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INCREMENTAL ELASTIC COEFFICIENTS OF AN ISOTROPIC MEDIUM IN FINITE STRAIN *)

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Summary

Incremental elastic coefficients are derived for an isotropic medium in a state of finite initial strain. The analysis is based on concepts and methods developed by the author in earlier publications [1] [2] [3] [5] which require only elementary procedures and bring to light the physical significance of the results. Remarkably simple formulas for the incremental shear coefficients are established. For comparison the same results are derived by an alternate procedure using Riemannian tensors and the calculation is shown to be much more elaborate. Application is made to the particular case of second order elasticity theory and expressions derived for the incremental coefficients including the correction terms of the first order in the initial strain. This provides a complete theory of first order correction for acoustic propagation under initial stress.

§ 1. *Introduction.* We consider an elastic continuum which is isotropic in the unstrained condition. By this is meant that the relations between the stresses and the finite strains are invariant under any solid rotation of the continuum.

The problem we are concerned with is that of deformation of such a medium in the vicinity of an initial state of finite strain. In particular we wish to derive relations between small incremental strains and incremental stresses. In some earlier work we have discussed in detail the nature of the incremental elastic coefficients [1] [2] [3]. This was done in complete generality and without reference to any particular property of elastic symmetry.

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In the present case since the medium is initially isotropic it is obvious that in a state of initial strain it will acquire the symmetry of the initial principal stress system. For incremental deformations it will behave as a medium of orthotropic symmetry. The three planes of elastic symmetry will coincide with the three orthogonal directions of the three principal initial stresses.

An important aspect of the problem resides in the definition of the incremental quantities. The small components of strain and rotation are defined without ambiguity in exactly the same way as in the classical theory of elasticity for small deformations. The choice of a definition for incremental stresses however is not unique. In the earlier work as well as in the present paper we have adopted a particular Cartesian definition. The incremental stress is referred to orthogonal axes which undergo the same local rotation as the medium. Hence the incremental components of stress are linear functions of the strain. In addition in the description of finite strain the elementary procedures of matrix algebra have been used instead of the concepts derived from the tensor theory.

There are considerable advantages in this approach as contrasted with the usual treatment of problems of finite strain. This is well illustrated by the two separate derivations of the incremental elastic coefficients given in the present paper. The first one contained in § 2 uses the elementary approach. The analytical steps are quite simple and illustrate quite clearly, the physical significance of the result. The alternate derivation in § 3 uses methods and concepts which belong essentially to the general tensor theory. A comparison shows that the latter is considerably more involved analytically. It leads to the required simple result only by noticing the cancellation of complicated expression as common factors. This fact should lead to some difficulty in more complicated problems where the simplifications are less apparent. Furthermore, the analytical steps do not provide any insight in the physical nature of the result.

The reader more familiar with the usual approach may well be disturbed by our use of symmetric matrix coefficients to describe finite strain. This is because the product of two such matrices is not necessarily symmetric and does not possess the group property. Another way of looking at this is by saying that if we apply two successive pure deformations we introduce a rotation in addition to a pure deformation. That this constitutes no difficulty is well illustrated by the derivation in § 2 where a small incremental pure strain is superimposed upon a pure finite elongation. Although each of these transformations is a pure strain without rotation the final result contains a rotation. It will be seen that this fact introduces no difficulty in the derivation.

In the last section we have discussed the particular case of second order elasticity for an isotropic medium. This is identical with the problem of deriving first order corrections to the elastic coefficients for the case of small initial strain. The theory was developed in detail by the author [3]), and the second order stress strain relations were derived very simply without using the concept of invariant. It was shown that this requires the introduction of five elastic coefficients. which are readily measurable *).

In § 4 the results of the present paper are combined with the previous second order theory. The first order correction of the elastic coefficients, due to initial strain is evaluated. The result is of special importance for the analysis of initial strain corrections in acoustics. These coefficients when inserted in the general theory of propagation of elastic wave in a medium under initial stress [5]) yield a complete mathematical treatment of the problem.

While the author's theory was developed more than twenty years ago it contains as a particular case what has more recently been referred to as "hypoelasticity". This becomes evident by considering the limiting case of vanishing increments of stress and displacement and dividing all equations by the time differential. In the limit all variables are replaced by time derivatives. The limiting Eulerian equations are formally identical with those obeyed by the author's incremental variables. A time dependent finite deformation may then be considered a continuous sequence of incremental deformations under initial stress. In addition to greater generality there are other advantages inherent in the author's Lagrangian viewpoint. These become apparent when solving specific problems of elastic and viscoelastic stability, in expressing boundary and initial conditions and interpreting the physical significance of the results. Furthermore the author has shown [1]) [2]) that the equations lead directly to the theory of elasticity of the second order and include the practically important case of small strains and large rotations.

*) This is in contrast with the coefficients l, m, n proposed by Murnaghan [4]).

§ 2. *General expressions for the incremental elastic coefficients.* Consider an elastic medium with arbitrary stress-strain relations of isotropic symmetry. Let the original unstressed state be denoted by (a) and the coordinates in this state by XYZ. Three principal stresses S_{11}, S_{22}, S_{33} are applied to the medium. The corresponding deformation produces deformation denoted as state (b). We assume the principal stresses to be directed along the coordinate axes. State (b) is the initial state of stress. Because of isotropy the principal directions of deformation are also parallel with the coordinate axes and the coordinates X, Y, Z in state (b) are

$$\begin{aligned} x &= a_{11}X, \\ y &= a_{22}Y, \\ z &= a_{33}Z. \end{aligned} \tag{2.1}$$

We now produce a state of deformation (c) by superposing a small incremental strain upon state (b). The coordinates in state (c) are

$$\begin{aligned} \xi &= (1 + e_{xx})\, x + e_{xy}y + e_{zx}z, \\ \eta &= e_{xy}x + (1 + e_{yy})\, y + e_{yz}z, \\ \zeta &= e_{zx}x + e_{yz}y + (1 + e_{zz})\, z. \end{aligned} \tag{2.2}$$

The stress increments are linearly related to these small strains by the relations

$$\begin{aligned} s_{11} &= B_{11}e_{xx} + B_{12}e_{yy} + B_{13}e_{zz}, \\ s_{22} &= B_{21}e_{xx} + B_{22}e_{yy} + B_{23}e_{zz}, \\ s_{33} &= B_{31}e_{xx} + B_{32}e_{yy} + B_{33}e_{zz}, \\ s_{23} &= 2Q_1e_{yz}, \\ s_{31} &= 2Q_2e_{zx}, \\ s_{12} &= 2Q_3e_{xy}. \end{aligned} \tag{2.3}$$

These coefficients reflect the orthotropic symmetry of the incremental coefficients in the state of initial stress. This is itself a consequence of the original isotropy of the medium. This writer has shown [2]) that the existence of a strain energy requires that the coefficients satisfy the following relations:

$$\begin{aligned} B_{12} - B_{21} &= S_{22} - S_{11}, \\ B_{23} - B_{32} &= S_{33} - S_{22}, \\ B_{31} - B_{13} &= S_{11} - S_{33}. \end{aligned} \tag{2.4}$$

The coefficients B_{ij} depend of course only on the incremental principal strain and must be measured by applying incremental stresses.

This is however not necessarily the case for the shear coefficients Q_1, Q_2, Q_3 and we will now derive the important property that in general these coefficients may be immediately calculated from the initial state of stress and the corresponding finite strain. In other words these may be expressed in terms of S_{11}, S_{22}, S_{33} and a_{11}, a_{22}, a_{33}.

To show this let us restrict the incremental strain of state (c) to a pure shear component e_{xy}. This is a two dimensional incremental strain where the coordinates in state (c) are

$$
\begin{aligned}
\xi &= x + e_{xy}y, \\
\eta &= e_{xy}x + y, \\
\zeta &= z.
\end{aligned}
\tag{2.5}
$$

Combining relations (2.1) and (2.5) we find:

$$
\begin{aligned}
\xi &= a_{11}X + e_{xy}a_{22}Y, \\
\eta &= e_{xy}a_{11}X + a_{22}Y, \\
\zeta &= a_{33}Z.
\end{aligned}
\tag{2.6}
$$

which defines the transformation from the unstressed state (a) to state (c). We need only consider the first two equations which reduce to a two-dimensional transformation. This transformation may be written

$$
\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}. \tag{2.7}
$$

This represents a solid rotation through an angle θ about the axis, followed by a pure deformation. We may evaluate the coefficients b_{ij} of the pure deformation by identifying the coefficients in the two transformations (2.6) and (2.7). This involves only very simple algebra. Since e_{xy} is small it is possible to express these coefficients by neglecting second order quantities. We find:

$$
\begin{aligned}
b_{11} &= a_{11}, \\
b_{22} &= a_{22}, \\
b_{12} &= \frac{a_{11}^2 + a_{22}^2}{a_{11} + a_{22}} e_{xy}.
\end{aligned}
\tag{2.8}
$$

The pure transformation b_{ij} represents principal strains b_{I} and b_{II}

along two principal directions I and II (fig. 1). The principal direction I lies at an angle α with the x axis. The magnitude of this angle to the first order is

$$\alpha = \frac{b_{12}}{b_{11} - b_{22}}, \tag{2.9}$$

or

$$\alpha = \frac{a_{11}^2 + a_{22}^2}{a_{11}^2 - a_{22}^2} e_{xy}. \tag{2.10}$$

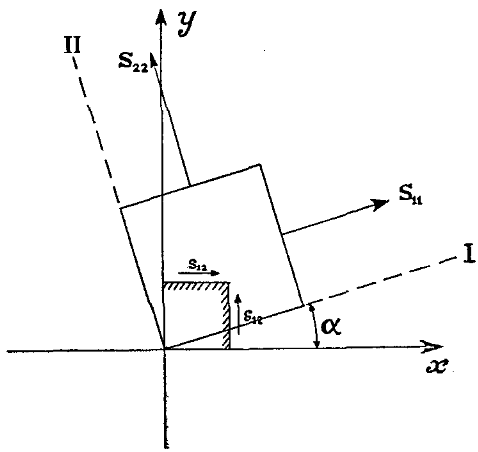

Fig. 1. Stress field after application of an incremental shear strain.

We also find that if we neglect second order quantities the magnitudes of the principal strains are

$$\begin{aligned} b_{\mathrm{I}} &= b_{11} = a_{11}, \\ b_{\mathrm{II}} &= b_{22} = a_{22}. \end{aligned} \tag{2.11}$$

In other words, to the first order the finite strain in state (c) is the same as in state (b) except that the principal directions are rotated through an angle α. Because of the assumption of isotropy and to the same approximation, the state of stress in state (c) must also be the same as in state (b) provided the principal directions are rotated through an angle α. If we resolve this state of stress on the coordinate axes x, y, we obtain a first order shear component

$$s_{12} = (S_{11} - S_{22})\, \alpha. \tag{2.12}$$

Substituting the value (2.10) of α this becomes:

$$s_{12} = (S_{11} - S_{22}) \frac{a_{11}^2 + a_{22}^2}{a_{11}^2 - a_{22}^2} e_{xy}. \tag{2.13}$$

Hence the value of the incremental shear coefficient:

$$2Q_3 = (S_{11} - S_{22}) \frac{a_{11}^2 + a_{22}^2}{a_{11}^2 - a_{22}^2}. \tag{2.14}$$

As announced its value depends only on the finite initial stress and does not involve any incremental physical properties. An identical derivation yields immediately the two other shear coefficients:

$$\begin{aligned} 2Q_1 &= (S_{22} - S_{33}) \frac{a_{22}^2 + a_{33}^2}{a_{22}^2 - a_{33}^2}, \\ 2Q_2 &= (S_{33} - S_{11}) \frac{a_{33}^2 + a_{11}^2}{a_{33}^2 - a_{11}^2}. \end{aligned} \tag{2.15}$$

These expressions are valid without reservation if the initial stresses are unequal. However they become indeterminate if any two of the principal stresses in the formulas are equal. For example let us assume that

$$\begin{aligned} S_{11} &= S_{22}, \\ a_{11} &= a_{22}. \end{aligned} \tag{2.16}$$

Obviously in this case the medium retains its isotropy around the z axis i.e. it becomes transverse isotropic. To find out the limiting value of the coefficients in this case we consider an increment Δa_{11} of the elongation a_{11}. This corresponds to an incremental strain

$$e_{xx} = \Delta a_{11}/a_{11}, \tag{2.17}$$

with $e_{yy} = e_{zz} = 0$.

The incremental stresses are:

$$s_{11} = \Delta S_{11}, \quad s_{22} = \Delta S_{22}, \quad s_{33} = \Delta S_{33}. \tag{2.18}$$

We may write:

$$\begin{aligned} \Delta(S_{11} - S_{22}) &= s_{11} - s_{22}, \\ a_{11}^2 - a_{22}^2 &= 2a_{11}\Delta a_{11}, \end{aligned} \tag{2.19}$$

and expression (2.14) becomes

$$2Q_3 = \frac{s_{11} - s_{22}}{e_{xx}}. \tag{2.20}$$

On the other hand subtracting the difference of the first two equations (2.3) (with $e_{yy} = e_{zz} = 0$) yields:

$$B_{11} - B_{12} = \frac{s_{11} - s_{22}}{e_{xx}} \,. \tag{2.21}$$

From (2.20) and (2.21) we derive:

$$B_{11} = B_{12} + 2Q_3. \tag{2.22}$$

Similarly applying an incremental strain Δa_{22} we find:

$$B_{22} = B_{21} + 2Q_3. \tag{2.23}$$

Also because $S_{11} = S_{22}$ relations (2.4) yield

$$B_{12} = B_{21}. \tag{2.24}$$

Hence

$$B_{11} = B_{22} = B_{12} + 2Q_3. \tag{2.25}$$

Symmetry about the z axis implies the following equalities:

$$B_{13} = B_{23},\ B_{31} = B_{32},\ Q_1 = Q_2 = Q. \tag{2.26}$$

With these results the incremental stress-strain relations (2.3) become

$$\begin{aligned}
s_{11} &= 2Q_3 e_{xx} + B_{12}(e_{xx} + e_{yy}) + B_{13} e_{zz},\\
s_{22} &= 2Q_3 e_{yy} + B_{12}(e_{xx} + e_{yy}) + B_{13} e_{zz},\\
s_{33} &= B_{31}(e_{xx} + e_{yy}) + B_{33} e_{zz},\\
s_{23} &= 2Q e_{yz},\\
s_{31} &= 2Q e_{zx},\\
s_{12} &= 2Q_3 e_{xy}.
\end{aligned} \tag{2.27}$$

We must still verify the last relation (2.4)

$$B_{31} - B_{13} = S_{11} - S_{33}. \tag{2.28}$$

Therefore if $S_{11} = S_{22}$ the stress-strain relations (2.27) contain five incremental coefficients. Furthermore in this case the coefficient Q_3 is defined by the incremental properties of the normal stresses.

§ 3. *Alternate derivation by the method of invariants.* For the purpose of comparison we shall now derive the same results by an alternate method which uses the concept of invariants of the general tensor theory.

Consider the homogeneous transformation

$$\begin{aligned}\xi &= a_{11}X + a_{12}Y + a_{13}Z,\\ \eta &= a_{21}X + a_{22}Y + a_{23}Z,\\ \zeta &= a_{31}X + a_{32}Y + a_{33}Z.\end{aligned} \tag{3.1}$$

The coordinates XYZ of a point in the unstrained original state (a) become ξ, η, ζ after deformation. The transformation (3.1) contains nine independent coefficients defining the superposition of a solid rotation and a pure deformation. It is well known that for an isotropic elastic medium a cube of unit volume in the original state acquires a strain energy W which is a function of three invariants,

$$W = W(I_1, I_2, I_3). \tag{3.2}$$

These invariants are functions of the nine coefficients in the transformation (3.1) and are defined as the coefficients in the polynomial [6] [7],

$$F(\Lambda^2) = \Lambda^6 - I_1\Lambda^4 + I_2\Lambda^2 - I_3, \tag{3.3}$$

where

$$F(\Lambda^2) = \begin{vmatrix} A_1 - \Lambda^2 & B_3 & B_2 \\ B_3 & A_2 - \Lambda^2 & B_1 \\ B_2 & B_1 & A_3 - \Lambda^2 \end{vmatrix}. \tag{3.4}$$

The elements in the determinant are:

$$\begin{aligned}A_1 &= a_{11}^2 + a_{21}^2 + a_{31}^2,\\ A_2 &= a_{12}^2 + a_{22}^2 + a_{32}^2,\\ A_3 &= a_{13}^2 + a_{23}^2 + a_{33}^2,\\ B_1 &= a_{12}a_{13} + a_{22}a_{32} + a_{32}a_{33},\\ B_2 &= a_{13}a_{11} + a_{23}a_{21} + a_{33}a_{31},\\ B_3 &= a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32}.\end{aligned} \tag{3.5}$$

These invariants I_1, I_2, I_3 remain unchanged when the transformation (3.1) represents rigid rotation.

We shall first consider a state (b) of finite initial strain defined by the transformation

$$\begin{aligned}\xi &= a_{11}X,\\ \eta &= a_{22}Y,\\ \zeta &= a_{33}Z.\end{aligned} \tag{3.6}$$

The principal directions of strain and the principal stress are parallel with the coordinate axes. The invariants become

$$
\begin{aligned}
I_1 &= A_1 + A_2 + A_3,\\
I_2 &= A_1A_2 + A_2A_3 + A_1A_2,\\
I_3 &= A_1A_2A_3,
\end{aligned}
\tag{3.7}
$$

with

$$
A_1 = a_{11}^2, \quad A_2 = a_{22}^2, \quad A_3 = a_{33}^2. \tag{3.8}
$$

The principal stresses are:

$$
\begin{aligned}
S_{11} &= \frac{1}{a_{22}a_{33}} \frac{\partial W}{\partial a_{11}},\\
S_{22} &= \frac{1}{a_{33}a_{11}} \frac{\partial W}{\partial a_{22}},\\
S_{33} &= \frac{1}{a_{11}a_{22}} \frac{\partial W}{\partial a_{33}},
\end{aligned}
\tag{3.9}
$$

Evaluating the partial derivatives we find:

$$
\begin{aligned}
S_{11} &= \frac{2a_{11}}{a_{22}a_{33}} \left[\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}(A_2 + A_3) + \frac{\partial W}{\partial I_3} A_2A_3 \right],\\
S_{22} &= \frac{2a_{22}}{a_{33}a_{11}} \left[\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}(A_3 + A_1) + \frac{\partial W}{\partial I_3} A_3A_1 \right],\\
S_{33} &= \frac{2a_{33}}{a_{11}a_{22}} \left[\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}(A_1 + A_2) + \frac{\partial W}{\partial I_3} A_1A_2 \right].
\end{aligned}
\tag{3.10}
$$

For our purpose, as will appear later we shall need the values of the derivatives $\partial W/\partial I_1$, $\partial W/\partial I_2$ and $\partial W/\partial I_3$ expressed by means of the stress components and the finite strains. We must therefore solve the system of equations (3.10) for these derivatives. We find:

$$
\begin{aligned}
\frac{\partial W}{\partial I_1} &= \frac{1}{D}[C_1A_1^2(A_2-A_3)+C_2A_2^2(A_3-A_1)+C_3A_2^3(A_1-A_2)],\\
\frac{\partial W}{\partial I_2} &= \frac{1}{D}[C_1A_1(A_3-A_2)+C_2A_2(A_1-A_3)+C_3A_3(A_2-A_1)],\\
\frac{\partial W}{\partial I_3} &= \frac{1}{D}[C_1(A_2-A_3)+C_2(A_3-A_1)+C_3(A_1-A_2)],
\end{aligned}
\tag{3.11}
$$

with

$$
D = A_3(A_2^2 - A_1^2) + A_2(A_1^2 - A_3^2) + A_1(A_3^2 - A_2^2) \tag{3.12}
$$

and

$$\begin{aligned} C_1 &= a_{11}a_{22}a_{33}\,\frac{S_{11}}{2A_1}\,, \\ C_2 &= a_{11}a_{22}a_{33}\,\frac{S_{22}}{2A_2}\,, \\ C_3 &= a_{11}a_{22}a_{33}\,\frac{S_{33}}{2A_3}\,. \end{aligned} \tag{3.13}$$

Let us further consider a state (c) defined by the transformation

$$\begin{aligned} \xi &= a_{11}X + a_{12}Y, \\ \eta &= a_{22}Y, \\ \zeta &= a_{33}Z. \end{aligned} \tag{3.14}$$

We assume a_{12} to be small. The state (c) is obtained from the state of initial stress (b) by adding a small shear displacement along the x direction (fig. 2). If γ denotes the small shear angle, then

$$a_{12} = a_{22}\gamma. \tag{3.15}$$

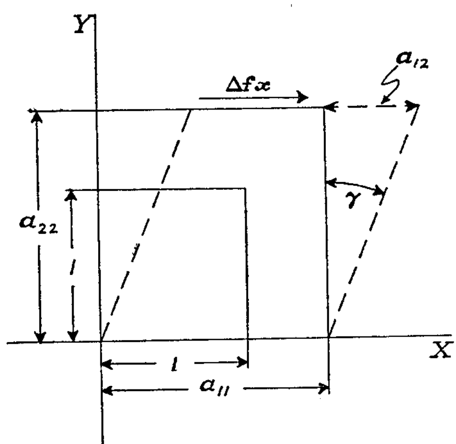

Fig. 2. Shear displacement superposed on a state of initial ~~in~~finite strain.

Denoting the coordinates in the initial state of stress by

$$\begin{aligned} x &= a_{11}X, \\ y &= a_{22}Y, \\ z &= a_{33}Z, \end{aligned} \tag{3.16}$$

the transformation (3.14) may be written as:

$$\begin{aligned} \xi &= x + e_{xy} y - \omega y, \\ \eta &= e_{xy} x + y + \omega x, \\ \zeta &= z, \end{aligned} \tag{3.17}$$

with

$$e_{xy} = -\omega = \tfrac{1}{2}\gamma = \frac{1}{2}\frac{a_{12}}{a_{22}}. \tag{3.18}$$

It defines the superposition of a small shear strain e_{xy} and a rotation ω. The deformation is obtained by applying a force Δf_x per unit area of state (b) to the face perpendicular to the Y axis. This tangential force is associated with an incremental shear stress s_{12} referred to axes which have undergone the same rotation ω as the material. From general equations expressing boundary forces in terms of stresses and derived in earlier work *) we may write

$$\Delta f_x = s_{12} - S_{22}\omega - S_{11} e_{xy}, \tag{3.19}$$

or

$$s_{12} = \Delta f_x + (S_{11} - S_{22})\, e_{xy}. \tag{3.20}$$

We must now calculate Δf_x from the strain energy. The invariants corresponding to the transformation (3.14) become:

$$\begin{aligned} I_1 &= A_1 + A_2 + A_3 + a_{12}^2, \\ I_2 &= A_1 A_3 + A_2 A_3 + A_1 A_2 + A_3 a_{12}^2, \\ I_3 &= A_1 A_2 A_3. \end{aligned} \tag{3.21}$$

The force is derived from the strain energy by the relation

$$a_{11} a_{22} \Delta f_x = \frac{\partial W}{\partial a_{12}} = 2\left(\frac{\partial W}{\partial I_1} + A_3 \frac{\partial W}{\partial I_2}\right) a_{12}. \tag{3.22}$$

Since a_{12} is small we may use in equation (3.22) values of the invariants in which we have substituted $a_{12} = 0$. Hence they are the same as in equations (3.7). The derivatives $\partial W/\partial I_1$ and $\partial W/\partial I_2$ are therefore expressed by equations (3.11). Substituting these values we obtain:

$$\frac{\partial W}{\partial I_1} + A_3 \frac{\partial W}{\partial I_2} = \frac{a_{11} a_{22} a_{33}}{2D} (A_1 A_2 - A_1 A_3 - A_2 A_3 + A_3^2)(S_{11} - S_{22}). \tag{3.23}$$

*) See equation (4.8) of reference 2.

An important simplification is introduced in this expression by noticing the identity

$$D = (A_1 A_2 - A_1 A_3 - A_2 A_3 + A_3^2)(A_1 - A_2). \tag{3.24}$$

Hence,

$$\frac{\partial W}{\partial I_1} + A_3 \frac{\partial W}{\partial I_2} = \frac{1}{2} \frac{a_{11} a_{22} a_{33}}{a_{11}^2 - a_{22}^2} (S_{11} - S_{22}). \tag{3.25}$$

We also write

$$a_{12} = 2a_{22} e_{xy}. \tag{3.26}$$

Substituting these last two expressions into equation (3.22) yields:

$$\Delta f_x = \frac{2a_{22}^2}{a_{11}^2 - a_{22}^2} (S_{11} - S_{22})\, e_{xy}. \tag{3.27}$$

Finally from equation (3.20) we derive:

$$s_{12} = \frac{a_{11}^2 + a_{22}^2}{a_{11}^2 - a_{22}^2} (S_{11} - S_{22})\, e_{xy}. \tag{3.28}$$

This result coincides with equation (2.13) obtained by a different method in the preceding section.

§ 4. *Second order elasticity and the corresponding incremental coefficients.* In the previous discussion we have considered an isotropic medium in the vicinity of a state of finite strain. We shall now turn our attention to a particular case of special interest where the state of initial strain is small but second order terms are taken into account in the expressions for the stress as a function of the initial strain.

We have already considered this problem in a paper published many years ago [3]) and it was shown that, for a body which is isotropic in the unstressed state, the second order stress-strain relations involve five elastic constants. The procedure similar to that followed in the present paper is elementary and does not require the use of invariants.

By using the results obtained in § 2 we may carry the analysis one step further and actually evaluate the incremental coefficients which in this case will be linear functions of the initial strain.

In order to do this let us briefly repeat the analysis of the earlier paper [3]). Consider a unit cube of the elastic medium in the unstressed state and let us apply normal forces τ_{11}, τ_{22}, τ_{33} to the faces. The lengths of the edges then become:

$$
\begin{aligned}
a_{11} &= 1 + \varepsilon_{11}, \\
a_{22} &= 1 + \varepsilon_{22}, \\
a_{33} &= 1 + \varepsilon_{33}.
\end{aligned}
\tag{4.1}
$$

The relation between stresses and deformation including the second order terms in the strain are:

$$
\begin{aligned}
\tau_{11} &= 2\mu\varepsilon_{11} + \lambda\varepsilon + D\varepsilon_{11}^2 + \\
&\quad + F(\varepsilon_{22}^2 + \varepsilon_{33}^2) + F'\varepsilon_{11}(\varepsilon_{22} + \varepsilon_{33}) + G\varepsilon_{22}\varepsilon_{33}, \\
\tau_{22} &= 2\mu\varepsilon_{22} + \lambda\varepsilon + D\varepsilon_{22}^2 + \\
&\quad + F(\varepsilon_{33}^2 + \varepsilon_{11}^2) + F'\varepsilon_{22}(\varepsilon_{33} + \varepsilon_{11}) + G\varepsilon_{33}\varepsilon_{11}, \\
\tau_{33} &= 2\mu\varepsilon_{33} + \lambda\varepsilon + D\varepsilon_{33}^2 + \\
&\quad + F(\varepsilon_{11}^2 + \varepsilon_{22}^2) + F'\varepsilon_{33}(\varepsilon_{11} + \varepsilon_{22}) + G\varepsilon_{11}\varepsilon_{22},
\end{aligned}
\tag{4.2}
$$

where

$$
\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}. \tag{4.3}
$$

The expression for τ_{11} is obtained by writing a general equation including all first and second order terms and then introducing the condition that it remains the same if we interchange ε_{22} and ε_{33}. Equations for τ_{22} and τ_{33} are then derived by cyclic permutation. In addition because of the existence of a strain energy the expression must satisfy the relations:

$$
\begin{aligned}
\frac{\partial\tau_{11}}{\partial\varepsilon_{22}} &= \frac{\partial\tau_{22}}{\partial\varepsilon_{11}}, \\
\frac{\partial\tau_{22}}{\partial\varepsilon_{33}} &= \frac{\partial\tau_{33}}{\partial\varepsilon_{22}}, \\
\frac{\partial\tau_{33}}{\partial\varepsilon_{11}} &= \frac{\partial\tau_{11}}{\partial\varepsilon_{33}}.
\end{aligned}
\tag{4.4}
$$

This requires:

$$
F' = 2F, \tag{4.5}
$$

and relations (4.2) become:

$$
\begin{aligned}
\tau_{11} &= 2\mu\varepsilon_{11} + \lambda\varepsilon + D\varepsilon_{11}^2 + \\
&\quad + F[\varepsilon_{22}^2 + \varepsilon_{33}^2 + 2\varepsilon_{11}(\varepsilon_{22} + \varepsilon_{33})] + G\varepsilon_{22}\varepsilon_{33}, \\
\tau_{22} &= 2\mu\varepsilon_{22} + \lambda\varepsilon + D\varepsilon_{22}^2 + \\
&\quad + F[\varepsilon_{33}^2 + \varepsilon_{11}^2 + 2\varepsilon_{22}(\varepsilon_{33} + \varepsilon_{11})] + G\varepsilon_{33}\varepsilon_{11}, \\
\tau_{33} &= 2\mu\varepsilon_{33} + \lambda\varepsilon + D\varepsilon_{33}^2 + \\
&\quad + F[\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\varepsilon_{33}(\varepsilon_{11} + \varepsilon_{22})] + G\varepsilon_{11}\varepsilon_{22}.
\end{aligned}
\tag{4.6}
$$

These relations now contain five elastic coefficients. Denote by σ_{11}, σ_{22}, σ_{33} the stresses, i.e. the forces per unit area after deformation. Hence we write:

$$\begin{aligned} \tau_{11} &= a_{22}a_{33}\sigma_{11}, \\ \tau_{22} &= a_{33}a_{11}\sigma_{22}, \\ \tau_{33} &= a_{11}a_{22}\sigma_{33}. \end{aligned} \tag{4.7}$$

Consider now a state of initial stress:

$$\begin{aligned} \sigma_{11} &= S_{11}, \\ \sigma_{22} &= S_{22}, \\ \sigma_{33} &= S_{33}. \end{aligned} \tag{4.8}$$

An incremental deformation along the same principal directions will generate stress increments which may be identified with the differentials

$$\begin{aligned} s_{11} &= \mathrm{d}\sigma_{11}, \\ s_{22} &= \mathrm{d}\sigma_{22}, \\ s_{33} &= \mathrm{d}\sigma_{33}. \end{aligned} \tag{4.9}$$

The strain increments are:

$$\begin{aligned} e_{xx} &= \frac{\mathrm{d}\varepsilon_{11}}{a_{11}} = \frac{\mathrm{d}a_{11}}{a_{11}}, \\ e_{yy} &= \frac{\mathrm{d}\varepsilon_{22}}{a_{22}} = \frac{\mathrm{d}a_{22}}{a_{22}}, \\ e_{zz} &= \frac{\mathrm{d}\varepsilon_{33}}{a_{33}} = \frac{\mathrm{d}a_{33}}{a_{33}}. \end{aligned} \tag{4.10}$$

Writing the total differentials of equations (4.7) after taking into account equations (4.8), (4.9) and (4.10) we find:

$$\begin{aligned} &s_{11} + S_{11}(e_{yy} + e_{zz}) = \\ &\quad = \frac{a_{11}}{a_{22}a_{33}} \frac{\partial \tau_{11}}{\partial \varepsilon_{11}} e_{xx} + \frac{1}{a_{33}} \frac{\partial \tau_{11}}{\partial \varepsilon_{22}} e_{yy} + \frac{1}{a_{22}} \frac{\partial \tau_{11}}{\partial \varepsilon_{33}} e_{zz}, \\ &s_{22} + S_{22}(e_{zz} + e_{xx}) = \\ &\quad = \frac{1}{a_{33}} \frac{\partial \tau_{22}}{\partial \varepsilon_{11}} e_{xx} + \frac{a_{22}}{a_{33}a_{11}} \frac{\partial \tau_{22}}{\partial \varepsilon_{22}} e_{yy} + \frac{1}{a_{11}} \frac{\partial \tau_{22}}{\partial \varepsilon_{33}} e_{zz}, \\ &s_{33} + S_{33}(e_{xx} + e_{yy}) = \\ &\quad = \frac{1}{a_{22}} \frac{\partial \tau_{33}}{\partial \varepsilon_{11}} e_{xx} + \frac{1}{a_{11}} \frac{\partial \tau_{33}}{\partial \varepsilon_{22}} e_{yy} + \frac{a_{33}}{a_{11}a_{22}} \frac{\partial \tau_{33}}{\partial \varepsilon_{33}} e_{zz}. \end{aligned} \tag{4.11}$$

Note that by virtue of equations (4.4) the coefficients on the right hand side of these equations constitute a symmetric matrix. Comparing with equations (2.3) we derive the elastic coefficients. Limiting the expression to first order terms we find:

$$\begin{aligned} B_{11} &= (2\mu + \lambda)(1 + \varepsilon_{11} - \varepsilon_{22} - \varepsilon_{33}) + 2D\varepsilon_{11} + 2F(\varepsilon_{22} + \varepsilon_{33}), \\ B_{22} &= (2\mu + \lambda)(1 + \varepsilon_{22} - \varepsilon_{33} - \varepsilon_{11}) + 2D\varepsilon_{22} + 2F(\varepsilon_{33} + \varepsilon_{11}), \\ B_{33} &= (2\mu + \lambda)(1 + \varepsilon_{33} - \varepsilon_{11} - \varepsilon_{22}) + 2D\varepsilon_{33} + 2F(\varepsilon_{11} + \varepsilon_{22}), \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} B_{23} + S_{22} = B_{32} + S_{33} &= \lambda(1 - \varepsilon_{11}) + 2F(\varepsilon_{22} + \varepsilon_{33}) + G\varepsilon_{11}, \\ B_{31} + S_{33} = B_{13} + S_{11} &= \lambda(1 - \varepsilon_{22}) + 2F(\varepsilon_{33} + \varepsilon_{11}) + G\varepsilon_{22}, \\ B_{12} + S_{11} = B_{21} + S_{22} &= \lambda(1 - \varepsilon_{33}) + 2F(\varepsilon_{11} + \varepsilon_{22}) + G\varepsilon_{33}. \end{aligned} \tag{4.13}$$

Note that the cross coefficients satisfy conditions (2.4) as should be.

It remains to evaluate the coefficients Q_1, Q_2, Q_3, for incremental shear. This can be done immediately by applying the results obtained in § 2. According to relations (4.7) the initial stresses are:

$$\begin{aligned} S_{11} &= \frac{\tau_{11}}{a_{22}a_{33}}, \\ S_{22} &= \frac{\tau_{22}}{a_{33}a_{11}}, \\ S_{33} &= \frac{\tau_{33}}{a_{11}a_{22}}. \end{aligned} \tag{4.14}$$

Substituting in (2.14) we find:

$$2Q_3 = \frac{(\tau_{11}a_{11} - \tau_{22}a_{22})(a_{11}^2 + a_{22}^2)}{a_{11}a_{22}a_{33}(a_{11} - a_{22})(a_{11} + a_{22})}. \tag{4.15}$$

To the first order and after cancellation of the common factor $\varepsilon_{11} - \varepsilon_{22}$ in numerator and denominator we derive:

$$2Q_3 = 2\mu + (\lambda+\mu+D-F)(\varepsilon_{11}+\varepsilon_{22}) + (\lambda-2\mu+2F-G)\,\varepsilon_{33}. \tag{4.16}$$

Similarly the other two coefficients are:

$$\begin{aligned} 2Q_1 &= 2\mu + (\lambda+\mu+D-F)(\varepsilon_{22}+\varepsilon_{33}) + (\lambda-2\mu+2F-G)\,\varepsilon_{11}, \\ 2Q_2 &= 2\mu + (\lambda+\mu+D-F)(\varepsilon_{33}+\varepsilon_{11}) + (\lambda-2\mu+2F-G)\,\varepsilon_{22}. \end{aligned} \tag{4.17}$$

Equations (4.12), (4.13), (4.16) and (4.17) express the incremental elastic coefficients of the isotropic elastic medium under initial stress as a function of the first order correction terms in the initial strain.

These coefficients were used in the general theory of propagation of elastic waves in a medium under initial stress [5]. By introducing the values (4.12), (4.13), (4.16) and (4.17) they furnish complete equations by which we can predict the first order effect of the initial stress on the acoustic propagation in the isotropic solid.

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