EXACT THEORY OF BUCKLING OF A THICK SLAB *)

by M. A. BIOT

New York, U.S.A.

Summary

The incremental elastic coefficients for rubbertype elasticity are inserted in some earlier theories by the author [1]) leading to an exact theory for the buckling of a thick slab in finite strain. The critical compression as a function of the wave length shows a continuous variation. The buckling is of the bending type at large wavelengths and becomes a shear type instability for shorter slabs. In the limiting case of vanishing wavelength the buckling degenerates into a surface instability. The formal behavior is analogous to the transition from bending waves to Rayleigh waves in the vibration of elastic plates. The range of validity of the classical Euler theory is discussed and stress distributions across the thickness are evaluated. The problem is also treated by an approximate variational procedure and it is shown that a very simple bending-sheartype deformation yields a remarkably accurate formula through the complete range of wavelengths.

§ 1. *Introduction.* The equations of elasticity for a medium under initial stress were applied by the author in 1938 to the buckling of a thick slab [1]) **). As can be seen from the recent development [2]) this early result is also an exact solution for the stability problem of a slab of rubber-like material in finite elasticity. Since the theory is mathematically very simple and is also not handicapped by approximations it becomes possible to clarify the physics of the phenomenon and to evaluate the range of validity of well known classical approximate treatments such as the Euler theory and variational procedures.

---

*) This work was supported by the Air Force Office of Scientific Research under contract No. AF 49(638)–837. The contents of this paper were included in AFOSR Report No. 1770 of November 1961.

**) The problem is also a particular case of the buckling of an embedded slab analyzed more recently in the context of viscoelasticity [5]) [6]).

The Euler formula is compared with the exact results and found to be valid down to slenderness ratios corresponding to a length equal to ten times the thickness in the compressed state.

As the slab becomes shorter the nature of the instability exhibits a gradual transition from a bending-type buckling to a shear buckling. The latter starts to appear at slenderness ratios for which results depart from the Euler theory. Finally for vanishing shortness the shear buckling itself degenerates asymptotically into a surface buckling already analyzed previously [2]). The phenomenon is formally analogous to the behavior of bending waves in a slab, which at very short wavelengths degenerate into Rayleigh waves propagating at each of the free surfaces.

Variational principles developed earlier [3]) [4]) for the theory of elasticity of a continuum under initial stress are applied to the problem of slab buckling in the last section. The introduction of a very simple, two parameter approximation for the buckling mode, into the variational process yields a remarkably accurate result which is valid throughout the complete range of slenderness ratios from zero to infinity.

§ 2. *The stability theory.* A rubber slab originally of length $\mathscr{L}_0$ and thickness h_0 is shown in fig. 1. The width in the direction perpendicular to the figure is infinite. The slab is then compressed to a length $\mathscr{L}$ and a width h by a compressive stress P in the direction of its axis (fig. 2). The compression is exerted by two frictionless and rigid blocs a and b. Overall slippage between the rubber and the blocs is prevented by attaching the rubber to the blocs at points A and B on the axis.

We shall investigate the stability of this slab for plane strain perturbations in the plane of the figure chosen as the x, y plane. The x axis is on the axis of the slab and the faces are located at ordinates $y = \pm h/2$. The components of the perturbation displacements in the x, y plane are denoted by u, v. The strain components are

$$e_{xx} = \frac{\partial u}{\partial x}\,, \quad e_{yy} = \frac{\partial v}{\partial y}\,, \tag{2.1}$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \tag{2.2}$$

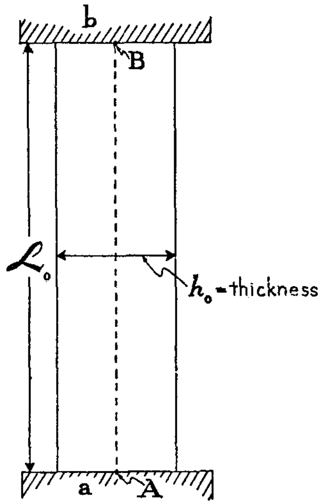

Fig. 1. Rubber slab in the original unstressed state. The slab is viewed across the thickness.

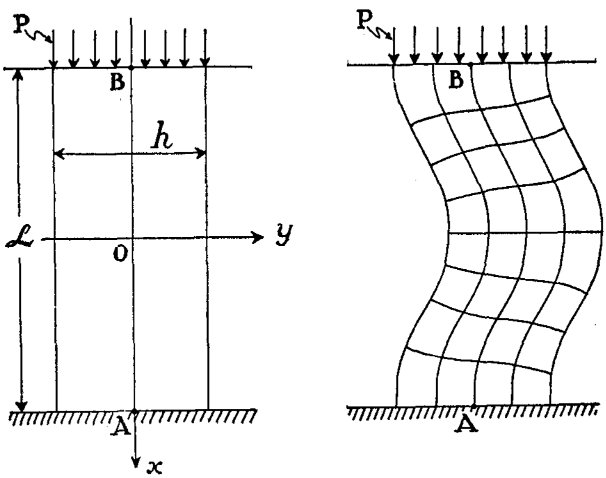

Fig. 2. Rubber slab in the initial state of compression and in the buckled state. The slab is viewed across the thickness.

The local rotation is

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{2.3}$$

The incremental stresses referred to locally rotated axes satisfy the equations

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P\frac{\partial \omega}{\partial y} = 0,$$
$$\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P\frac{\partial \omega}{\partial x} = 0. \tag{2.4}$$

We have shown [2]) that for a rubbertype material in a state of initial strain these incremental stresses are related to the perturbation strain components by the relations

$$\begin{aligned} s_{11} - s &= 2\mu e_{xx}, \\ s_{22} - s &= 2\mu e_{yy}, \\ s_{12} &= 2\mu e_{xy}. \end{aligned} \tag{2.5}$$

The incremental modulus μ depends on the initial strain. The condition of incompressibility is

$$e_{xx} + e_{yy} = 0. \tag{2.6}$$

For the state of initial strain we may consider two possibilities. It may itself be a state of plane strain derived by compressing the slab in the x direction and restraining any change of length in a direction perpendicular to the x, y plane. In this case the finite stress-strain relation is given by [2])

$$P = \mu_0(1/\lambda^2 - \lambda^2) \tag{2.7}$$

and the incremental modulus is [2])

$$\mu = \tfrac{1}{2}\mu_0(\lambda^2 + 1/\lambda^2). \tag{2.8}$$

The quantity λ ($\lambda < 1$) measures the finite compressive strain. In this case the length and thickness of slab in the compressed state are respectively

$$\mathscr{L} = \mathscr{L}_0\lambda, \quad h = h_0/\lambda. \tag{2.9}$$

On the other hand the initial strain may correspond to a case where the slab is free to expand perpendicularly to the x, y plane. In this case the only stress is the compression P and the finite stress-strain relation is [2])

$$P = \mu_0(1/\lambda - \lambda^2), \tag{2.10}$$

with an incremental modulus [2])

$$\mu = \tfrac{1}{2}\mu_0(\lambda^2 + 1/\lambda). \tag{2.11}$$

The length and width of the plate become

$$\mathscr{L} = \mathscr{L}_0\sqrt{\lambda}, \quad h = h_0/\sqrt{\lambda}. \tag{2.12}$$

The parameter μ_0 is the shear modulus in the unstressed state ($\lambda = 1$). The condition of incompressibility of the material is satisfied by putting

$$u = -\,\partial\phi/\partial y, \quad v = \partial\phi/\partial x, \tag{2.13}$$

Introducing this function ϕ into equations (2.1), (2.2), (2.3), (2.4) and (2.5) after suitable eliminations leads to the equations

$$\begin{aligned} \frac{\partial s}{\partial x} - \left(\mu + \frac{P}{2}\right)\frac{\partial}{\partial y}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) &= 0, \\ \frac{\partial s}{\partial y} + \left(\mu - \frac{P}{2}\right)\frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) &= 0. \end{aligned} \tag{2.14}$$

Elimination of s yields

$$\left(\mu - \frac{P}{2}\right)\frac{\partial^4\phi}{\partial x^4} + 2\mu\frac{\partial^4\phi}{\partial x^2\partial y^2} + \left(\mu + \frac{P}{2}\right)\frac{\partial^4\phi}{\partial y^4} = 0. \tag{2.15}$$

This equation may also be written with factorized operator

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[\left(\mu - \frac{P}{2}\right)\frac{\partial^2\phi}{\partial x^2} + \left(\mu + \frac{P}{2}\right)\frac{\partial^2\phi}{\partial y^2}\right] = 0. \tag{2.16}$$

We are looking for a solution of equations (2.14) which is sinusoidal along x. Such a solution is readily found by making use of equation (2.16). We find:

$$\begin{aligned} \phi &= \frac{1}{l^2}\,(C_2 \cosh ly + C_2 \cosh kly) \sin lx, \\ s &= C_2 Pk \sin kly \cos lx. \end{aligned} \tag{2.17}$$

where C_1 and C_2 denote undetermined constants and

$$k^2 = \frac{\mu - P/2}{\mu + P/2}\,. \tag{2.18}$$

We shall now introduce the boundary conditions that the faces of the slab are stress free. We have shown that this is expressed by the equations

$$\begin{aligned} e_{xy} &= 0, \\ s + 2\mu e_{yy} &= 0, \end{aligned} \tag{2.19}$$

to be verified at $y = \pm h/2$. Because of the symmetry of the solution (2.17) this needs only be verified at one of the faces. Substituting the solution (2.17) into the boundary conditions (2.19) we obtain

$$\begin{aligned} 2C_1 \cosh \gamma + C_2(1 + k^2) \cosh k\gamma = 0, \\ C_1(1 + k^2) \sinh \gamma + 2C_2 k \sinh k\gamma = 0, \end{aligned} \tag{2.20}$$

with

$$\gamma = \tfrac{1}{2} lh. \tag{2.21}$$

Eliminating the constants C_1 and C_2 yields the characteristic equation

$$4k \tanh k\gamma - (1 + k^2)^2 \tanh \gamma = 0. \tag{2.22}$$

This is a relation between k and γ which corresponds to instability.

§ 3. *Numerical discussion of the characteristic equations.* We put

$$\zeta = \frac{P}{2\mu}, \tag{3.1}$$

hence

$$k^2 = \frac{1 - \zeta}{1 + \zeta}.$$

Equation (2.22) may then be written

$$(1 + \zeta)^2 k \tanh k\gamma - \tanh \gamma = 0. \tag{3.2}$$

Since ζ is a function of the compression P, and γ is a parameter representing the wavelength, equation (3.2) represents a functional relationship between the buckling wavelength and the axial compression in the slab.

Equation (3.2) is a particular case of results obtained in earlier work by the author [1]) [5]) [6]) *).

The value of ζ determined by equation (3.2) has been plotted as a function of γ in fig. 3.

*) Due to a misprint the characteristic equation derived in [1]) should read

$$\frac{1 + (1 - 2\nu)(\sigma/2G)^2}{(1 + \sigma/2G)^2} = \frac{k \tanh \frac{1}{2}akt}{\tanh \frac{1}{2}at}.$$

For $\nu = \frac{1}{2}$ it becomes identical with (3.2) above.

By putting $\gamma = \infty$ in equation (3.2) we obtain the equation

$$(1 + \zeta)^2 k - 1 = 0, \tag{3.3}$$

whose real root is the asymptotic value

$$\zeta = 0.839. \tag{3.4}$$

This asymptotic case corresponds to the surface instability of the half space ($h = \infty$) already discussed previously [2] [5] [6].

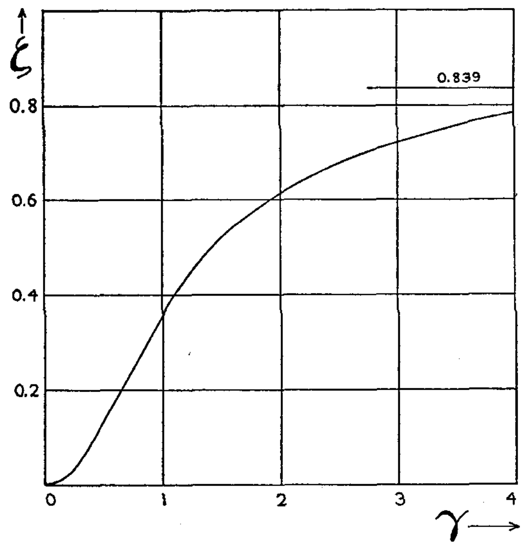

Fig. 3. Value of ζ as a function of γ under buckling conditions as determined by the characteristic equation (3.2).

For small values of γ we may expand the hyperbolic functions of the characteristic equation (3.2) in power series. Including terms up to the third power in γ we may write

$$\begin{aligned} \tanh k\gamma &= k\gamma - \tfrac{1}{3}k^3\gamma^3, \\ \tanh \gamma &= \gamma - \tfrac{1}{3}\gamma^3. \end{aligned} \tag{3.5}$$

Substituting these approximations in equation (3.2) we obtain

$$\zeta = \tfrac{1}{3}\gamma^2(2 - \zeta), \tag{3.6}$$

or

$$\zeta = \frac{2\gamma^2}{3 + \gamma^2}\,. \tag{3.7}$$

We shall discuss the physical significance of this result in the next section. The parameter ζ is essentially a function of the finite initial strain. If the initial compression is one of plane strain we express P and μ by means of equations (2.7) and (2.8), hence

$$\zeta = \frac{1 - \lambda^4}{1 + \lambda^4}\,. \tag{3.8}$$

If the initial compression allows for free expansion in a direction normal to the load we must use equations (2.10) and (2.11). Hence

$$\zeta = \frac{1 - \lambda^3}{1 + \lambda^3}\,. \tag{3.9}$$

Since

$$0 < \lambda < 1, \tag{3.10}$$

we also conclude that in either case

$$0 < \zeta < 1. \tag{3.11}$$

Therefore k remains a real quantity.

The solution obtained here represents a sinusoidal bending of the slab in the x, y plane. If $\mathscr{L}$ denotes the wavelength we may write

$$\gamma = \pi h/\mathscr{L}. \tag{3.12}$$

Hence γ is inversely proportional to the ratio of buckling wavelength to the thickness of the slab. We may think of the solution as representing the buckling of a slab of length $\mathscr{L}$ compressed between two rigid and frictionless surfaces but attached to these surfaces at two points A and B on the axis as shown in fig. 2.

In the state of initial compression the slenderness corresponding to values $\gamma = 1$, $\gamma = 2$, and $\gamma = 3$ is illustrated in fig. 4. The general behavior of the slab may be described by stating that the compressive load increases gradually as the wavelength decreases. As the compression reaches a critical value corresponding to $\zeta = 0.839$ the wavelength becomes vanishingly small and the buckling degenerates into surface ripples.

§ 4. *Physical significance of the results.* Consider the approximate equation (3.7) which was obtained as the first approximation by a series expansion of the characteristic equation in powers of γ.

Since ζ is of the order γ^2 we may further neglect ζ in the factor $(2 - \zeta)$. We obtain

$$\zeta = \tfrac{2}{3}\gamma^2. \tag{4.1}$$

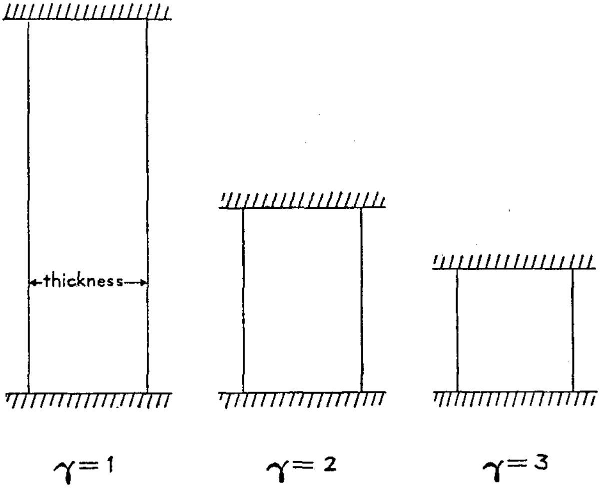

Fig. 4. Slab slenderness in the state of initial compression for $\gamma = 1, 2, 3$.

This approximate value is plotted in fig. 5 at the same time as the exact value. As can be seen equation (4.1) provides a good approxi-

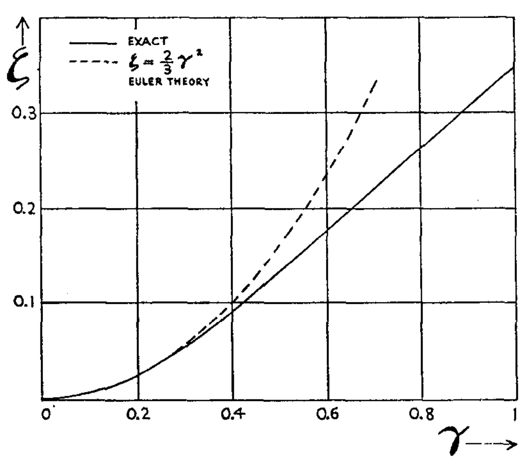

Fig. 5. Portion of the graph of fig. 3 near the origin, and comparison with the Euler theory.

mation to the exact curve up to values

$$\gamma = 0.3. \tag{4.2}$$

This corresponds to a slenderness represented by a length to thickness ratio

$$\mathscr{L}/h \simeq 10. \tag{4.3}$$

The approximate equation (4.1) is identical with the result obtained from the Euler theory of buckling of slender plates. This is easily verified as follows: putting $s_{22} = 0$ in the stress strain relations (2.5) and taking into account condition (2.6) for incompressibility, we derive:

$$s_{11} = 4\mu e_{xx}. \tag{4.4}$$

Hence the coefficient 4μ plays the role of an incremental Young's modulus for plane strain. The equation for the deflection v_0 of a thin plate of thickness h under an axial compressive stress P is

$$4\mu \cdot \frac{h^3}{12} \frac{\mathrm{d}^2 v_0}{\mathrm{d}x^2} + Phv_0 = 0. \tag{4.5}$$

Substituting a sinusoidal deflection

$$v_0 = V \cos lx, \tag{4.6}$$

we obtain the characteristic equation

$$P = \tfrac{1}{3}\mu l^2 h^2, \tag{4.7}$$

which is identical to equation (4.1). The Euler theory is therefore valid down to $\mathscr{L}/h$ ratios of about ten.

It is of considerable interest to examine the stress distribution. The stress components may of course be expressed immediately in terms of the constants C_1 and C_2 by introducing expressions (2.17) into the stress strain relations (2.5). It is also convenient to express the stresses in terms of the normal deflection of the plate at the free surface. The surface deflection (at $y = h/2$) may be written

$$v = V \cos lx. \tag{4.8}$$

Hence from (2.13), (2.17) and (4.8) we derive:

$$lV = -C_1 \cosh \gamma - C_2 \cosh k\gamma. \tag{4.9}$$

We find the values of C_1 and C_2 in terms of V by adding to equation (4.9) one of the boundary conditions (2.20).

The "bending stress" s_{11} is given by

$$\frac{s_{11}}{2\mu l V \cos lx} = \frac{4k}{1-k^2} \tanh k\gamma \left[\frac{\sinh \gamma\eta}{\sinh \gamma} - k^2 \frac{\sinh k\gamma\eta}{\sinh k\gamma} \right], \tag{4.10}$$

with

$$\eta = (2/h)\, y. \tag{4.11}$$

The distribution across the thickness is shown in fig. 6 by plotting the factor

$$F_{11} = \frac{\sin \gamma\eta}{\sinh \gamma} - k^2 \frac{\sinh k\gamma\eta}{\sinh k\gamma}, \tag{4.12}$$

as a function of η for two values of γ, namely:

$$\gamma = 1.95,\ 3.05. \tag{4.13}$$

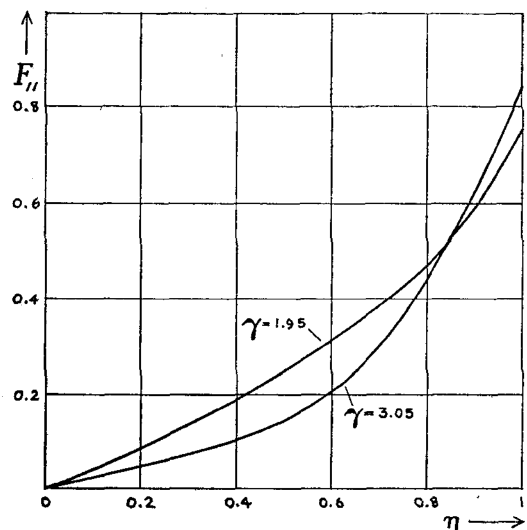

Fig. 6. Distribution of "bending stress" s_{11} as represented by the factor F_{11} for $\gamma = 1.95$ and $\gamma = 3.05$.

The maximum value of s_{11} occurs at $\eta = 1$. Its value $(s_{11})_{max}$ is given by the relation

$$\frac{(s_{11})_{max}}{2\mu l V \cos lx} = 4k \tanh k\gamma. \tag{4.14}$$

For $\gamma \ll 1$ hence for large wavelengths this equation reduces to

$$\frac{(s_{11})_{max}}{2\mu l V \cos lx} = 4\gamma. \tag{4.15}$$

This expression is the same as that obtained from thin plate theory. The shear stress is given by

$$\frac{s_{12}}{2\mu l V \sin lx} = -\frac{1+k^2}{1-k^2}\left(\frac{\cosh k\gamma\eta}{\cosh k\gamma} - \frac{\cosh \gamma}{\cosh \gamma}\right). \tag{4.16}$$

Its distribution across the thickness is shown in fig. 7 by plotting the factor

$$F_{12} = \frac{\cosh k\gamma\eta}{\cosh k\gamma} - \frac{\cosh \gamma\eta}{\cosh \gamma}, \tag{4.17}$$

for the same values (4.13) of γ.

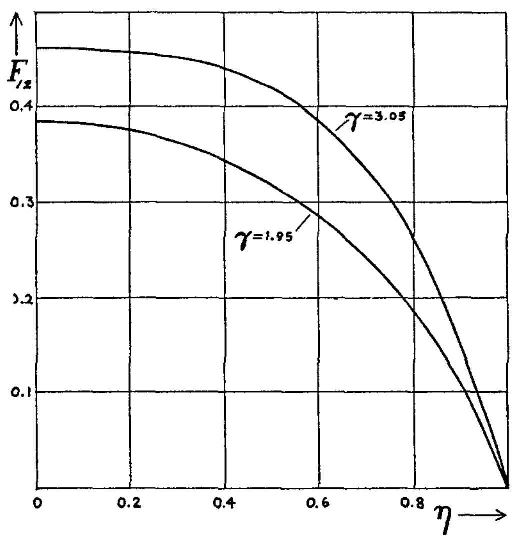

Fig. 7. Distribution of shear stress s_{12} as represented by the factor F_{12} for $\gamma = 1.95$ and $\gamma = 3.05$.

The maximum value of s_{12} occurs at the center $(y = 0)$ and is given by

$$\frac{(s_{12})_{max}}{2\mu l V \sin lx} = -\frac{1+k^2}{1-k^2}\left(\frac{1}{\cosh k\gamma} - \frac{1}{\cosh \gamma}\right). \tag{4.18}$$

For $\gamma \ll 1$ it becomes:

$$\frac{(s_{12})_{max}}{2\mu lV \sin lx} = -\gamma^2. \tag{4.19}$$

Finally the stress component s_{22} is given by

$$\frac{s_{22}}{2\mu lV \cos lx} = \frac{4k}{1-k^2} \tanh k\gamma \left[\frac{\sinh k\gamma\eta}{\sinh k\gamma} - \frac{\sinh \gamma\eta}{\sinh \gamma}\right]. \tag{4.20}$$

For small values of γ this reduces to

$$\frac{s_{22}}{2\mu lV \cos lx} = \tfrac{2}{3}\gamma^3(\eta - \eta^3). \tag{4.21}$$

The maximum value occurs at $\eta = 1/\sqrt{3}$ and its value is given by

$$\frac{(s_{22})_{max}}{2\mu lV \cos lx} = \frac{4}{9\sqrt{3}}\gamma^3. \tag{4.22}$$

We notice that for large wavelength the relative orders of magnitude of the stresses obey the relations

$$\begin{aligned} s_{12} &\cong s_{11}\gamma, \\ s_{22} &\cong s_{11}\gamma^2. \end{aligned} \tag{4.23}$$

The stresses s_{12} and s_{22} are respectively of first and second order relative to the stress s_{11}.

§ 5. *Approximate variational method.* We have shown that the equations of elasticity of a medium under initial stress may be derived by a variational method [3] [4]. In the present case of plane strain and an initial compression P in the x direction the incremental energy density used in the variational procedure becomes

$$\Delta V = \tfrac{1}{2}t_{11}e_{xx} + \tfrac{1}{2}t_{22}e_{yy} + t_{12}e_{xy} - P(e_{xy}\omega + \tfrac{1}{2}\omega^2). \tag{5.1}$$

The stresses t_{ij} are components referred to initial areas. They are related to the stress components s_{ij} by the relations

$$\begin{aligned} t_{11} &= s_{11} - Pe_{yy}, \\ t_{22} &= s_{22}, \\ t_{12} &= s_{12} + \tfrac{1}{2}Pe_{xy}. \end{aligned} \tag{5.2}$$

Substituting in the energy density (5.1) and taking into account equations (2.1), (2.2), (2.5) and (2.6) we obtain

$$\Delta V = 2Me_{xx}^2 + 2Le_{xy}^2 - \tfrac{1}{2}P\left(\frac{\partial v}{\partial x}\right)^2, \tag{5.3}$$

where

$$M = \mu + \frac{P}{4}, \qquad L = \mu + \frac{P}{2}. \tag{5.4}$$

The elastic coefficients L and M have a simple physical significance which will be discussed in more detail in a forthcoming publication. The coefficient $4M$ refers to the incremental force in the x direction and per unit initial area produced by an elongation e_{xx} in the x direction (fig. 8). The coefficient L measures the tangential force applied to the faces of a thin strip cut along x and associated with a slip deformation (fig. 8). We may refer to L as a "slide modulus". Note that these coefficients are defined in the vicinity of a state of initial compressive stress P.

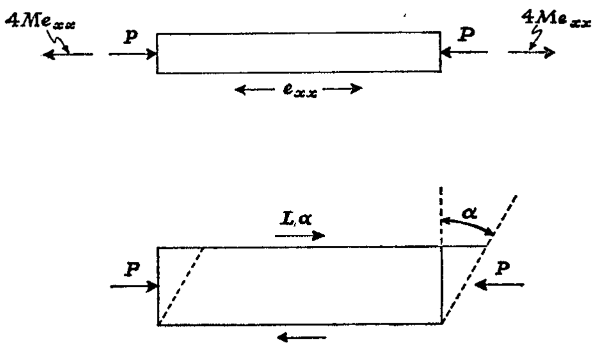

Fig. 8. Physical significance of coefficients M and L.

The energy of a length $\mathscr{L}$ of the slab is

$$W = \int_0^{\mathscr{L}} dx \int_{-h/2}^{+h/2} \Delta V \, dy. \tag{5.6}$$

In order to apply the variational method let us assume a mode of deformation such that an initial cross section remains plane. Hence the displacements are

$$u = u_1 y \sin lx, \qquad v = v_0 \cos lx \tag{5.7}$$

where u_1 and v_0 are to be determined. An additional assumption is introduced when performing the integration in equation (5.6). The term $2Le_{xy}^2$ is proportional to the product $s_{12}e_{xy}$. The approximation (5.7) yields a constant value of s_{12} over the cross section. Since actually this is not the case it is more accurate to average out the integral by assuming s_{12} to be constant in the interval

$$-K\frac{h}{2} < y < K\frac{h}{2}, \tag{5.8}$$

and zero outside $(0 < K < 1)$. This amounts to integrating the terms $2Le_{xy}^2$ between the limits $-Kh/2$ and $Kh/2$. With this procedure the value of W is given by

$$2W/\mathscr{L} = \tfrac{1}{6}Ml^2h^3u_1^2 + \tfrac{1}{2}LhK(u_1 - lv_0)^2 - \tfrac{1}{2}Phl^2v_0^2. \tag{5.9}$$

The variational principle yields the equations

$$\partial W/\partial u_1 = 0, \quad \partial W/\partial v_0 = 0, \tag{5.10}$$

or

$$\begin{aligned} \tfrac{1}{3}Ml^2h^2u_1 + LK(u_1 - lv_0) &= 0, \\ LK(u_1 - lv_0) + Plv_0 &= 0. \end{aligned} \tag{5.11}$$

Eliminating u_1 and v_0 we find the characteristic equation

$$\tfrac{1}{3}l^2h^2M(1 - P/KL) = P. \tag{5.12}$$

Introducing the variables γ and ζ defined by equations (2.21) and (3.1) with the definition (5.4) for M and L the characteristic equation (5.12) becomes:

$$\gamma^2 = \frac{3\zeta}{2+\zeta}\,\frac{1}{\left[1 - \dfrac{2\zeta}{K(1+\zeta)}\right]}. \tag{5.13}$$

If we choose the value $K = 0.91$ this equation yields a curve which cannot be distinguished from the exact one when plotted in fig. 3. It is of course remarkable that the simple procedure used here leads to a result which is valid throughout the complete range of buckling wavelengths. This includes the horizontal asymptote for large γ where the phenomenon degenerates into a surface buckling.

In the analysis we have assumed an incompressible material.

However the variational method is not restricted to this case. For a compressible material we may introduce the approximation $t_{22} = 0$. The energy density (5.1) then reduces to the same expression (5.3). The property of compressibility is contained in the particular value of the coefficient M. For example in a material which is unstressed in the initial state the coefficient $4M$ becomes $E/(1 - \nu^2)$ where E is Young's modulus and ν is Poisson's ratio. The result of the present analysis indicates that the same high order of accuracy is to be expected if we apply the variational method to evaluate the buckling of a compressible slab.

Received 1st May, 1962.

REFERENCES

1) Biot, M. A., Theory of Elasticity with Large Displacements and Rotations. Proc. Fifth Int. Cong. Appl. Mech. 1938.
2) Biot, M. A., Appl. sci. Res. **A12** (1963) 168.
3) Biot, M. A., Phil. Mag. Sec. 7, **XXVII** (1939) 468.
4) Biot, M. A., Z. Angew. Math. Mech. **20** (1940) 89.
5) Biot, M. A., Stability Problems of Inhomogeneous Viscoelastic Media, Proc. IUTAM colloquium on Non-homogeneity in Elasticity and Plasticity, Warsaw 1958, p. 311–321, Pergamon Press.
6) Biot, M. A., Quart. Appl. Math. **XVII** (1959) 185.