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STABILITY OF MULTILAYERED CONTINUA INCLUDING THE EFFECT OF GRAVITY AND VISCOELASTICITY

BY

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ABSTRACT

The theory of stability of multilayered continua is extended to include the effect of gravity and the case of viscoelastic materials. It is also applied to obtain numerical solutions for the buckling of the anisotropic plate in finite elasticity with free boundaries or embedded in an infinite medium. In the multilayered system it is shown that the effect of gravity forces may be included by a very simple process leading to a matrix multiplication scheme for the solution of the characteristic stability problem and to a new variational principle. By the correspondence principle the theory is immediately extended to the stability problem of multilayered viscoelastic media, and a general theorem is derived for the conditions under which only real values are possible for the characteristic exponents of the stability problem. As an example of gravity instability, two cases are solved numerically for purely viscous or elastic layers. The theory yields the solution for a large class of problems of technological and geophysical interest.

1. INTRODUCTION

In a previous paper (1)[2] the general equations for the mechanics of a continuum under initial stress were applied to the problem of stability of an elastic anisotropic medium in finite strain. The results included the problem of stability of the single elastic plate or the multilayered medium.

In the present paper, numerical solutions are evaluated and plotted for the case of an elastic medium, and the theory of stability of multilayered continua is further extended to include the case of viscoelastic materials and the effect of gravity. The particular case of buckling of the anisotropic elastic plate is analyzed numerically in Section 2. In the same section a numerical solution is also derived for the buckling of the same plate embedded in an infinite medium either isotropic or anisotropic.

The influence of gravity forces acting in a direction normal to the layers may be introduced by a very simple and general procedure already developed for particular cases (2, 3). This is derived in Section 3. It is shown that for incompressible media the system is equivalent to an analog model where gravity is replaced by surface forces acting at the interfaces or free surfaces of the layers and proportional to the normal displacements. They may be interpreted as a buoyancy effect,

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

depending on density differences. All equations obtained for the gravity-free cases are, therefore, readily extended to the heavy medium. The recurrence and matrix equations are also extended to this case and provide methods of programming the numerical solution of stability problems for a large number of layers when gravity forces enter into play.

A variational principle including the effect of gravity is also derived by considering the total potential of the analog model.

Application of the general principle of correspondence leads to equations for the stability of layered viscoelastic media which are formally identical with those of the elastic case. This is discussed in Section 4 and general thermodynamic properties derived previously for the viscoelastic operators of a medium under initial stress are discussed for the particular case of the incompressible orthotropic medium considered in the present analysis. The state of initial stress assumed in each layer is of the same type as in the elastic case.

The question of the nature of the characteristic exponent of the stability problem is examined in Section 5. It is shown that the modes of instability are proportional to a real exponential function of time under conditions which are less restrictive than those imposed by thermodynamic principles.

Section 6 deals with the stability in a system where the initial stress is due only to gravity forces and is purely hydrostatic. Two examples of interest in geology are solved numerically. They are discussed in the context of purely viscous and isotropic media. The problem in this case becomes identical with that of gravity instability of layered incompressible Newtonian fluids. The results are also applicable to purely elastic media and show the existence of critical values for the appearance of buckling under initial hydrostatic gravity stresses.

2. BUCKLING OF A FREE AND EMBEDDED ANISOTROPIC PLATE

The first application of the general theory (1) to be considered is the stability of a single elastic anisotropic plate of thickness h with free

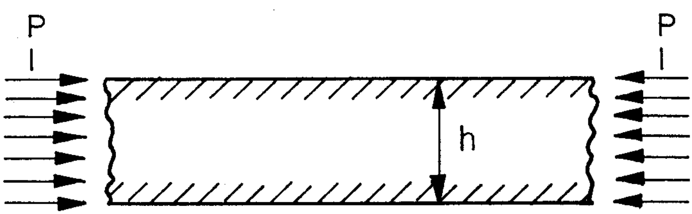

FIG. 1. Free layer subject to a uniform compression P.

defined by two elastic coefficients N and Q for incremental plane strain. surfaces subject to a uniform compression P in a direction parallel with the plate (Fig. 1). The material is assumed incompressible and

Alternate elastic coefficients (4, 5) used in the theory are

$$L = Q + \tfrac{1}{2}P \qquad M = N + \tfrac{1}{4}P. \tag{1}$$

L, the slide modulus, represents a shearing rigidity for tangential forces applied in a direction parallel with the faces of the plate; M represents a "tangent modulus" for normal stress in the same direction. These elastic coefficients were discussed in (11) and (12).

Recalling the result for the case of a single layer (1), next consider a bending deformation. Such a deformation is antisymmetric in the thickness co-ordinate. The tangential and normal stresses at the top, distributed sinusoidally along the distance parallel with the plate, are

$$\Delta f_x = \tau \sin lx \qquad \Delta f_y = q \cos lx. \tag{2}$$

The displacement components at the top surface are

$$u = U \sin lx \qquad v = V \cos lx. \tag{3}$$

The surface stresses and the displacement are related by (1, Eqs. 34)

$$\begin{aligned} \frac{\tau}{lL} &= a_{11}U + a_{12}V \\ \frac{q}{lL} &= a_{12}U + a_{22}V. \end{aligned} \tag{4}$$

The coefficients are

$$\begin{aligned} a_{11} &= \frac{\beta_1^2 - \beta_2^2}{x_1 - x_2} \\ a_{12} &= \frac{(\beta_1^2 + 1)x_2 - (\beta_2^2 + 1)x_1}{x_1 - x_2} \\ a_{22} &= a_{11}x_1x_2 \end{aligned} \tag{5}$$

with

$$\begin{aligned} x_1 &= \beta_1 \tanh \beta_1\gamma \\ x_2 &= \beta_2 \tanh \beta_2\gamma \\ \gamma &= \tfrac{1}{2}lh. \end{aligned} \tag{6}$$

The parameters β_1 and β_2 are the positive values of

$$\begin{aligned} \beta_1 &= \sqrt{m + \sqrt{m^2 - k^2}} \\ \beta_2 &= \sqrt{m - \sqrt{m^2 - k^2}} \\ m &= \frac{2N - Q}{Q + \frac{1}{2}P} \qquad k^2 = \frac{Q - \frac{1}{2}P}{Q + \frac{1}{2}P}. \end{aligned} \tag{7}$$

The nature of the irrational expressions β_1 and β_2 will be discussed later in the context of their physical interpretation. For the time being assume that β_1 and β_2 are real and positive.

Putting $\tau = 0$ in Eqs. 4 and eliminating U yield

$$\frac{q}{lL} = \frac{a_{11}a_{22} - a_{12}^2}{a_{11}} V. \tag{8}$$

Introducing the values of Eqs. 5 for a_{ij} and cancelling out the common factor $x_1 - x_2$, yields

$$\frac{a_{11}a_{22} - a_{12}^2}{a_{11}} = \frac{(\beta_1^2 + 1)^2 x_2 - (\beta_2^2 + 1)^2 x_1}{\beta_1^2 - \beta_2^2}. \tag{9}$$

For a free layer, $q = 0$. Hence, the characteristic equation is

$$(\beta_1^2 + 1)^2 x_2 - (\beta_2^2 + 1)^2 x_1 = 0, \tag{10}$$

which may be considered as a relation between $\zeta = P/2Q$ and $\gamma = lh/2$, with N/Q as a parameter. The root ζ of Eq. 10 is plotted in Figs. 2 and 3 for $N/Q = 1$, 3, and 10. The ratio N/Q is a measure of the anisotropy

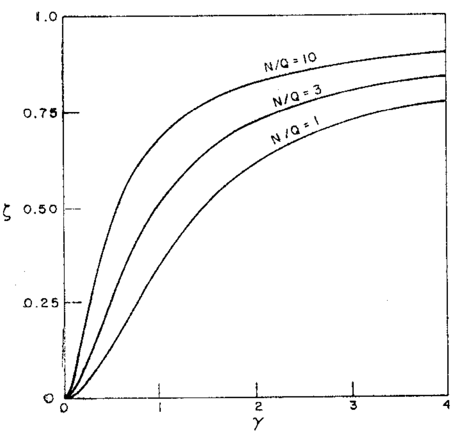

FIG. 2. Stability parameter ζ as a function of γ for the free layer of Fig. 1.

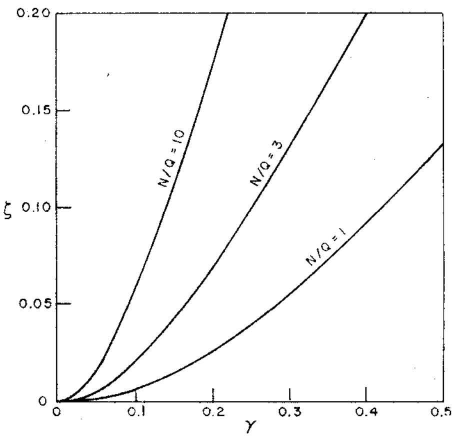

FIG. 3. Stability parameter ζ as a function of γ for the free layer of Fig. 1.

of the material under initial stress. The case $N/Q = 1$ corresponds to an isotropic medium and its solution is discussed in (4, 5, and 11).

For large wave lengths γ is a small quantity, and the hyperbolic tangents may be replaced by their power series, retaining only the terms in γ and γ^3. After cancellation of the factor $(\beta_2^2 - \beta_1^2)\gamma$, Eq. 10

[3] In these equations, s_{ij} is the incremental stress referred to locally rotated axes. The displacement components are u and v, the local rotation is ω, and the plane strain components are e_{xx}, e_{yy}, and e_{xy}.

becomes

$$k^2 - 1 + \tfrac{2}{3}(k^2 + m)\gamma^2 = 0. \tag{11}$$

Substituting the values of Eq. 7 for k^2 and m and solving for ζ give

$$\zeta = \frac{2}{3}\frac{N}{Q}\frac{\gamma^2}{1 + \frac{1}{3}\gamma^2}. \tag{12}$$

For small γ this becomes

$$\zeta = \frac{2}{3}\frac{N}{Q}\gamma^2, \tag{13}$$

which shows the parabolic behavior of the curves of Fig. 3 for small values of γ. Equation 13 coincides with the result found by Euler's theory of buckling for a thin plate, provided $4N$ is used as the elastic modulus. This can be verified by writing Eq. 13 in the form

$$P = 4N\frac{l^2h^2}{12}. \tag{14}$$

Hence, for large wave lengths the instability appears as a buckling of the plate through bending under axial compression.

As the wave length decreases there is a gradual change from bending to shear buckling. At the larger values of γ—that is, for wave lengths which are small relative to the thickness—the phenomenon degenerates into a surface instability. When γ tends to infinity, the hyperbolic tangents in Eq. 10 tend to unity, and the characteristic equation becomes

$$(\beta_2^2 + 1)^2\beta_1 - (\beta_1^2 + 1)^2\beta_2 = 0. \tag{15}$$

After cancellation of the factor $\beta_1 - \beta_2$, Eq. 15 may be written

$$2k(m + 1) + k^2 - 1 = 0. \tag{16}$$

Substituting the values of Eq. 7 for m and k and solving for N/Q yields

$$\frac{N}{Q} = \tfrac{1}{2}\zeta\left(\sqrt{\frac{1 + \zeta}{1 - \zeta}} - 1\right). \tag{17}$$

This equation yields the value of ζ corresponding to the horizontal asymptotes of Fig. 2 for $\gamma = \infty$. It is a function of N/Q, and when N/Q varies between 1 and infinity, the asymptotic value of ζ goes from 0.839 to 1. Its value is tabulated in (13).

Equation 10 has been solved under the assumption that the roots β_1

and β_2 are real. Note that

$$m^2 - k^2 = \frac{1}{\left(Q + \frac{P}{2}\right)^2}\left[4N(N - Q) + \tfrac{1}{4}P^2\right]. \tag{18}$$

Hence, if it is assumed that $N/Q > 1$, then $m > 0$ and $m^2 - k^2 > 0$. This corresponds to Case 1 of the discussion of the roots in (1). There are two subcases:

1a. For $\zeta < 1$ the roots β_1 and β_2 are real, and this corresponds to the branches plotted in Fig. 7.

1b. For $|\zeta| > 1$ the root β_2 is pure imaginary, which corresponds to internal instability. Putting $\beta_2 = i\xi$, the value of x_2 remains real

$$x_2 = \beta_2 \tanh \beta_2\gamma = - \xi \tan \xi\gamma. \tag{19}$$

Solutions of Eq. 10 for this case are represented by an infinite number of branches in the plot of ζ *versus* γ. They are not shown in Fig. 2. If plotted, they would lie outside the range $-1 < \zeta < 1$, and represent the phenomenon of internal instability in the presence of free boundaries.

The second example concerns the stability of a single layer embedded in an infinite medium (Fig. 4). Since the embedding medium is the same on top and bottom, only the antisymmetric deformation need be considered. The layer of thickness h is under an initial compressive stress P acting in a direction parallel with the layer. To simplify the analysis, two assumptions are made: (1) assume perfect slip between the layer and the medium; and (2) assume that the initial stress is zero or

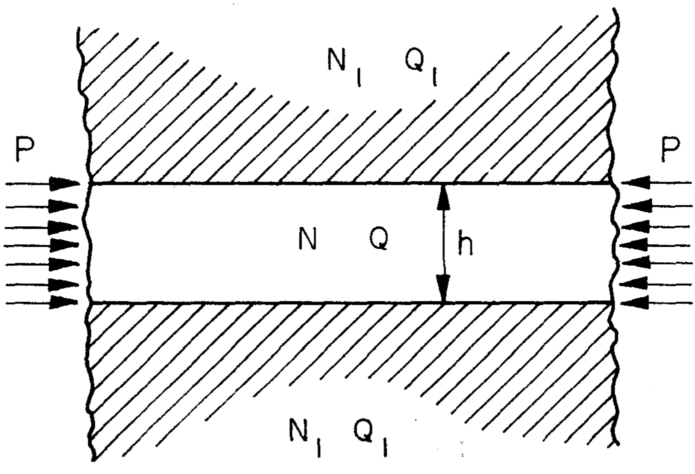

FIG. 4. Single layer embedded in an infinite medium.

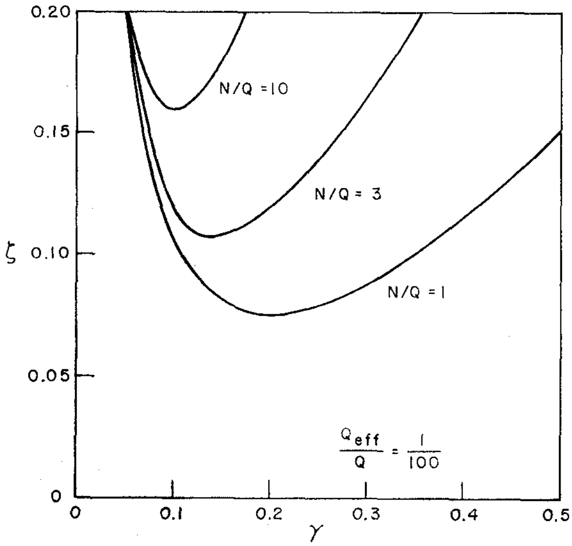

FIG. 5. Stability parameter ζ as a function of γ for the embedded layer of Fig. 4 (with $Q_{eff}/Q = 1/100$).

negligible in the embedding medium. Hence, the tangential force τ is put equal to zero, and the relation between the normal force q and the normal deflection V is then given by Eq. 8. It has been shown (6, 7) that the correction due to perfect adherence is very small for the case of the single layer. The layer is anisotropic and characterized by the two elastic coefficients N and Q. The embedding medium also anisotropic, is characterized by the elastic coefficients N_1, Q_1.

To derive the characteristic equation for this problem, the properties of the half space must be examined. In the absence of initial stress, the coefficients a_{ij} for the half space are obtained by putting $P = 0$ in Eqs. 68 of (1):

$$\begin{aligned} a_{11} &= a_{22} = 2\sqrt{N_1/Q_1} \\ a_{12} &= 0 \end{aligned} \tag{20}$$

and the slide modulus becomes $L_1 = Q_1$. Substituting these quantities in Eqs. 76 of (1) for the upper half space, yields

$$q = -2\sqrt{N_1Q_1}lV. \tag{21}$$

This result was discussed in (13). In this case the half space behaves as an isotropic medium of effective shear modulus

$$Q_{eff} = \sqrt{N_1Q_1}. \tag{22}$$

The characteristic equation is obtained by equating the values of q in

Eqs. 8 and 21 at the upper interface. Introducing the value

$$L = Q + \frac{P}{2} = Q(1 + \zeta) \tag{23}$$

for the slide modulus of the layer yields the characteristic equation

$$\frac{2Q_{\text{eff}}}{Q} + (1 + \zeta)\, \frac{(\beta_1^2 + 1)^2 x_2 - (\beta_2^2 + 1)^2 x_1}{\beta_1^2 - \beta_2^2} = 0. \tag{24}$$

This equation establishes a functional relationship between $\zeta = P/2Q$ and $\gamma = lh/2$ for given values of the two parameters, N/Q and Q_{eff}/Q. Plots of ζ *versus* γ are shown in Fig. 5 for $Q_{\text{eff}}/Q = 1/100$ and for $N/Q = 1$, 3, and 10.

If the coefficients N and Q were independent of the initial stress, the buckling load P and the corresponding wave length would be given by the minimum value $\zeta_{\min}$ of ζ in Fig. 5. Actually the coefficients N and Q depend on the initial stress P and the complete family of curves ζ *versus* γ must be considered as a two-parameter system, depending on N/Q and Q_{eff}/Q. Obviously as the finite strain is increased, buckling will occur at the point where the horizontal of ordinate ζ touches the stability curve. Hence, the buckling condition is written

$$\frac{P}{2Q} = \zeta_{\min}\left(\frac{N}{Q}, \frac{Q_{\text{eff}}}{Q}\right). \tag{25}$$

On the right-hand side of this equation the minimum value of ζ is considered as a function of the two variables N/Q and Q_{eff}/Q. It is, therefore, an intrinsic equation for the critical buckling stress P. As in the case of the free layer, Eq. 24 contains branches in the range $|\zeta| > 1$ which correspond to internal instability.

The particular case of the isotropic medium, $N/Q = 1$, has been the object of a previous analysis for the case of perfect interfacial slip (5) and perfect adherence (6). In the latter case the branches in the region $|\zeta| > 1$ corresponding to internal instability were also plotted.

For small wave length, hence large values of γ, the plots of Fig. 5 become asymptotic to horizontal lines, which represent an interfacial instability. A more complete plot showing this asymptotic value is given in (6) for the case $N = Q$. The phenomenon of interfacial instability was discussed separately in another paper (14).

3. INTRODUCTION OF GRAVITY FORCES—ANALOG MODEL

The effect of gravity may be introduced quite simply into the formulation (2, 3).

This section will present a general derivation of this procedure. In a homogeneous region of incompressible medium of mass density ρ, with the acceleration of gravity g acting in the negative y direction, the initial stress components are

$$\begin{aligned} S_{11} &= -P + \rho g y + C \\ S_{22} &= \rho g y + C \\ S_{12} &= 0, \end{aligned} \tag{26}$$

where $P = S_{22} - S_{11}$. Both P and C are assumed constant in the homogeneous region. The body force in this case is

$$\begin{aligned} X &= 0 \\ Y &= -g. \end{aligned} \tag{27}$$

Substituting these expressions in Eqs. A1 of (1) gives

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - \rho g \frac{\partial v}{\partial x} - P \frac{\partial \omega}{\partial y} &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - \rho g \frac{\partial v}{\partial y} - P \frac{\partial \omega}{\partial x} &= 0. \end{aligned} \tag{28}$$

The derivation of the second equation takes account of the condition for incompressibility, replacing e_{xx} by $\partial v/\partial y$.[3] If the fictitious stress components,

$$\begin{aligned} s_{11}' &= s_{11} - \rho g v \\ s_{22}' &= s_{22} - \rho g v \\ s_{12}' &= s_{12}, \end{aligned} \tag{29}$$

are introduced, the Eqs. 28 become

$$\begin{aligned} \frac{\partial s_{11}'}{\partial x} + \frac{\partial s_{12}'}{\partial y} - P \frac{\partial \omega}{\partial y} &= 0 \\ \frac{\partial s_{12}'}{\partial x} + \frac{\partial s_{22}'}{\partial y} - P \frac{\partial \omega}{\partial x} &= 0. \end{aligned} \tag{30}$$

Putting $s' = s - \rho g v$, the stress-strain relations become

$$\begin{aligned} s_{11}' - s' &= 2N e_{xx} \\ s_{22}' - s' &= 2N e_{yy} \\ s_{12}' &= 2Q e_{xy}. \end{aligned} \tag{31}$$

With this fictitious stress system, Eqs. 30 and 31 are identical with those in which gravity is equal to zero (1); however, there is a difference in the boundary conditions. The boundary forces, expressed by Eqs. 89 of (1), become

$$\Delta f_x = s_{12}' + Pe_{xy} - S_{22}\frac{\partial v}{\partial x} \qquad (32)$$

$$\Delta f_y = s_{22}' + \rho g v + S_{22}\frac{\partial u}{\partial x}$$

when the fictitious stresses are substituted.

Consider the terms

$$\Delta' f_x = s_{12}' + Pe_{xy} \qquad (33)$$

$$\Delta' f_y = s_{22}' + \rho g v.$$

Their physical significance is the same as in the gravity-free case. They represent the incremental stress components along directions tangent and normal to the deformed surface and referred to unit areas after deformation (1). An important property of the stress components $\Delta' f_x$ and $\Delta' f_y$ is their continuity across an interface between two layers of different material. This is immediately evident for the case of perfect adherence, since these components represent stress at the same point. On the other hand, for the case of perfect slip $\Delta' f_x = 0$, but $\Delta' f_y$ represents normal stresses at different points of the interface; however, the differential slip is a small quantity of the first order so that $\Delta' f_y$ may still be considered continuous if second order quantities are neglected.

Next, introduce the stress,

$$\Delta'' f_x = s_{12}' + Pe_{xy} = \tau \sin lx \qquad (34)$$

$$\Delta'' f_y = s_{22}' = q \cos lx,$$

which represent the boundary stresses τ and q in the gravity-free case. The stresses (Eqs. 33) may then be written

$$\Delta' f_y = \tau \sin lx \qquad (35)$$

$$\Delta' f_y = (q + \rho g V) \cos lx.$$

These results are directly applicable to multilayered systems by expressing the continuity of the stresses of τ and $q + \rho g V$ at an interface. Note that the values of τ and q at the top and bottom faces of a layer are given by the equations derived for the case of zero gravity.

First consider the case of perfect adherence. Proceeding as in the derivation of equations for the multilayered system (1) and expressing the stress continuity at the interface adjacent to the j^{th} and $(j + 1)^{\text{th}}$

layers give

$$- lL_j \frac{\partial I_j}{\partial U_{j+1}} = lL_{j+1} \frac{\partial I_{j+1}}{\partial U_{j+1}} \tag{36}$$

$$- lL_j \frac{\partial I_j}{\partial V_{j+1}} + \rho_j g V_{j+1} = lL_{j+1} \frac{\partial I_{j+1}}{\partial V_{j+1}} + \rho_{j+1} g V_{j+1},$$

which may be written as

$$\frac{\partial}{\partial U_{j+1}} (L_j I_j + L_{j+1} I_{j+1}) = 0 \tag{37}$$

$$\frac{\partial}{\partial V_{j+1}} (L_j I_j + L_{j+1} I_{j+1}) + \frac{1}{l} (\rho_{j+1} - \rho_j) g V_{j+1} = 0.$$

Similar equations are obtained for the interfaces adjacent to a semi-infinite medium. The equations may be given a more compact form by introducing a gravity invariant

$$\mathcal{G} = \frac{1}{2l} \sum_{j=0}^{n} (\rho_{j+1} - \rho_j) g V_{j+1}^2 \tag{38}$$

where ρ_0 and ρ_{n+1} are the mass densities of the upper and lower half spaces, respectively. The stability equations of the multilayered system are now

$$\frac{\partial \mathcal{J}}{\partial U_j} = 0 \qquad \frac{\partial}{\partial V_j} (\mathcal{J} + \mathcal{G}) = 0 \tag{39}$$

where

$$\mathcal{J} = LI_l + \sum^{j} L_j I_j + L' I_u'. \tag{40}$$

(See Eq. 102 of 1.) The terms LI_l and $L'I_u'$ represent the contribution of the lower and upper half space. They are omitted if the corresponding upper or lower surface is free.

Equations 39 may also be written as a variational principle

$$\delta(\mathcal{J} + \mathcal{G}) = 0 \tag{41}$$

with arbitrary variations δU_j, δV_j.

The physical significance of this expression is apparent since it represents the total energy of the analog model.

These results may be expressed intuitively by stating that the effect of gravity may be replaced by a force $(\rho_{j+1} - \rho_j) g V$ per unit area normal to the interface and positive when acting downward. It is proportional to the vertical displacement V, and is in the nature of an elastic restoring force with a "spring constant" equal to $(\rho_{j+1} - \rho_j) g$. Hence, it is stabilizing or destablizing, depending on the sign of the density difference. At a free surface the effect of gravity is represented by a normal stress $\rho g V$ acting downward.

The replacement of gravity by elastic interfacial and surface forces leads to a system which is mathematically equivalent to the actual one. It may therefore be considered as an *analog model.*

The present analysis is an exact derivation from the general theory of a result which is intuitively evident if the fictitious interfacial forces are identified with buoyancy forces.

The case of layers with perfect slip at the interface is treated in a similar way.

The procedure of matrix multiplication developed earlier may be extended to include the effect of gravity. This is readily done by addition and subtraction of terms in Eqs. 108 of (1) and writing them in the form

$$\begin{bmatrix} \tau_1 \\ q_1 + \rho g V_1 \\ l U_1 \\ l V_1 \end{bmatrix} = \mathfrak{N} \begin{bmatrix} \tau_2 \\ q_2 + \rho g V_2 \\ l U_2 \\ l V_2 \end{bmatrix}. \tag{42}$$

The matrix is now

$$\mathfrak{N} = \begin{bmatrix} B_1 & B_2 & LB_5 & LC_6' \\ C_3 & C_4 & -LC_6 & LC_7 \\ \frac{1}{L}B_8 & \frac{1}{L}B_9 & B_1 & -C_3' \\ -\frac{1}{L}B_9 & \frac{1}{L}B_{10} & -B_2 & C_4' \end{bmatrix}. \tag{43}$$

The additional coefficients are

$$\begin{aligned} C_3 &= B_3 - \frac{\rho g}{lL} B_9 & C_4 &= B_4 + \frac{\rho g}{lL} B_{10} \\ C_3' &= B_3 + \frac{\rho g}{lL} B_9 & C_4' &= B_4 - \frac{\rho g}{lL} B_{10} \\ C_6 &= B_6 + \frac{\rho g}{lL} B_2 & C_6' &= B_6 - \frac{\rho g}{lL} B_2. \\ C_7 &= B_7 + \left(\frac{\rho g}{lL}\right)^2 B_{10} \end{aligned} \tag{44}$$

Because of the continuity of $q + \rho g V$ at an interface, the same procedure of matrix multiplication may be used as in Eq. 112 of (1). Note that the boundary condition at a stress-free surface is now $q + \rho g V = 0$.

The recurrence equations (39) or the matrix multiplication process using the matrix (43) are well suited to programming of automatic computers for the numerical solution of problems including gravity instability which involve a large number of layers.

4. VISCOELASTICITY AND CORRESPONDENCE PRINCIPLE

The general validity of a correspondence principle applicable to viscoelastic media under initial stress has been shown (8). Equations for viscoelastic stability are formally the same as for elastic media and are obtained by replacing the incremental elastic coefficients by operators.

The form of these operators was derived from the principles of irreversible thermodynamics (9, 10) and later generalized to the case of a medium under initial stress (7). Elastic coefficients defined in terms of incremental forces per unit initial area may be chosen instead of stresses. Such coefficients were used in the previous sections. For example, if $s_{22} = S_{22} = 0$, the stress-strain relations may be written

$$\begin{aligned} t_{11} &= 4Me_{xx} \\ t_{21}' &= 2Le_{xy}. \end{aligned} \tag{45}$$

These coefficients yield the incremental axial force t_{11} and the tangential force t_{21}' per unit initial area for a strip of material under the initial axial stress S_{11}. If the medium is in a state of thermodynamic equilibrium under the initial stress, the operators corresponding to such coefficients as M and L have the same form as for a medium initially stress-free (7). Hence, for a viscoelastic material, Eqs. 45 are replaced by

$$\begin{aligned} t_{11} &= 4\bar{M}e_{xx} \\ t_{21}' &= 2\bar{L}e_{xy} \end{aligned} \tag{46}$$

with

$$\begin{aligned} \bar{M} &= \int_0^\infty \frac{p}{p+r} M(r)dr + M + M'p \\ \bar{L} &= \int_0^\infty \frac{p}{p+r} L(r)dr + L + L'p. \end{aligned} \tag{47}$$

In these expressions, let

$$p = \frac{d}{dt} \tag{48}$$

(t = time variable). When p is an algebraic real or complex quantity, Eqs. 46 represent the stress-strain relations between variables which vary with time proportionally to the factor exp (pt). Thermodynamic principles also require that $M(r)$, M, M', and $L(r)$, L, L' be all positive quantities.

According to Eqs. 1, the coefficients N and Q correspond to the operators

$$\begin{aligned} \bar{N} &= \bar{M} - \tfrac{1}{4}P \\ \bar{Q} &= \bar{L} - \tfrac{1}{2}P \end{aligned} \tag{49}$$

and the operational stress-strain relations are

$$\begin{aligned} s_{11} - s &= 2\bar{N}e_{xx} \\ s_{22} - s &= 2\bar{N}e_{yy} \\ s_{12} &= 2\bar{Q}e_{xy}. \end{aligned} \tag{50}$$

As an example, consider a laminated medium constituted by thin alternating layers of elastic and viscous incompressible materials. The elastic layers carry the initial compressive stress P_1. Their elastic properties are defined by the coefficients N_1 and Q_1 and they occupy a fraction α_1 of the thickness. The soft layers are constituted by a viscous material of viscosity η occupying a fraction α_2 of the total thickness. The viscous layers are assumed to be free of initial stress. The operators for the viscous layers are

$$\bar{L}_2 = \bar{N}_2 = \bar{Q}_2 = p\eta. \tag{51}$$

Applying the correspondence principle to Eqs. 32 and 33 of (1), the operators of the composite medium are derived,

$$\begin{aligned} \bar{N} &= N_1\alpha_1 + p\eta\alpha_2 \\ \bar{L} &= \frac{L_1\eta p}{\alpha_1\eta p + \alpha_2 L_1}. \end{aligned} \tag{52}$$

The slide modulus L_1 of the elastic layer is

$$L_1 = Q_1 + \tfrac{1}{2}P_1 \tag{53}$$

and the initial stress in the composite medium is

$$P = \alpha_1 P_1. \tag{54}$$

The operators (52) correspond to a Kelvin and Maxwell model, respectively.

As another example, consider the initially stress-free elastic half space. As shown by Eq. 21, the deflection under surface forces is the same as if it were isotropic with an effective elastic coefficient

$$Q_{\text{eff}} = \sqrt{N_1 Q_1}. \tag{55}$$

If the half space is purely viscous, it is represented by two operators,

$$\bar{N}_1 = p\eta_1 \qquad (56)$$
$$\bar{Q}_1 = p\eta_2.$$

Applying the correspondence principle, the coefficient Q_{eff} becomes the operator

$$\bar{Q}_{eff} = p\sqrt{\eta_1\eta_2}. \qquad (57)$$

This shows that the surface of an anisotropic viscous medium behaves as an isotropic one of Newtonian viscosity

$$\eta = \sqrt{\eta_1\eta_2}. \qquad (58)$$

These considerations may be applied to the stability problem of single or multilayered viscoelastic media. By the correspondence principle the results obtained previously for the elastic medium may be immediately extended to viscoelasticity. Consider the case represented in Fig. 4 for the single embedded layer under a given initial compressive stress P and perfect interfacial slip. For viscoelastic materials we must replace the elastic coefficients of the layer by the operators $\bar{N}$ and $\bar{Q}$. Similarly, the half space is characterized by the effective operator $\bar{Q}_{eff}$. These operators are functions of the real quantity p. For given value of p we may calculate the value of ζ as a function of γ using the same characteristic equation as for the elastic case. This yields a plot which belongs to the family of curves of Fig. 5. Consider now the ordinate $\zeta = P/2\bar{Q}$. The horizontal line through this ordinate may intersect the curve for two values γ_1 and γ_2 of γ. This shows the existence of two wave lengths in the layer such that their amplitudes are proportional to exp (pt). Repeating this process for other values of p its value may be plotted as a function of γ. The value of γ for which p is a maximum defines a dominant wave length. It is the wave length of folding of the layer whose amplitude grows at the fastest rate. Obviously at this point $\gamma_1 = \gamma_2$. Therefore, the maximum value of p must satisfy the same equation (25) as for the elastic case where the coefficients are replaced by the operators.

$$\frac{P}{2Q^*} = \zeta_{min}\left(\frac{\bar{N}}{\bar{Q}}, \frac{\bar{Q}_{eff}}{\bar{Q}}\right). \qquad (59)$$

The dominant wave length of an embedded layer for the case of isotropic viscosity has been discussed and evaluated (5).

5. CONDITIONS FOR REAL VALUES OF THE CHARACTERISTIC EXPONENTS OF VISCOELASTIC INSTABILITY

The characteristic equation for stability must be solved for p as a function of the wave length. There are usually several or an infinite number of such roots represented by multiple branches in the p *versus* γ plot.

The characteristic roots p must be real if the principles of linear thermodynamics are satisfied (10). This was derived from the property that for thermodynamic systems initially in an equilibrium state the stability problem is defined by two quadratic forms. One of these is the dissipation function which by nature is positive definite. Hence, the characteristic roots p are real quantities. In the case of a system under initial stress the other quadratic form which represents the generalized free energy may be indefinite, indicating an unstable equilibrium, and some of the characteristic roots yield increasing exponentials of time.

This is formally identical with the properties of the characteristic roots represented by the square of the frequency in stability and vibration problems of conservative dynamical systems.

A less restrictive and more formal condition for the existence of real characteristic roots may be obtained by considering the particular situation analyzed in the present paper. In this case the properties of the medium depend on the vertical coordinate y. The material is incompressible and is characterized by two operators $\bar{N}(y)$ and $\bar{Q}(y)$ with axes of orthotropic symmetry along the vertical and horizontal directions. The principal initial stresses are oriented in the same directions and expressed by

$$\begin{aligned} S_{11} &= -P(y) + S_{22}(y) \\ S_{22} &= S_{22}(y). \end{aligned} \tag{60}$$

They are functions only of the vertical coordinate y.

With a constant gravity field of acceleration g, and a mass density distribution $\rho(y)$ a function only of y, the equilibrium condition for the initial stress is

$$\frac{dS_{22}}{dy} - \rho g = 0. \tag{61}$$

The equilibrium equations (A1 of 1) for the incremental stress may be written in the form

$$\begin{aligned} &\frac{\partial \Delta_{11}}{\partial x} + \frac{\partial \Delta_{12}}{\partial y} = 0 \\ &\frac{\partial \Delta_{12}}{\partial x} + \frac{\partial \Delta_{22}}{\partial y} + gv\frac{d\rho}{dy} - P\frac{\partial^2 v}{\partial x^2} = 0, \end{aligned} \tag{62}$$

where[4]

$$\begin{aligned}\Delta_{11} &= s_{11} - Pe_{yy} - \rho g v \\ \Delta_{12} &= s_{12} + Pe_{xy} \\ \Delta_{22} &= s_{22} - \rho g v.\end{aligned} \tag{63}$$

The condition of incompressibility,

$$e_{xx} + e_{yy} = 0, \tag{64}$$

has been used in deriving these equations. Assume that the characteristic equation for stability has complex conjugate roots p and p^* with corresponding solutions represented by the variables u, u^*, v, v^*, etc. Add Eqs. 62, after multiplying the first by u^* and the second by v^*, and integrate the result over the area S. This gives

$$\iint_S \left(u^* \frac{\partial \Delta_{11}}{\partial x} + u^* \frac{\partial \Delta_{12}}{\partial y} + v^* \frac{\partial \Delta_{12}}{\partial x} + v^* \frac{\partial \Delta_{22}}{\partial y} + g v^* v \frac{d\rho}{dy} - P v^* \frac{\partial^2 v}{\partial x^2} \right) dxdy = 0. \tag{65}$$

Assume a solution which is periodic along x and choose a domain S limited by two vertical lines at a distance of one wave length and two horizontal lines of ordinates y_1 and y_2. Integrating by parts, Eq. 65 becomes, after substituting the values in Eq. 50 for the stresses and taking into account the condition of incompressibility,

$$\left[\int (u^* \Delta_{12} + v^* \Delta_{22}) dx \right]_{y_1}^{y_2} = \iint_S \left(4\bar{M} e_{xx} e_{xx}^* + 4\bar{L} e_{xy} e_{xy}^* + g v v^* \frac{d\rho}{dy} - P \frac{\partial v}{\partial x} \frac{\partial v^*}{\partial x} \right) dxdy. \tag{66}$$

Because of the periodicity of the solution, the line integral along the vertical sides cancels out and reduces to the term on the left-hand side. If the displacements vanish at the horizontal boundaries, the line integral is equal to zero. This is the case for rigid adhering boundaries and for an infinite or semi-infinite medium with vanishing displacements at infinity. For a rigid boundary with perfect slip it is also equal to zero because v^* and Δ_{12} (which, according to Eq. 63, represents the tangential force) both vanish. Finally, for a free boundary, $s_{22} = 0$ and $\Delta_{12} = 0$. Hence,

$$\int (u^* \Delta_{12} + v^* \Delta_{22}) dx = -\rho g \int v^* v dx. \tag{67}$$

In all these cases the derivation of Eq. 66 may be repeated, replacing all

[4] The variables are defined as in Eq. 28.

quantities by their complex conjugates. Subtracting the two results from each other gives

$$\iint_S [(\bar{M} - \bar{M}^*)e_{xx}e_{xx}^* + (\bar{L} - \bar{L}^*)e_{xy}e_{xy}^*]dxdy = 0. \tag{68}$$

This equation cannot be verified if the imaginary parts of $\bar{M}$ and $\bar{L}$ are of the same sign. *Hence, if this is the case, the characteristic roots p must be real.* This is also equivalent to the condition that the imaginary parts of $\bar{N}$ and $\bar{Q}$ be of the same sign. As shown by expressions (57), this condition is satisfied for the operators derived from thermodynamics.

It is of interest to point out that Eq. 51 furnishes implicitly a proof of the uniqueness and stability of the viscoelastic solution under given perturbing forces for the case where $\bar{M}$ and $\bar{L}$ are positive functions of p while $d\rho/dy$ and P are negative or zero. If the field u, v represents the difference between two solutions corresponding to the same perturbing forces, it satisfies Eq. 66 considered as a Laplace transform with real values of p. It can only be satisfied for $u = v = 0$. Hence, in this case the solution is unique and stable.

The conditions for the existence of real characteristic roots are illustrated by the problem of the surface instability of the viscoelastic half space (5). When the characteristic equation is rationalized, it is found to be cubic, with two of the roots complex. However, they are spurious, and it was shown that only the real root satisfies the condition of vanishing disturbance at infinity.

6. GRAVITY INSTABILITY OF VISCOUS AND ELASTIC MEDIA

The particular case when the initial stress is purely hydrostatic leads to instability due only to the gravity forces. The basic equations are obtained by putting $P = 0$ into the previous result. The simplest way to derive the equations is to use the concept of analog model. This analog model behaves like the actual medium under initial stress.

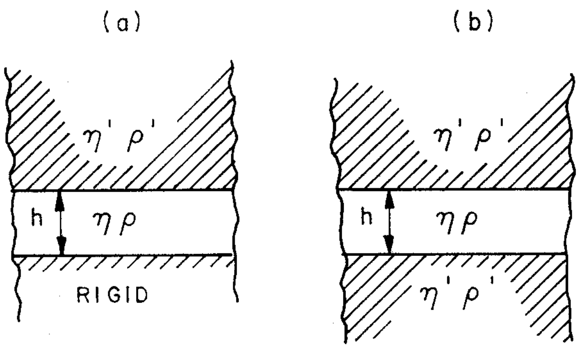

FIG. 6. Gravity instability of a viscous layer. (*a*) on top of a rigid base and surmounted by an infinite viscous medium; (*b*) between two infinite identical viscous media.

It is obtained by considering a stress-free medium and replacing the gravity force by surface forces acting at the interface between the layers. The coefficients a_{ij} and b_{ij} for the stress-free layers are then given by Eqs. 85 and 86 of (1). Consider the medium to be incompressible, purely viscous and isotropic. In this case the operators become

$$\bar{M} = \bar{L} = \bar{N} = \bar{Q} = \eta p \tag{69}$$

where η is the viscosity coefficient.

As an example, consider a viscous layer of thickness h, viscosity η and density ρ lying on a rigid base (Fig. 6a). On top of this layer lies an infinitely thick medium of viscosity η' and density $\rho'(\rho' > \rho)$. Perfect adherence is assumed at the interfaces. Therefore, the displacement vanishes at the bottom side of the layer and the problem may be formulated in terms of two variables U and V representing the displacement components at the upper interface.

Consider the upper half space. In the absence of initial stress and for an isotropic medium the coefficients (Eqs. 20) are

$$\begin{aligned} a_{11} &= a_{22} = 2 \\ a_{12} &= 0. \end{aligned} \tag{70}$$

The stress in the upper half space at the interface is

$$\begin{aligned} \tau' &= -\,2l\eta' p U \\ q' &= -\,2l\eta' p V. \end{aligned} \tag{71}$$

In the layer the displacement at the bottom face vanishes. Hence, applying Eq. 59 of (1), the stresses in the layer at the top face are

$$\begin{aligned} \tau &= l\eta p(A_1 U + A_2 V) \\ q &= l\eta p(A_2 U + A_3 V). \end{aligned} \tag{72}$$

The coefficients are

$$\begin{aligned} A_1 &= \tfrac{1}{2}(a_{11} + b_{11}) \\ A_2 &= \tfrac{1}{2}(a_{12} + b_{12}) \\ A_3 &= \tfrac{1}{2}(a_{22} + b_{22}). \end{aligned} \tag{73}$$

The values of a_{ij} and b_{ij} are those of the *isotropic* and *stress-free layer* expressed by Eqs. 85 and 86 of (1). They are functions only of $\gamma = \frac{1}{2}lh$. In the analog model, a surface force $(\rho - \rho')qV$ must be applied at the interface such that

$$\begin{aligned} \tau' &= \tau \\ q' &= q + (\rho - \rho')gV. \end{aligned} \tag{74}$$

Combining Eqs. 71, 72, and 74 yields

$$(2 + \kappa A_1)U + \kappa A_2 V = 0$$

$$\kappa A_2 U + \left[2 + \kappa A_3 + \frac{\sigma}{2\gamma}\right] V = 0 \tag{75}$$

with the parameters

$$\kappa = \frac{\eta}{\eta'} \qquad \sigma = \frac{(\rho' - \rho)gh}{\eta' p}. \tag{76}$$

The characteristic equation is obtained by equating to zero the determinant of Eqs. 75. It may be written

$$\sigma = 2\gamma\left[2 + \kappa A_3 - \frac{\kappa^2 A_2{}^2}{2 + \kappa A_1}\right]. \tag{77}$$

For a given value of the viscosity ratio κ, σ is a function only of γ.

This value of σ as a function of γ for $\kappa = 1/1000$ has been plotted as curve *a* in Fig. 7. The value of σ goes through a minimum. This minimum represents a maximum value for p and corresponds to a dominant wave length. The dominant wave length is

$$\mathcal{L}_d = \frac{\pi h}{\gamma_d}, \tag{78}$$

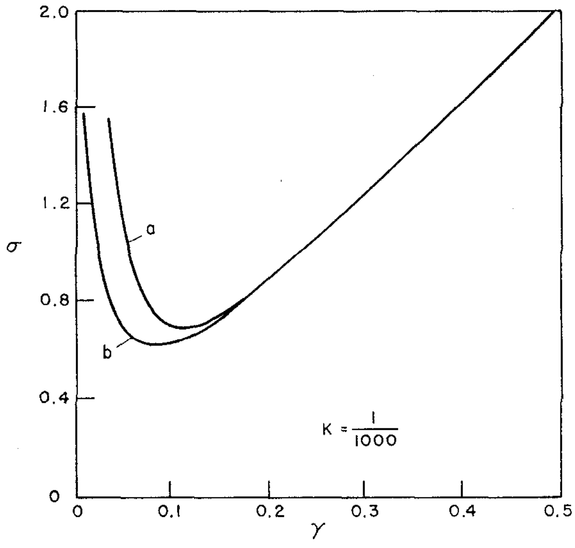

Fig. 7. Parameter σ of gravity instability as a function of γ for the case $\kappa = 1/1000$. Curve *a*—For the layer on a rigid base (Fig. 6*a*). Curve *b*—For the embedded layer (Fig. 6*b*).

where γ_d is the value of γ corresponding to the minimum value of σ. This minimum is shown in Table I as a function of κ along with corresponding values of γ_d and $\frac{\mathfrak{L}_d}{h}$.

TABLE I.—*Minimum Value of σ and corresponding value γ_d and $\frac{\mathfrak{L}_d}{h}$.*

κ	σ_{min}	γ_d	$\mathfrak{L}_d/h$
1/1000	0.687	0.114	27.6
1/100	1.49	0.243	12.9
1/10	3.36	0.495	6.35

Equations for the layer embedded in an infinite medium represented in Fig. 6*b* are obtained in a similar way from the analog model. Using Eqs. 59 of (1) for the layer gives

$$\begin{aligned} -\frac{2}{\kappa}U_1 &= A_1U_1 + A_2V_1 - A_4U_2 + A_5V_2 \\ \left(-\frac{2}{\kappa} + \frac{\sigma}{2\gamma}\right)V_1 &= A_2U_1 + A_3V_1 - A_5U_2 + A_6V_2 \\ \frac{2}{\kappa}U_2 &= A_4U_1 + A_5V_1 - A_1U_2 + A_2V_2 \\ \left(\frac{2}{\kappa} + \frac{\sigma}{2\gamma}\right)V_2 &= -A_5U_1 - A_6V_1 + A_2U_2 - A_3V_2. \end{aligned} \tag{79}$$

The coefficients A_j are functions of γ only and are given by Eq. 60, 85, and 86 of (1). Equating to zero the determinant of these equations yields a quadratic equation for σ. Its solution is shown as curve *b* in Fig. 7 for $\kappa = 1/1000$.

An interesting property of Eqs. 79 is their invariance when U_1, V_1, U_2, V_2 are replaced by U_2, $-V_2$, U_1, $-V_1$ and σ by $-\sigma$. This shows that the characteristic equation contains only σ^2. Hence, the roots σ are equal and opposite in sign. There are two characteristic modes for the system (79). The unstable mode is associated with a positive value of p and its amplitude is proportional to the increasing exponential exp (pt). In this mode and for $\rho' > \rho$ the amplitude of the upper interface is the largest. The other mode, obtained by performing the substitution indicated above, is stable and its amplitude is proportional to the decreasing exponential exp $(-pt)$. If the layer density is higher than in the surrounding medium, that is, for $\rho > \rho'$, it is easily seen that the latter mode is the unstable one.

In general, the characteristic equation for p will be of a degree equal to the total number of non-rigid interfaces plus the free surfaces. To each root corresponds a modal solution which is either stable or unstable.

By the correspondence principle the present solution is, of course, immediately applicable to gravity instability of elastic media. The operators ηp and $\eta' p$ must be replaced by the corresponding shear moduli μ and μ'. The parameter κ becomes

$$\kappa = \frac{\mu}{\mu'}. \tag{80}$$

For the case of a rigid base, the interface will buckle as soon as the density difference satisfies the equation

$$\frac{(\rho' - \rho)gh}{\mu'} = \sigma_{\min}, \tag{81}$$

where $\sigma_{\min}$ is given by Table I. For values of $(\rho' - \rho)gh/\mu'$ smaller than the value in Eq. 81, the layer is stable.

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misprint equation (79)