SURFACE INSTABILITY OF RUBBER IN COMPRESSION

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Summary

The previously derived expressions for the incremental elastic coefficients of an isotropic medium under initial stress are applied to rubbertype elasticity. As a corollary an exact theory is obtained for the surface instability of such material under compression. It is found that in plane strain the incremental properties remain isotropic and are characterized by a single strain-dependent modulus. In three-dimensional strain the elastic properties are found to coincide with those of the elastic medium introduced by Green to illustrate the properties of electromagnetic propagation. The apparent rigidity of the surface as a function of strain is evaluated and is shown to result from the combined effect of the variation of rigidity modulus and a membrane effect due to the initial stress. At a critical compression the two effects act in opposite directions and the apparent surface rigidity vanishes causing incipient instability.

The phenomenon is formally analogous to Rayleigh waves. Attention is also called to the existence of interfacial instability at a surface of discontinuity of two elastic media under initial stress in analogy with Stoneley waves.

§ 1. *Introduction.* The expressions for the incremental coefficients previously derived [7]) yield some remarkable results when applied to rubber. We shall assume that the material obeys the standard stress-strain relations established by Treloar [5]) for rubbertype elasticity. It is found that in two-dimensional strain the incremental elastic properties remain isotropic under finite initial strain. Many earlier results derived for this case by the author are exact solutions of stability in finite elasticity. Our purpose here is to discuss such

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solutions. The case of the instability of the free surface of an elastic half-space which was analysed some time ago in preliminary form [4] [12] [13] is treated here in more detail. The appearance of surface modes of buckling is mathematically analogous to Rayleigh waves.

There are other problems to which the same results are immediately applicable. We shall contend ourselves with a brief mention of two cases. One of these cases is concerned with the instability arising at a surface of discontinuity between two elastic media of different rigidity under initial stress. This corresponds to the mathematically analogous phenomenon represented by Stoneley waves. The existence of such interfacial instability was shown as a particular case of a problem treated elsewhere [11]. The problem is that of instability of a plate embedded in an indefinite medium. At wavelengths which are small compared to the thickness of the plate the buckling degenerates into an instability which is localized at the interface.

The problem of instability of an elastic half-space whose rigidity varies exponentially with depth was also discussed in detail in the more general context of viscoelasticity [12]. The medium is under the combined action of gravity and a horizontal compression. The results of that paper lead to a complete and exact solution for the buckling of the inhomogeneous rubberlike half-space under finite initial strain with an exponential distribution of the elastic modulus.

The author's equations have also been applied by Buckens [14] to the problem of Rayleigh waves propagation in a prestressed semi-infinite medium. Surface instability in this case is established by the vanishing of the surface wave velocity.

A general theory of elasticity for a continuum under initial stress was developed by the author and presented in a series of earlier papers initiated in 1934 [1] [2] [3] [10]. This theory constitutes a rigorous analysis of small incremental deformations in the vicinity of a state of initial stress. An important feature of our treatment is the use of stress components referred to orthogonal axes which undergo the same local rotation as the medium.

The reader should not confuse this exact theory with approximate formulations introducing the assumption that the rotation is large in relation to the strain. While this assumption may easily be incorporated in the general theory as discussed earlier by the author, it is not necessary and no such approximation is introduced here.

More recently the theory was applied to the problem of surface instability of an elastic half-space with an initial compression in a direction parallel with the surface [4] [13]. It was found that the surface may become unstable and exhibit a spontaneous waviness beyond a critical value of the compression stress.

The theory developed in the more recent papers is directly applicable to the case of rubber elasticity. This follows from the property established hereafter that in plane strain a material such as rubber retains its isotropy for incremental deformations in the vicinity of a state of finite strain. We also derive the interesting reciprocal theorem namely that a material which remains isotropic in plane strain must obey the finite stress-strain relation of rubber. The proof is based on formulas for the incremental elastic coefficients which were obtained in the preceding paper [7].

In the more general case a material of rubbertype elasticity becomes anisotropic for incremental strains but retains its isotropy for a state of plane strain increments in each of the planes defined by the principal initial stresses.

Stress-strain relations of this type were already discussed by Green in 1839 with reference to an analogy between transverse elastic waves and the propagation of light in a crystal [8].

It is a remarkable fact that in plane strain analysis rubbertype elasticity leads to incremental stress-strain relations of the same type as for the unstressed medium. They retain their isotropy and the elastic properties are characterized by a single shear modulus which is a simple function of the initial state of strain. The effect of the initial stress appears explicitly only in the equilibrium equations of the stress field through terms which are proportional to the gradient of the rotation.

§ 2 presents a short outline of the previously developed theory of instability of the elastic half-space [4]. The incremental elastic coefficients for rubbertype elasticity are evaluated in § 3. Expressions for the incremental shear coefficient as a function of the finite initial strain are obtained. These coefficients are introduced into the stability theory of the elastic half-space and the results are discussed in § 4.

The critical value of the compression corresponds to a finite strain of the order of one half. In addition to the instability we have also computed the change of shear modulus with the strain and the apparent change of surface rigidity as a function of the compression strain. At the critical strain the apparent surface rigidity vanishes.

§ 2. *General equations of stability.* The general equations for the incremental deformations of a continuum under initial stress were derived by the author in previous publications [1] [2] [3]. More recently they were applied to the stability of the viscoelastic half-space [4] [12]. We shall consider the result for the half-space in the particular case of an incompressible elastic medium.

The surface of the half-space is at $y = 0$ and the y axis is directed positively inside the solid. A uniform compression P is acting in the solid in the x direction, i.e. in a direction parallel with the free surface. The deformation is assumed two-dimensional and is represented by the displacement field u, v in the x, y plane. These are incremental displacements which are zero in the state of uniform compression considered as the initial state. These displacements are associated with incremental strain components

$$e_{xx} = \frac{\partial u}{\partial x}\,, \quad e_{yy} = \frac{\partial v}{\partial y}\,, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \tag{2.1}$$

and a rotation

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{2.2}$$

The incremental stress components s_{11}, s_{22}, s_{12} are defined relative to axes which rotate locally through an angle ω. We have shown that they satisfy the equilibrium conditions *):

$$\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P\frac{\partial \omega}{\partial y} &= 0, \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P\frac{\partial \omega}{\partial x} &= 0. \end{aligned} \tag{2.3}$$

The equations relating the incremental stress deviator to the strain

*) See equation (3.2) [4] or the more general equations to be found in the earlier work by the author [2] [3].

are written:

$$\begin{aligned} s_{11} - s &= 2\mu e_{xx}, \\ s_{22} - s &= 2\mu e_{yy}, \\ s_{12} &= 2\mu e_{xy}. \end{aligned} \tag{2.4}$$

We shall see that these stress-strain relations are the correct ones for the particular rubber-like material considered hereafter. It is found that the incremental properties exhibit a two-dimensional isotropy as implied in relations (2.4) with an incremental shear modulus μ which is a function of the initial compression.

The boundary conditions express the condition that the surface $y = 0$ is a free surface. They are *):

$$\begin{aligned} e_{xy} &= 0, \\ s_{22} &= 0. \end{aligned} \tag{2.5}$$

We must also add the condition of incompressibility

$$e_{xx} + e_{yy} = 0. \tag{2.6}$$

These equations may be considered as three equations for the quantities u, v, s with the boundary conditions (2.5). A solution which is sinusoidal along x and decreases exponentially with depth is

$$\begin{aligned} u &= -\sin lx(A\,\mathrm{e}^{-ly} + Ck\,\mathrm{e}^{-lky}), \\ v &= -\cos lx(A\,\mathrm{e}^{-ly} + C\,\mathrm{e}^{-lky}). \end{aligned} \tag{2.7}$$

We have put

$$\begin{aligned} k &= \sqrt{\frac{1-\zeta}{1+\zeta}}, \\ \zeta &= \frac{P}{2\mu}. \end{aligned} \tag{2.8}$$

The problem was solved in reference 4. The solution of the present case is derived by putting Poisson's ratio equal to $\nu = \frac{1}{2}$ in the result. This leads to the characteristic equation:

$$\zeta^3 + 2\zeta^2 - 2 = 0. \tag{2.9}$$

When this equation is satisfied a non-zero solution exists for the incremental displacements u and v. This corresponds therefore to

*) See equations (3.12) [4].

an instability in the sense that the medium is in neutral equilibrium for infinitesimal increments of deformation. The positive root of the cubic is

$$\zeta = \frac{P}{2\mu} = 0.839. \tag{2.10}$$

For a given value of μ this defines a critical compressive load P. For this compression the surface may develop sinusoidal waves. The characteristic equation being independent of the wavelength all wavelengths are equally unstable. The physical significance of this will be discussed below.

§ 3. *Incremental elastic coefficients for rubber under finite initial strain.* We shall now proceed to show that the incremental stress-strain relations (2.4) are the correct ones for rubber and we shall derive the dependence of μ upon the initial strain.

The finite stress-strain relations for a rubberlike material were derived by Treloar *) from statistical thermodynamics. He found that the isothermal free energy of rubber is

$$W = \tfrac{1}{2}NkT(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \tag{3.1}$$

per unit unstrained volume. A unit cube with principal directions of strain along its edges becomes a parallelepiped of sides λ_1, λ_2, λ_3 after deformation. These quantities are therefore the principal elongations. Boltzman's constant is represented by k, N denotes the number of molecular polymer chains per unit volume and T is the absolute temperature. The quantity W also represents the classical strain energy of the theory of elasticity. We put

$$\mu_0 = NkT, \tag{3.2}$$

then

$$W = \tfrac{1}{2}\mu_0(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3). \tag{3.3}$$

It will be shown that μ_0 is the shear modulus of rubber for small deformations near the original unstressed state.

It is assumed also that the rubber may be treated as incompressible. Hence the following constraint for constant volume must be satisfied:

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{3.4}$$

*) An outline of the results based on his earlier work is given by Treloar [5]. Further experimental confirmation was recently obtained by Cifferi and Flory [6].

Suppose that we apply stresses S_{11}, S_{22} in two perpendicular directions, the third principal stress being zero. Because of incompressibility the elongation λ_3 may be considered as a function of λ_1 and λ_2. Hence W is also a function of λ_1 and λ_2:

$$W = \tfrac{1}{2}\mu_0\left(\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1\lambda_2} - 3\right). \tag{3.5}$$

The stresses are:

$$\begin{aligned} S_{11} &= \frac{1}{\lambda_2\lambda_3}\frac{\partial W}{\partial \lambda_1} = \lambda_1 \frac{\partial W}{\partial \lambda_1}, \\ S_{22} &= \frac{1}{\lambda_1\lambda_3}\frac{\partial W}{\partial \lambda_2} = \lambda_2 \frac{\partial W}{\partial \lambda_2}, \\ S_{33} &= 0. \end{aligned} \tag{3.6}$$

We may superimpose a hydrostatic stress C without producing strain. The stresses then become:

$$\begin{aligned} S_{11} &= \lambda_1 \frac{\partial W}{\partial \lambda_1} + C, \\ S_{22} &= \lambda_2 \frac{\partial W}{\partial \lambda_2} + C, \\ S_{33} &= C. \end{aligned} \tag{3.7}$$

Substituting the value (3.5) for W we find:

$$S_{11} - S_{22} = \mu_0(\lambda_1^2 - \lambda_2^2). \tag{3.8}$$

Similarly we derive:

$$\begin{aligned} S_{22} - S_{33} &= \mu_0(\lambda_2^2 - \lambda_3^2), \\ S_{33} - S_{11} &= \mu_0(\lambda_3^2 - \lambda_1^2). \end{aligned} \tag{3.9}$$

The incremental stresses s_{11}, s_{22}, s_{33} are obtained by differentiating these expressions.

$$s_{11} - s_{22} = \mathrm{d}S_{11} - \mathrm{d}S_{22} = 2\mu_0(\lambda_1\,\mathrm{d}\lambda_1 - \lambda_2\,\mathrm{d}\lambda_2). \tag{3.10}$$

Since

$$\begin{aligned} e_{xx} &= \mathrm{d}\lambda_1/\lambda_1, \\ e_{yy} &= \mathrm{d}\lambda_2/\lambda_2, \end{aligned} \tag{3.11}$$

we have

$$s_{11} - s_{22} = 2\mu_0(\lambda_1^2 e_{xx} - \lambda_2^2 e_{yy}), \tag{3.12}$$

and similarly

$$\begin{aligned} s_{22} - s_{33} &= 2\mu_0(\lambda_2^2 e_{yy} - \lambda_3^2 e_{zz}), \\ s_{33} - s_{11} &= 2\mu_0(\lambda_3^2 e_{zz} - \lambda_1^2 e_{xx}). \end{aligned} \tag{3.13}$$

The shear coefficients $Q_1 Q_2 Q_3$ follow directly from equations (2.14) and (2.15) of the preceding paper [7]) by putting

$$\begin{aligned} a_{11} &= \lambda_1, \\ a_{22} &= \lambda_2, \\ a_{33} &= \lambda_3, \end{aligned} \tag{3.14}$$

and substituting the values (3.8), (3.9) for the initial stresses. This yields the following relations for the shear stresses:

$$\begin{aligned} s_{23} &= \mu_0(\lambda_2^2 + \lambda_3^2)\, e_{yz}, \\ s_{31} &= \mu_0(\lambda_3^2 + \lambda_1^2)\, e_{zx}, \\ s_{12} &= \mu_0(\lambda_1^2 + \lambda_3^2)\, e_{xy}. \end{aligned} \tag{3.15}$$

These expressions determine the significance of μ_0 as the shear modulus in the vicinity of the original unstressed state, i.e. for $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

Let us now consider a two-dimensional incremental strain in the x, y plane, i.e. we put:

$$e_{zz} = 0. \tag{3.16}$$

Because of incompressibility we may write:

$$e_{xx} + e_{yy} = 0. \tag{3.17}$$

Taking into account this last condition the incremental stress components s_{11}, s_{22} and s_{12} in the x, y plane are related to the strain by the relations:

$$\begin{aligned} s_{11} - s_{22} &= 2\mu_0(\lambda_1^2 + \lambda_2^2)\, e_{xx}, \\ s_{22} - s_{11} &= 2\mu_0(\lambda_1^2 + \lambda_2^2)\, e_{yy}, \\ s_{12} &= \mu_0(\lambda_1^2 + \lambda_2^2)\, e_{xy}. \end{aligned} \tag{r.18}$$

If we take into account the condition (3.17) for incompressibility these relations are equivalent to

$$\begin{aligned} s_{11} - s &= 2\mu e_{xx}, \\ s_{22} - s &= 2\mu e_{yy}, \\ s_{12} &= 2\mu e_{xy}, \end{aligned} \tag{3.19}$$

with

$$\mu = \tfrac{1}{2}\mu_0(\lambda_1^2 + \lambda_2^2). \tag{3.20}$$

These relations coincide with equations (2.4). Results of the instability theory are therefore rigorously applicable to rubberlike materials whose finite strain energy is given by (3.3).

We note the remarkable property that for two-dimensional incremental strain in any of the planes of symmetry of the initial stress the material remains isotropic. It is then defined by a single incremental modulus μ whose magnitude depends on the initial strain. There are three such moduli corresponding to each plane of symmetry of the initial stress.

The existence of anisotropic media which exhibit stress-strain relations with properties which have just been described were derived in the classical literature on the theory of elasticity by Green[8] *). He discusses the analogy between the propagation of transverse elastic waves in such media and the propagation of light in crystals. Green's work refers of course to initially unstressed media. Therefore his conclusions on wave propagation do not apply to the present case for which the propagation equations contains additional terms arising from the initial stresses [10]. The additional terms due to the initial stress are the same as in equations (2.3). The stress-strain relations discussed by Green allow for compressibility of the medium. For the particular case of incompressibility they become identical with our relations (3.12), (3.13) and (3.15). If the initial finite strain is two-dimensional we put:

$$\lambda_3 = 1 \tag{3.21}$$

and

$$\lambda_1 = \lambda, \;\; \lambda_2 = 1/\lambda. \tag{3.22}$$

The initial stresses parallel with the plane of deformation satisfy the relation

$$S_{11} - S_{22} = \mu_0(\lambda^2 - 1/\lambda^2). \tag{3.23}$$

If this deformation is produced by a single force applied in the x direction we put $S_{22} = 0$ and

$$S_{11} = \mu_0(\lambda^2 - 1/\lambda^2). \tag{3.24}$$

The incremental shear modulus for this case is

$$\mu = \tfrac{1}{2}\mu_0(\lambda^2 + 1/\lambda^2). \tag{3.25}$$

*) A brief discussion of Green's results can be found in Love's Theory of Elasticity [9].

If the initial strain is three-dimensional and results from the application of a uniaxial stress S_{11} in the x direction we put

$$S_{22} = S_{33} = 0 \tag{3.26}$$

and

$$\begin{aligned} \lambda_1 &= \lambda, \\ \lambda_2 &= \lambda_3 = 1/\lambda^{\frac{1}{2}}. \end{aligned} \tag{3.27}$$

The finite stress-strain relation is then

$$S_{11} = \mu_0(\lambda^2 - 1/\lambda) \tag{3.28}$$

and the incremental shear modulus

$$\mu = \tfrac{1}{2}\mu_0(\lambda^2 + 1/\lambda). \tag{3.29}$$

It is also of interest to examine the condition required for a material to remain isotropic for incremental stresses in plane strain. Consider a plane strain deformation parallel with the x, y plane, corresponding to the elongations (3.21) and (3.22). Assume that it is produced by a stress S_{11} while $S_{22} = 0$. Then S_{11} is a function of λ, the elongation in the x direction,

$$S_{11} = S_{11}(\lambda). \tag{3.30}$$

The incremental normal stress is

$$s_{11} = \lambda \frac{\mathrm{d}S_{11}}{\mathrm{d}\lambda} e_{xx}. \tag{3.31}$$

On the other hand applying equation (2.13) of the previous paper [7]) the incremental shear stress is

$$s_{12} = S_{11} \frac{\lambda^4 + 1}{\lambda^4 - 1} e_{xy}. \tag{3.32}$$

The condition of isotropy for incremental stresses requires

$$\lambda \frac{\mathrm{d}S_{11}}{\mathrm{d}\lambda} = 2S_{11} \frac{\lambda^4 + 1}{\lambda^4 - 1}. \tag{3.33}$$

This is a differential equation for $S_{11}(\lambda)$ whose solution is expression (3.24). Hence the condition of incremental isotropy in two-dimensional strain requires that the material obeys the finite stress-strain relation of rubber-type elasticity *).

This property is undoubtedly related to the statistical thermodynamics of long chain molecules.

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*) It can be shown that a Mooney material satisfies the condition of incremental isotropy in the three principal planes of finite triaxial initial strain.

§ 4. *Discussion of surface instability.* We have shown in the preceding section that for rubber-type elasticity the incremental stress-strain relations are given by equations (2.4). Hence it is possible to apply the previously developed theory for surface instability and in particular the characteristic equation (2.10).

There are two significant particular cases to consider. In one case a state of uniform initial compression P is attained by two-dimensional strain i.e. by maintaining constant the dimension in a certain direction perpendicular to the compression. In the other the compression P is applied as a uniaxial stress with no lateral restraint.

Let us consider first the case of two-dimensional compression. Expression (3.25) for the incremental shear modulus is a function of λ and invariant when transforming λ into $1/\lambda$. It is a minimum for $\lambda = 1$.

The surface of the rubber is unstable when the characteristic equation (2.10) is verified. We note that this equation involves two variables, the compressive stress P and the incremental modulus μ which both increase with the deformation. It is therefore not evident a priori that there exists a physical solution of the equation i.e. a critical load. That this is actually the case is verified as follows: From equation (3.24) the compression $P = -S_{11}$ is

$$P = \mu_0(1/\lambda^2 - \lambda^2). \tag{4.1}$$

Introducing this value of P and expression (3.25) for μ into equation (2.10) we find:

$$\zeta = \frac{P}{2\mu} = \frac{1 - \lambda^4}{1 + \lambda^4} = 0.839. \tag{4.2}$$

This equation is satisfied for

$$\lambda = 0.543. \tag{4.3}$$

Hence for a compressive strain slightly greater than one half the surface of the rubber is unstable. The critical value of the compression is

$$P = 3.08\mu_0. \tag{4.4}$$

Since all wavelengths are equally unstable the instability should appear in the form of wrinkles whose size is controlled by several factors not considered in the theory. Such factors are for example the inhomogeneities and surface irregularities of the medium. The size of the wrinkles is also influenced by second order effects associated with large local strains or large slopes present in the wrinkles. Also as pointed out below for a medium which is restrained at finite depth instability occurs only for wavelengths smaller than a certain limiting value.

Expression (3.25) for the incremental shear modulus indicates that the rigidity increases with the deformation. On the other hand the apparent surface rigidity must vanish at the critical value (4.4) of the compression. It is of interest to evaluate the variation of the surface rigidity as a function of the compression by considering a normal sinusoidally distributed load

$$q = q_0 \cos lx. \tag{4.5}$$

This is a force perpendicular to the initial plane surface and of magnitude q per unit initial area. We should remember that we have considered a two-dimensional strain in the plane so that the surface load (4.5) corresponds to constant values along lines perpendicular to this plane. The surface deflection is also sinusoidal and represented by

$$v = V \cos lx. \tag{4.6}$$

The relation between q_0 and V was evaluated in a previous paper [4]) and found to be

$$V = \frac{q_0}{2\mu l\varphi}\,, \tag{4.7}$$

where

$$\varphi = \frac{1}{\zeta}\,[k(1+\zeta)^2 - 1]. \tag{4.8}$$

In the unstressed state, i.e. for $P = \zeta = 0$, we derive $\varphi = 1$ and $\mu = \mu_0$. Hence for this case the normal deflection of the surface is

$$V = \frac{q_0}{2\mu_0 l}\,. \tag{4.9}$$

When a compression P is superimposed the vertical surface deflection is obtained by the same formula as for the initially unstressed isotropic medium except for the replacement of μ_0 by an "effective" rigidity modulus $\mu\varphi$. By Fourier analysis it is seen that in the present case of plane strain the conclusion holds for any arbitrary distribution of the surface load along x. Note that we refer here only to the normal deflection under normal loads and that the procedure does not extend to the simultaneous evaluation of tangential displacements at the surface.

The variation of the "effective" surface rigidity modulus with the initial stress is represented by the factor

$$\frac{\mu\varphi}{\mu_0} = \tfrac{1}{2}(\lambda^2 + 1/\lambda^2)\,\varphi. \tag{4.10}$$

Numerical values of this factor as a function of λ are shown in table I. Values of the relative rigidity modulus μ/μ_0 are also given.

TABLE I

The incremental modulus μ and the effective surface rigidity factor $\mu\varphi/\mu_0$ as a function of the initial strain λ.

λ	μ/μ_0	$\mu\varphi/\mu_0$
2.00	2.12	2.27
1.50	1.35	1.52
1.20	1.06	1.19
1.00	1.00	1.00
.90	1.02	.90
.80	1.09	.76
.70	1.27	.57
.60	1.57	.28
.543	1.84	0

Values $\lambda > 1$ correspond to an initial state of tension and $\lambda < 1$ to an initial compression. It can be seen that the effective surface rigidity decreases with increasing compression and vanishes at the critical value of λ given by (4.3). At this critical value we find $\varphi = 0$ and this equation has the same real root as the cubic (2.9). Under tension the effective surface rigidity increases but is somewhat larger than that due to the increase of the rigidity modulus alone. The apparent stiffening is due to the initial tension which tends to act like a membrane stress. This effect acts in the opposite way when the initial stress is a compression. The rigidity μ is of course a minimum equal to μ_0 for $\lambda = 1$, i.e. when there is no initial stress.

It is of interest to note the value

$$k = \lambda^2. \tag{4.11}$$

Hence k is always real *in the solutions* (2.7) *and the deformation* vanishes exponentially with the distance from the surface.

The second case where P is a uniaxial compression is solved by using the same characteristic equation except that μ is now given by expression (3.29). While the compression P is obtained from equation (3.28) where $P = -S_{11}$. Hence:

$$P = \mu_0(1/\lambda - \lambda^2). \tag{4.12}$$

The characteristic equation becomes:

$$\zeta = \frac{P}{2\mu} = \frac{1 - \lambda^3}{1 + \lambda^3} = 0.839. \tag{4.13}$$

The root λ of this equation is:

$$\lambda = 0.444. \tag{4.14}$$

The critical value of the compression is:

$$P = 2.05\mu_0. \tag{4.15}$$

Surface instability in this case requires a higher compressional strain λ than for plane strain deformation but the critical compression is smaller. The value of k is $k = \lambda^{\frac{3}{2}}$. Again it is always real.

In both cases discussed here the value of k is $k = 0.295$ at the critical compression. Referring to the solution (2.7) which contains an exponential factor exp $(-lky)$ we see *that the disturbance due* to surface instability does not penetrate to a depth greater than approximately three times the wavelengths. In a slab of finite thickness the stability of larger wavelengths will be influenced by the thickness. As a consequence, if one of the boundaries is restrained, instability at the other boundary will only appear for wavelengths which are smaller than about one third the thickness.

The general case of arbitrary initial stress presents no difficulty since the characteristic equation (2.10) remains unchanged.

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