earlier. For an isotropic medium, where $k_{ij} = k\delta_{ij}$, it coincides with the result of Hubbert [11].

The equations written here are expressed in terms of the final coordinate $\xi_i$. After incremental deformations and all other values in the equations are at the displaced point $\xi_i$. For our purpose however we must introduce the initial coordinates $x_i$.

Let us consider the equilibrium value of $\phi$ which is put equal to a constant $\phi_0$:

$$\phi(\xi) = \psi(\xi) + U(\xi) = \phi_0. \tag{4.13}$$

Since $\xi$ may be replaced by $x$ in this equation we may also write

$$\psi(x) + U(x) = \phi_0. \tag{4.14}$$

On the other hand for nonequilibrium the value of $\phi$ at point $\xi$ is

$$\phi'(\xi) = \psi'(\xi) + U(\xi). \tag{4.15}$$

The difference is

$$\phi'(\xi) - \phi_0 = \Delta\psi + \Delta U, \tag{4.16}$$

where

$$\begin{aligned} \Delta\psi &= \psi'(\xi) - \psi(x), \\ \Delta U &= V(\xi) - U(x). \end{aligned} \tag{4.17}$$

The quantity $\Delta\psi$ is the same as that defined in the stress-strain relations (3.23). It represents the *increment of $\psi$ at a point attached to the solid.* Substituting the value (4.16) of $\phi'(\xi)$ instead of $\phi$ into equation (4.9) we find

$$-\rho_f \frac{\partial}{\partial \xi_i}(\Delta\psi + \Delta U) = \eta_{ij}\dot{w}_j. \tag{4.18}$$

The derivatives are expressed by means of the coordinates $\xi_i$ after deformation of the porous medium. For our purpose we must express them in terms of the initial coordinates $x_i$. Therefore we write

$$\frac{\partial}{\partial \xi_i}(\Delta\psi + \Delta U) = \frac{\partial x_j}{\partial \xi_i}\frac{\partial}{\partial x_j}(\Delta\psi + \Delta U). \tag{4.19}$$

Since $\Delta\psi + \Delta U$ is of the first order we put

$$\frac{\partial x_j}{\partial \xi_i} = \delta_{ij} \tag{4.20}$$

and to the same order we may write

$$\frac{\partial}{\partial \xi_i}(\Delta\psi + \Delta U) = \frac{\partial}{\partial x_i}(\Delta\psi + \Delta U). \tag{4.21}$$

Hence equation (4.18) becomes

$$-\rho_f \frac{\partial}{\partial x_i}(\Delta\psi + \Delta U) = \eta r_{ij} w_j. \tag{4.22}$$

We may further linearize $\Delta V$ by first order Taylor expansion,

$$\Delta V = \frac{\partial V}{\partial x_j} u_j = -X_j u_j. \tag{4.23}$$

Inserting this expression into equation (4.22) yields

$$-\rho_f \frac{\partial}{\partial x_i} (\Delta\psi - X_j u_j) = \eta r_{ij} \dot{w}_j. \tag{4.24}$$

We note that for a constant fluid density the equation becomes

$$-\frac{\partial}{\partial x_i} (p_f - \rho_f X_j u_j) = \eta r_{ij} \dot{w}_j. \tag{4.25}$$

The quantity in the bracket represents the excess pressure over the hydrostatic equilibrium value at the displaced point.

**5. General field equations.** The field equations for the mechanics of a porous medium under initial stress are obtained by combining the preceding results. Since $\Delta\rho$ is the mass of fluid which has entered through the pores into an element initially of unit volume we may write

$$\Delta\rho = -\frac{\partial}{\partial x_i} (\rho_f w_i). \tag{5.1}$$

From the definition (3.20) of $\zeta$ this becomes

$$\Delta\rho = \rho_f \zeta. \tag{5.2}$$

With this value inserted in the equation we write the equilibrium condition (2.23) and the generalized Darcy relation (4.24)

$$\begin{aligned} &\frac{\partial}{\partial x_j} (s_{ij} + S_{ij} e + S_{kj} \omega_{ik} - S_{ik} e_{kj}) + \rho \Delta X_i + \rho_f X_i \zeta = 0, \\ &-\rho_f \frac{\partial}{\partial x_i} (\Delta\psi - X_j u_j) = \eta r_{ij} \dot{w}_j. \end{aligned} \tag{5.3}$$

In addition we need the stress-strain relations (3.23)

$$\begin{aligned} s_{ij} &= B_{ij}^{\mu\nu} e_{\mu\nu} + M_{ij} \zeta, \\ \rho_f \Delta\psi &= M_{ij} e_{ij} + M\zeta. \end{aligned} \tag{5.4}$$

Substituting these values of $s_{ij}$ and $\Delta\psi$ into equations (5.3) we derive six field equations for the displacements $u_i$ and $w_i$ ,

$$\begin{aligned} &\frac{\partial}{\partial x_j} (B_{ij}^{\mu\nu} e_{\mu\nu} + M_{ij} \zeta) + \frac{\partial}{\partial x_j} (S_{ij} e + S_{kj} \omega_{ik} - S_{ik} e_{kj}) + \rho \Delta X_i + \rho_f X_i \zeta = 0, \\ &-\rho_f \frac{\partial}{\partial x_i} \left[ \frac{1}{\rho_f} (M_{ij} e_{ij} + M\zeta) \right] + \rho_f \frac{\partial}{\partial x_i} (X_j u_j) = \eta r_{ij} \dot{w}_j. \end{aligned} \tag{5.5}$$

If we wish to use the form (2.24) of the equilibrium condition the first equation (5.5) is replaced by

$$\frac{\partial}{\partial x_j}(B_{ij}^{\mu\nu}e_{\mu\nu} + M_{ij}\zeta) + \rho\Delta X_i + (\rho_f\zeta - \rho e)X_i - \rho\omega_{ik}X_k + S_{jk}\frac{\partial \omega_{ik}}{\partial x_j} + S_{ik}\frac{\partial \omega_{jk}}{\partial x_j} - e_{jk}\frac{\partial S_{ik}}{\partial x_j} = 0. \tag{5.6}$$

In the particular case where the fluid is of uniform density in the initial state the stress-strain relations (3.23) become equation (3.16) with the definition

$$\zeta = -\frac{\partial w_i}{\partial x_i} \tag{5.7}$$

for $\zeta$ and the incremental fluid pressure $p_f$. In this case the second field equation (5.5) is also simplified to

$$-\frac{\partial}{\partial x_i}(M_{ij}e_{ij} + M\zeta) + \rho_f\frac{\partial}{\partial x_i}(X_ju_j) = \eta r_{ij}\dot{w}_j. \tag{5.8}$$

The condition of uniform density requires either that the fluid be incompressible or that the body force be zero. In the latter case the initial fluid pressure is also uniform. In the absence of a body force the field equations are further simplified by disappearance of all terms containing $X_i$. In this case using the form (5.6) the field equations become

$$\begin{aligned} &\frac{\partial}{\partial x_j}(B_{ij}^{\mu\nu}e_{\mu\nu} + M_{ij}\zeta) + S_{jk}\frac{\partial \omega_{ik}}{\partial x_j} + S_{ik}\frac{\partial \omega_{jk}}{\partial x_j} - e_{jk}\frac{\partial S_{ik}}{\partial x_j} = 0, \\ &-\frac{\partial}{\partial x_i}(M_{ij}e_{ij} + M\zeta) = \eta r_{ij}\dot{w}_j. \end{aligned} \tag{5.9}$$

For uniform initial stress the term $\partial S_{ik}/\partial X_j$ further vanishes in these equations.

**The case of variable permeability.** In many problems where the pores are very small or exhibit strong anisotropy the variation of the porosity cannot be assumed small in the mathematical sense. In this case the tensor $r_{ij}$ must be made dependent on the deformation and local rotation and the equations become nonlinear. It is convenient in this case to write the flow-rate equation (4.24) by using the permeability tensor $k_{ij}$.

$$-\rho_f k_{ij}\frac{\partial}{\partial x_j}(\Delta\psi - X_k u_k) = \eta\dot{w}_i. \tag{5.10}$$

The values of $k_{ij}$ are those at the displaced point and they are referred to unrotated axes. By analogy with equation (2.18) for the stress we may write it

$$k_{ij} = K_{ij} + K_{\mu j}\omega_{i\mu} + K_{i\mu}\omega_{j\mu}, \tag{5.11}$$

where $K_{ij}$ is the permeability referred to locally rotated axes. We have already discussed earlier the dependence of $K_{ij}$ on the strain tensor [9], [10]. For example if the initial permeability is isotropic

$$K_{ij} = K\delta_{ij}. \tag{5.12}$$

The permeability after deformation may be written

$$K_{ij} = 2\beta_1(e)e_{ij} + \delta_{ij}\beta_2(e), \tag{5.13}$$

where $\beta_1(e)$ and $\beta_2(e)$ are functions of $e$, such that

$$\beta_2(0) = K. \tag{5.14}$$

A realistic expression for the permeability is obtained by inserting the condition that for a critical change of volume $e = e_c$ the porosity vanishes and $K_{ij}$ drops to zero. This is obtained by choosing the functions $\beta_1(e)$ and $\beta_2(e)$ such that

$$\beta_1(e_c) = \beta_2(e_c) = 0. \tag{5.15}$$

Simple functions for $\beta_1$ and $\beta_2$ should be adequate to approximate the empirical data.

Strictly speaking, the permeability should depend also on the fluid content. For example, in equations (5.13) we may write

$$\begin{aligned} \beta_1 &= \beta_1(e, \zeta), \\ \beta_2 &= \beta_2(e, \zeta), \end{aligned} \tag{5.16}$$

where $\beta_1\beta_2$ are now functions of $e$ and $\zeta$.

It is, of course, possible to write more general functional relations of this type by applying general theorems of tensor invariance. However it is doubtful whether such additional complications are required in actual applications.

**Instantaneous rate equations.** As a particular case of the previous results it is of interest to derive equations which govern the instantaneous time derivatives of the variables at a fixed point $x_i$. This is readily obtained by dividing the equations for incremental variables by the time interval $\Delta t$. Since the equations are all linear in the increments the variables tend toward the time derivatives when $\Delta t$ tends to zero. For example, we write

$$\lim \quad (u_i/\Delta t) = v_i, \tag{5.17}$$

where $v_i$ is the velocity of the solid at point $x_i$. We also put

$$\begin{aligned} \lim \quad (s_{ij}/\Delta t) &= \mathfrak{s}_{ij}, \\ \lim \quad (\Delta X_i/\Delta t) &= \frac{\partial X_i}{\partial x_j} v_j, \\ \lim \quad (\Delta\rho/\Delta t) &= \frac{\partial}{\partial x_i}(\rho_f \dot{w}_i), \end{aligned} \tag{5.18}$$

$$\mathcal{E}_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

$$\Omega_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right),$$

$$\mathcal{E} = \mathcal{E}_{ij}\delta_{ij}.$$

With these definitions the limiting equations (5.3) are

$$\frac{\partial}{\partial x_j}\left(\mathcal{S}_{ij} + \sigma_{ij}\mathcal{E} + \sigma_{kj}\Omega_{ik} - \sigma_{ik}\mathcal{E}_{kj}\right) + \rho\frac{\partial x_i}{\partial x_j}v_j + X_i\frac{\partial}{\partial x_i}(\rho_f \dot{w}_i) = 0. \tag{5.19}$$

The initial stress $S_{ij}$ is now replaced by $\sigma_{ij}$ the instantaneous stress value at point $x_i$ and time $t$. Similarly the stress-strain relations (3.23) may be written in terms of time derivatives. We put

$$\begin{aligned} \lim\,(\zeta/\Delta t) &= \frac{1}{\rho_f}\frac{\partial}{\partial x_i}(\rho_f \dot{w}_i) = \mathfrak{z}, \\ \lim\,(\Delta\psi/\Delta t) &= \frac{1}{\rho_f}\frac{DP}{Dt}. \end{aligned} \tag{5.20}$$

Equations (3.23) become

$$\begin{aligned} \mathcal{S}_{ij} &= B_{ij}^{\mu\nu}\mathcal{E}_{\mu\nu} + M_{ij}\mathfrak{z}, \\ \frac{DP}{Dt} &= M_{ij}\mathcal{E}_{ij} + M\mathfrak{z}. \end{aligned} \tag{5.21}$$

In these equations $P$ is the fluid pressure field at time $t$. The operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i} \tag{5.22}$$

is the time derivative at a point attached to the solid.

With rate variables at point $x_i$ Darcy's law is the same as equations (4.9) in which we write $x_i$ instead of $\xi_i$ . Using the permeability $k_{ij}$ instead of $r_{ij}$ the equation is written

$$-k_{ij}\left(\frac{\partial P_f}{\partial x_j} + \rho_f\frac{\partial V}{\partial x_j}\right) = \eta\dot{w}_i. \tag{5.23}$$

A rate equation is also immediately derived from equation (5.11) for the permeability *i.e.*

$$\frac{Dk_{ij}}{Dt} = \mathcal{K}_{ij} + k_{\mu j}\Omega_{i\mu} + k_{i\mu}\Omega_{j\mu}, \tag{5.24}$$

where $\mathcal{K}_{ij}$ is the rate of change of the permeability referred to locally transported and rotating axes. The value of $\mathcal{K}_{ij}$ may be put equal to a linear function of $\mathcal{E}_{ij}$ and $\mathfrak{z}$ *i.e.*

$$\mathcal{K}_{ij} = G_{ij}^{\mu\nu}\mathcal{E}_{\mu\nu} + H_{ij}\mathfrak{z} \tag{5.25}$$

The interest of the rate equations lie in the fact that in principle they are applicable to finite deformations.

**6. Variational and thermodynamic principles.** The possibility of expressing the fundamental equations by means of a variational principle is of considerable importance for a number of reasons. It is of course useful in deriving approximate solutions by the use of generalized coordinates. Another important application of variational principles is the formulation of the equations in curvilinear coordinates without having to use the particular methods of the tensor calculus. In addition the variational principle brings the theory within the fold of the general thermodynamics of irreversible processes as formulated by this writer. Such a variational principle is readily obtained in this case by a straightforward generalization of the similar principle derived earlier by the writer for the continuum under initial stress [3], [4], [5]. We put

$$\Delta \mathcal{V} = \tfrac{1}{2}(t_{ij}e_{ij} + \rho_f \Delta\psi\zeta) + S_{ij}\mathfrak{F}_{ij} - \rho_f X_i u_i \zeta \tag{6.1}$$

with

$$\mathfrak{F}_{ij} = \tfrac{1}{2}(e_{i\mu}\omega_{\mu j} + e_{j\mu}\omega_{\mu i}) + \tfrac{1}{2}\omega_{i\mu}\omega_{j\mu} \tag{6.2}$$

The variational principle is

$$\begin{aligned} \delta \iiint_V \Delta\mathcal{V}\, dV + \iiint_V \eta r_{ij}\dot{w}_j\ \delta w_i\, dV = \iiint_V \rho\ \Delta X_i\ \delta u_i \\ + \iint_S (\Delta f_i\ \delta u_i - \rho_f \Delta\Psi\, n_i\ \delta w_i)\, dS \end{aligned} \tag{6.3}$$

with

$$\Delta\Psi = \Delta\psi - X_j u_j . \tag{6.4}$$

The independent variables to be varied in the equation are the six components $\delta u_i$ and $\delta w_i$ .

We may verify the variation principle (6.3) by evaluating the variations. We note that the bracketed term in expression (6.1) is a quadratic form with symmetric coefficients. Hence the variation is written

$$\tfrac{1}{2}\delta(t_{ij}e_{ij} + \rho_f\Delta\psi\zeta) = t_{ij}\delta e_{ij} + \rho_f\Delta\psi\delta\zeta. \tag{6.4}$$

Furthermore inserting the value (3.12) for $t_{ij}$ we verify that

$$t_{ij}\delta e_{ij} + S_{ij}\delta\mathfrak{F}_{ij} = (s_{ij} + S_{ij}e + S_{kj}\omega_{ik} - S_{ik}e_{ki})\frac{\partial}{\partial x_j}\delta\mu_i \tag{6.5}$$

also

$$\delta(\rho_f X_i u_i \zeta) = \rho_f X_i u_i \delta\zeta + \rho_f X_i \zeta \delta u_i. \tag{6.6}$$

Introducing relations (6.4)(6.5) and (6.6) into the variational principle (6.3) it is verified identically by performing the usual integrations by parts and using the field equations (5.3).

The principle may be formulated in a more compact and significant form which relates it to thermodynamics. We write

$$\Delta X_i = \frac{\partial X_i}{\partial x_j} u_i = -\frac{\partial^2 U}{\partial x_i \partial x_j} u_j \tag{6.7}$$

and put

$$\mathcal{P} = \iiint_V \left( \Delta \mathcal{V} + \tfrac{1}{2}\rho \frac{\partial^2 U}{\partial x_i \partial x_j} u_i u_j \right) dV. \tag{6.8}$$

The variational principle then becomes

$$\delta \mathcal{P} + \iiint_V \eta r_{ij} \dot{w}_i \, \delta w_j \, dV = \iint_S (\Delta f_i \, \delta u_i - \rho_f \, \Delta \Psi \, n_i \delta w_i) \, dS. \tag{6.9}$$

This may be further simplified by introducing an operational dissipation function

$$\hat{D} = \tfrac{1}{2} p \eta r_{ij} w_i w_j. \tag{6.10}$$

The variation principle becomes

$$\delta \mathcal{P} + \delta \hat{D} = \int_S (\Delta f_i \, \delta u_i - \rho_f \, \Delta \Psi \, n_i \delta w_i) \, dS. \tag{6.11}$$

In the variation the time differential operator

$$p = \frac{\partial}{\partial t} \tag{6.12}$$

is treated as an algebraic quantity. The variational equation (6.11) brings out its obvious relation to the principles of the nonequilibrium thermodynamics as developed in very general form by the writer [9], [14], [15] and based on Onsager's relations [12], [13].

By using a generalized coordinate representation it leads to Lagrangian equations for the stability and consolidation problem.

**Application to nonlinear problems.** It is worth noting that the variational method is applicable to nonlinear problems, where the permeability is assumed to be a function of the strain and the fluid content. This is of particular interest in consolidation problems where drastic changes in pore size and even closing of the pores is associated with radical changes in the consolidation process.

**7. Porous medium with viscoelastic properties.** We shall now consider a medium with viscoelastic properties under initial stress. In this case the initial stress may or may not be associated with a steady deformation rate. A strict application of linear thermodynamics to this case requires that the medium be initially at rest and in thermostatic equilibrium in the initial state of stress.

However, the results which are derived here are still approximately valid when the initial state is one of steady flow provided the total deformations remain small during the time interval considered.

The thermodynamics of irreversible processes as formulated by the writer is applicable to perturbation around a state of initial stress, and it was shown that a viscoelastic correspondence may be used to derive the relations between the incremental stresses and the strain. This is done by substituting operators in the elastic stress-strain relations. As pointed out in an earlier paper [16] use of the thermodynamic operators under initial stress is permissible in stress-strain relations which involve the components $t_{ij}$ . Hence the stress-strain relations become

$$\begin{aligned} t_{ij} &= \hat{C}_{ij}^{\mu\nu} e_{\mu\nu} + \hat{M}_{ij}\zeta, \\ \rho f \, \Delta\psi &= \hat{M}_{ij} e_{ij} + \hat{M}\zeta. \end{aligned} \tag{7.1}$$

The 28 operators in these equations constitute a symmetric matrix and the thermodynamic principles show that they are of the general form

$$\hat{C}_{ij}^{\mu\nu} = p \int_0^\infty \frac{C_{ij}^{\mu\nu}(r)}{p + r} \, dr + C_{ij}^{\mu\nu} + pC_{ij}^{\prime\mu\nu} \tag{7.2}$$

with similar expressions for $\hat{M}_{ij}$ and $\hat{M}$. The coefficients also satisfy certain conditions of positiveness which are the same as in the initially unstressed medium and are discussed in more detail in earlier publications.

All other equations derived in the previous section remain the same, and all results are immediately extended to the viscoelastic medium by substituting the operators in place of the elastic coefficients. This generalization applies to the variational principle by introducing an operational expression for $\mathfrak{V}$. The principle becomes

$$\delta\hat{\mathcal{V}} + \delta\hat{D} = \iint_S (\Delta f_i \, \delta u_i - \rho_f \, \Delta\Psi n_i \, \delta w_i) \, dS, \tag{7.3}$$

where $\hat{\mathcal{V}}$ is obtained by substituting operators for the elastic coefficients in expression (6.8).

The nonlinear problems of variable permeability may also be solved for the case of viscoelasticity by application of the variational principle.

**Real characteristic exponents.** In stability problems the characteristic equation must be solved for $p$. For each root $p$ there is a mode where all deformations are proportional to exp $(pt)$. From thermodynamic principles it was pointed out [14] that all roots $p$ must be real. This will be the case if all operators obey the thermodynamic restrictions. In this case only real values of $p$ are obtained.

**Special case of a viscoelastic medium isotropic in finite strain.** Of particular interest is that of a medium which is isotropic in finite strain for infinitely slow deformations. Let us assume that for slow deformations it behaves as a purely elastic isothermal system. In the state of initial stress it acquires an orthotropic symmetry whose axes coincides with the principal directions of the initial stress. The operational stress-strain relations in this case are formally identical with equations (3.24) and are obtained by substituting operators for the elastic coefficients. However, in this case, because of the finite isotropy of the medium the coefficients $Q_1$, $Q_2$, $Q_3$ acquire a special form. For example

$$\hat{Q}_1 = \int_0^\infty \frac{p}{p+r} Q_1(r)\, dr + Q_1 + pQ_1'. \tag{7.4}$$

For slow deformation ($p = 0$) this operator reduces to the coefficient $Q_1$. The latter must coincide with the values (3.28) already derived for finite isotropy *i.e.*

$$Q_1 = \tfrac{1}{2}(S_{22} - S_{33}) \frac{\lambda_2^2 + \lambda_3^2}{\lambda_2^2 - \lambda_3^2}. \tag{7.5}$$

The other two values $Q_2$, $Q_3$ are also given by equations (3.28). The initial extension ratios $\lambda_1$, $\lambda_2$, $\lambda_3$ and initial stress values $S_{11}$, $S_{22}$, $S_{33}$ in these expressions are those associated with infinitely slow quasi-static finite strain.

## Appendix

In this appendix we shall briefly outline two areas of application which do not strictly belong to the main subject of this paper but are closely related to it.

One of these applications is the theory of Thermoelasticity of a continuum under initial stress. The other is the dynamics and acoustic propagation theory for a porous medium under initial stress. Since these developments are of interest in related fields and are immediately derived from the present paper the main results will be summarily sketched. They will be discussed in more detail in separate publications.

**Thermoelasticity under initial stress.** This time we are dealing with a true continuum under initial stress in thermodynamic equilibrium. The perturbation from this initial state is represented by incremental stresses and an increment of local temperature $\theta$ above the initial uniform level $T_n$. The analogy between thermoelasticity and the mechanics of porous medium developed earlier [17] for the initially stress-free case may be applied to the case of a medium under initial stress. The analogy applies to the case of a porous medium saturated with a weightless fluid initially of uniform density. In this case the value of $\zeta$ and $\rho_f\, \Delta\psi$ become

$$\zeta = -\frac{\partial w_i}{\partial x_i}, \tag{A.1}$$

$$\rho_f \Delta\psi = p_f.$$

In the analogy the entropy density is represented by $\zeta$ and the incremental temperature by $p_f$. The entropy displacement is represented by $w_i$. The field equations are obtained by putting $\rho_f = 0$ in equations (5.6) and (5.9). The tensor $\eta r_{ij}$ now represents $\lambda_{ij} T_r$ where $\lambda_{ij}$ is the thermal resistivity. With this analogy all equations derived for the porous medium are valid including the variational principle. This analogy is a consequence of the universal character of noneqilibrium thermodynamics whose general formulation covers both phenomena.

The analogy also applies to acoustic propagation in a thermoelastic continuum under initial stress by using the results outlined in the following paragraph for the dynamics of a porous medium.

**Acoustic propagation in a porous medium under initial stress.** Adding the inertia terms leads immediately to the equations which govern the dynamics of porous media under initial stress. Following the procedure developed in a recent paper [18] the equilibrium equations (2.23) are replaced by

$$\text{(A.2)} \quad \frac{\partial}{\partial x_j}(s_{ij} + S_{ij}e + S_{kj}\omega_{ik} - S_{ik}e_{kj}) + \rho\Delta X_i + X_i\Delta\rho = \rho\frac{\partial^2 u_i}{\partial t^2} + \rho_f\frac{\partial^2 w_i}{\partial t^2}.$$

The flow rate equation (4.24) becomes

$$\text{(A.3)} \quad -\rho_f\frac{\partial}{\partial x_i}(\Delta\psi - X_j u_j) = \rho_f\frac{\partial^2 u_i}{\partial t^2} + \hat{Y}_{ij}\frac{\partial \omega_j}{\partial t^2},$$

where $\hat{Y}_{ij}$ is the viscodynamic symmetric tensor operator introduced and discussed in detail in the quoted paper [18]. The variational principle as generalized from equation (7.3) is expressed by

$$\text{(A.4)} \quad \delta\hat{\mathcal{P}} + \delta\hat{\mathfrak{J}} = \iint_S (\Delta f_i\, \delta u_i - \rho_f\, \Delta\Psi\, n_i \delta w_i)\, dS,$$

where

$$\text{(A.5)} \quad \hat{\mathfrak{J}} = \tfrac{1}{2}\iiint_V (p^2\rho u_i u_i + p^2\rho_f u_i w_i + p\hat{Y}_{ij} w_i w_j)\, dV.$$

This is an operational invariant which embodies the viscodynamic properties of the medium while $\hat{\mathcal{P}}$ represents the viscoelastic properties. This separation in two basic invariants represents the essentially distinctive feature of the mechanics of porous media.

Field equations obtained from these results govern the general phenomena of dynamic stability and seismic propagation in porous media. Attention is called to the possibility of including the thermoelastic dissipation of the porous medium by the use of suitable viscoelastic operators provided we neglect the effect of the temperature change on the fluid density and its associated coupling with the gravity field.

## References

[1] M. A. Biot, General theory of three-dimensional consolidation, *Journal of Applied Physics*, **12** (1941) 155-164.

[2] M. A. Biot, Theory of Elasticity with Large Displacements and Rotations, *Proc. Fifth Inter. Congress of Applied Mechanics*, 1938 (pp. 118-121).

[3] M. A. BIOT, Theorie de l'Elasticite du Second Ordre avec Application a la Theorie du Flambage, *Ann. Soc. Scient. de Bruxelles*, **LIX** (1939) 104-112.
[4] M. A. BIOT, Nonlinear Theory of Elasticity and the Linearized Case for a Body under Initial Stress, *Phil. Mag.*, **XXVII**, 468.
[5] M. A. BIOT. Elastizitäts Theorie Zweiter Ordnung mit Anwendungen, *Zeit. fur Ang. Math., und Mech.*, **29** (1940) 89-99.
[6] M. A. BIOT, The Influence of Initial Stress on Elastic Waves, *Journal of Applied Physics*, **11** (1940) 522-530.
[7] M. A. BIOT, Incremental Elastic Coefficients of an Isotropic Medium in Finite Strain, AFOSR Report No. 1772, November 1961. (To be published in *Applied Scient. Res.*, 1963).
[8] M. A. BIOT, Internal Buckling under Initial Stress in Finite Elasticity. (To be published in *Proc. Roy. Soc.*, 1963).
[9] M. A. BIOT, Theory of Deformation of a Porous Viscoelastic Anisotropic Solid, *Journal of Applied Physics*, **27** (1956) 459-467.
[10] M. A. BIOT, Mechanics of Deformation and Acoustic Propagation in Porous Media, *Journal of Applied Physics*, **33** (1962) 1482-1498.
[11] M. KING HUBBERT, The Theory of Ground Water Motion, *The Journal of Geology*, **XLVIII** (1940) 785-944.
[12] ONSAGER L., Reciprocal Relations in Irreversible Processes I, *Phys. Rev.*, **37** (1931) 405-426.
[13] ONSAGER L., Reciprocal Relations in Irreversible Processes II, *Phys. Rev.*, **38** (1931) 2265-2279.
[14] M. A. BIOT, Linear Thermodynamics and the Mechanics of Solids, *Proc. of the Third U. S. National Congress on Applied Mechanics* (Amer. Soc. Mech. Eng. New York 1958), pp. 1-18.
[15] M. A. BIOT, Theory of Stress-Strain Relations in Anisotropic Viscoelasticity and Relaxation Phenomena, *Jour. Applied Physics*, **25** (1954) 1385-1391.
[16] M. A. BIOT, On the Instability and Folding Deformation of a Layered Viscoelastic Medium in Compression, *Journal of Applied Mechanics ser. E*, **26** (1959) 393-400.
[17] M. A. BIOT, Thermoelasticity and Irreversible Thermodynamics, *Journal of Applied Physics*, **27** (1956) 240-253.
[18] M. A. BIOT, Generalized Theory of Acoustic Propation in a Porous Dissipative Media, *Journal of the Acoustical Society of America*, **54** (1962) 1254–1264.

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