Continuum Dynamics of Elastic Plates and Multilayered Solids Under Initial Stress

M. A. BIOT

Offprint from

JOURNAL OF MATHEMATICS AND MECHANICS

Vol. 12, No. 6, November, 1963, pp. 793–810

*Continuum Dynamics of Elastic Plates and Multilayered Solids Under Initial Stress**

M. A. BIOT

Communicated by I. N. SNEDDON

Abstract. General solutions are developed for the continuum dynamics of elastic plates and multilayered media under initial stress. By the use of specially suited analytical devices they are obtained in a remarkably simple form which brings out their mathematical structure and their physical significance. The results are derived for orthotropic materials and include the special case of a material isotropic in finite strain. They provide basic solutions, incorporating the effect of initial stress, in problems of vibrations and acoustics of sandwich plates and seismic propagation in stratified rock. The problem of elastic stability of plates and multilayered media is solved as a particular case by putting the frequency equal to zero. This extends to compressible media the analysis of internal, interfacial, and surface instability and other related phenomena developed earlier by the writer in the context of incompressibility. By viscoelastic correspondence the result also provides complete solutions for the dynamics of viscoelastic plates and multilayered viscoelastic media under initial stress.

1. Introduction. A continuum theory for the dynamics of elastic plates under initial stress is of considerable interest in many fields. It is essential in order to evaluate the influence of initial stresses on the acoustic properties of the plate, and should provide a rigorous foundation for problems of buckling and dynamic instability of plate structures. By extension to multilayered media constituted by a superposition of elastic layers the theory becomes applicable to problems of vibration and acoustics of sandwich structures and to seismic propagation in stratified sedimentary rock in the presence of initial stress.

It is our purpose here to derive general equations for the dynamics of elastic plates and multilayered continua under initial stress.

While the results are rigorous and very general the analytical expressions involved are remarkably simple in view of the complexity of the phenomena involved.

* This work was sponsored by the Shell Development Co.

The derivation is based on the theory of elasticity of a medium under initial stress established by this writer in a series of papers published more than twenty years ago (see for example [1] and [2]).

The medium may be isotropic or orthotropic. The initial stress and the elastic properties are assumed to have one plane of symmetry parallel with the plate.

The theory is also applicable to an isotropic medium in a state of finite initial strain. The treatment is elementary and does not require the use of tensor theory or invariants.

The present application to the dynamics of plates parallels very closely the analysis of the stability of incompressible multilayered elastic media in a recent paper [3]. In fact the experience obtained in solving that less complicated case has been crucial by leading the way to important and drastic analytical simplifications of the present results, and by providing their physical interpretation. (The author has been assisted by Dr. A. Winzer in the analytical work.)

The problem of the single plate is analyzed by considering separately the excitation of symmetric and antisymmetric deformations in sections 3 and 4. The general case is obtained by superposition in section 5. The result is expressed by means of six distinct matrix elements which play a fundamental role in the theory. It leads directly to a compact formulation of the dynamics of the multilayered systems expressed by recurrence equations as shown in section 6. This also provides a matrix multiplication scheme, similar to the method suggested by Thomson [4], and further developed by Haskell [5] which is particularly suitable for numerical work when a large number of layers is involved. The limiting case of an incompressible medium is derived in section 7, and the close relationship between oscillations and modes of instability is briefly discussed in section 8.

By putting the frequency equal to zero the present theory becomes the stability theory of elastic multilayered media, thereby generalizing the results obtained earlier for the case of incompressibility. All the special features of internal buckling, surface and interfacial instability discussed previously in the context of incompressibility are, of course, contained in the present more general results.

Viscoelastic Correspondence. The principle of viscoelastic correspondence developed by the writer in several papers in 1954–55 is applicable to the continuum under initial stress. All the results obtained in the present paper are presented in the context of elasticity, but are considerably more general. They are immediately applicable to viscoelastic media by replacing the elastic coefficients by corresponding operators (for example see reference [6]). These operators have been derived from thermodynamics. Strictly speaking this is valid only for a medium initially at rest in the state of initial stress, but for all practical purposes this restriction may generally be overlooked.

2. Dynamics of a plate under initial stress. Consider an elastic plate of thickness h in a state of uniform initial stress S_{11} (Fig. 1). Choose the y axis to be normal to the plate with the two faces coinciding with the planes $y = \pm h/2$ and the x axis oriented along the initial stress S_{11}.

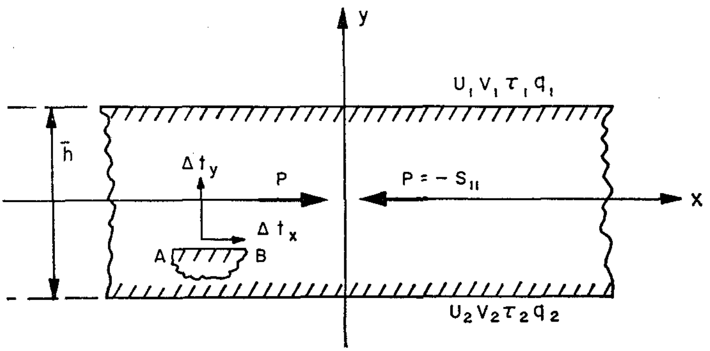

FIGURE 1. Plate under initial stress P viewed across the thickness.

Consider plane incremental deformations represented by the x, y displacements u, v. The incremental strain is defined by

$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{yy} = \frac{\partial v}{\partial y}, \tag{2.1}$$
$$e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

It was shown ([1], [2]) that the incremental stress s_{ij} is related to this strain by the relations

$$\begin{aligned} s_{11} &= B_{11}e_{xx} + B_{12}e_{yy}, \\ s_{22} &= B_{21}e_{xx} + B_{22}e_{yy}, \\ s_{12} &= 2Qe_{xy}. \end{aligned} \tag{2.2}$$

The incremental stress s_{ij} is referred to axes which have undergone a rotation

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{2.3}$$

The elastic coefficients satisfy the relation

$$B_{12} = B_{21} + P, \tag{2.4}$$

where

$$P = -S_{11}. \tag{2.5}$$

The notation P is used here in conformity with the symbols used in previous work.

The incremental stresses must satisfy the dynamical equations [2],

$$(2.6)\qquad \begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P\frac{\partial \omega}{\partial y} &= \rho\frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P\frac{\partial \omega}{\partial x} &= \rho\frac{\partial^2 v}{\partial t^2}, \end{aligned}$$

where ρ is a constant density of the material. For harmonic oscillations all quantities are proportional to the factor exp $(i\alpha t)$ which contains the time t.

We may omit this factor in the solution and write equations (2.6) in the form

$$(2.7)\qquad \begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P\frac{\partial \omega}{\partial y} + \alpha^2\rho u &= 0, \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P\frac{\partial \omega}{\partial x} + \alpha^2\rho v &= 0. \end{aligned}$$

We shall consider a forced harmonic oscillation or a wave propagation such that the deformation of the plate is sinusoidal along x.

In order to introduce the boundary conditions we need another result from the general theory. This involves certain force components Δf_x and Δf_y which are defined as follows.

Consider an area AB inside the plate (Fig. 1). In the state of initial stress but prior to the application of any incremental deformation this area is a plane surface parallel with the faces of the plate. When incremental stresses are applied, Δf_x and Δf_y represent the x and y components of the stress at the surface AB acting on the material lying *below* this surface. It was found ([1], [3]) that these components are related to the incremental stress s_{ij} by the relations

$$(2.8)\qquad \begin{aligned} \Delta f_x &= s_{12} + Pe_{xy}, \\ \Delta f_y &= s_{22}. \end{aligned}$$

These equations provide a complete formulation of the dynamical problem.

Isotropic medium. The equations include the case of an isotropic medium in finite strain. We have shown ([7], [8]) that in this case the incremental coefficient is given by

$$(2.9)\qquad Q = \tfrac{1}{2}P\,\frac{\lambda_2^2 + \lambda_1^2}{\lambda_2^2 - \lambda_1^2},$$

where λ_1 and λ_2 represent the finite extension ratios of the initial state in directions parallel and normal to the plate.

3. Single plate–Antisymmetric case. We shall first consider an antisymmetric solution which represents flexural deformations of the plate (Fig. 2a). In addition we shall consider solutions which are sinusoidal along x. Hence we put

$$
\begin{aligned}
u &= U(y) \sin lx, \\
v &= V(y) \cos lx.
\end{aligned} \tag{3.1}
$$

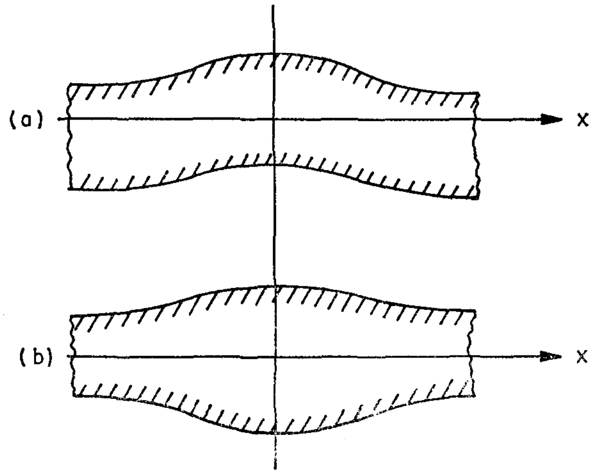

FIGURE 2. Antisymmetric (a) and symmetric (b) deformations of a plate.

The functions $U(y)$ and $V(y)$ must satisfy the condition of antisymmetry,

$$
\begin{aligned}
U\left(\frac{h}{2}\right) &= -U\left(-\frac{h}{2}\right) = U_a , \\
V\left(\frac{h}{2}\right) &= V\left(-\frac{h}{2}\right) = V_a .
\end{aligned} \tag{3.2}
$$

By definition U_a and V_a are the displacements at the upper face. We must find a solution of equations (2.1), (2.2), (2.7), satisfying condition (3.2). Such a solution is

$$
\begin{aligned}
u &= C_1 \sinh \beta_1 ly + C_2 \sinh \beta_2 ly, \\
v &= C_1' \cosh \beta_2 ly + C_2' \cosh \beta_2 ly.
\end{aligned} \tag{3.3}
$$

The quantities appearing in this solution are defined as follows. We have put

$$
\begin{aligned}
\Omega &= B_{11} - \alpha^2 \rho / l^2, \\
L &= Q + \tfrac{1}{2} P.
\end{aligned} \tag{3.4}
$$

The latter expression is the slide modulus L which has been introduced and extensively discussed in previous work ([3], [8]). The constants of integration satisfy the relations

$$
C_i' = -\frac{\Omega - L\beta_i^2}{(B_{21} + L)\beta_i} C_i . \tag{3.5}
$$

Substitution of the general solution (3.3) into the differential equations (2.1), (2.2) and (2.7) leads to a characteristic equation which is quadratic in β^2, *i.e.*

$$\beta^4 - 2m\beta^2 + k^2 = 0. \tag{3.6}$$

The coefficients are

$$\begin{aligned} 2m &= \frac{1}{LB_{22}}\left[\Omega B_{22} - L\left(2B_{21} + P + \frac{\alpha^2\rho}{l^2}\right) - B_{21}^2\right], \\ k^2 &= \frac{\Omega}{LB_{22}}\left(L - P - \frac{\alpha^2\rho}{l^2}\right). \end{aligned} \tag{3.7}$$

The two roots β_1^2 and β_2^2 of equation (3.6) yield the values of β_1 and β_2 appearing in the solution (3.3).

The corresponding force components Δf_x and Δf_y are obtained from equations (2.8). They are of the form

$$\begin{aligned} \Delta f_x &= \tau(y) \sin lx, \\ \Delta f_y &= q(y) \cos lx. \end{aligned} \tag{3.8}$$

Because of the antisymmetry they satisfy the condition

$$\begin{aligned} \tau\left(\frac{h}{2}\right) &= \tau\left(-\frac{h}{2}\right) = \tau_a , \\ q\left(\frac{h}{2}\right) &= -q\left(-\frac{h}{2}\right) = q_a . \end{aligned} \tag{3.9}$$

The values τ_a and q_a therefore represent the stress at the top of the plate. They are the driving forces causing the oscillation of the plate. They are applied at the top and bottom surface so as to generate an antisymmetric flexural motion. The values of U_a , V_a , τ_a , q_a contain two undetermined constants C_1 and C_2 . Elimination of these two constants yields the values of τ_a and q_a in terms of the displacements of the surface. We derive

$$\begin{aligned} \frac{\tau_a}{lL} &= a_{11}U_a + a_{12}V_a , \\ \frac{q_a}{lL} &= a_{12}U_a + a_{22}V_a . \end{aligned} \tag{3.10}$$

Note the symmetry of the matrix. In writing the values of the coefficients we put

$$\begin{aligned} \gamma &= \tfrac{1}{2}lh, \\ z_1 &= \beta_1 \tanh \beta_1\gamma, \\ z_2 &= \beta_2 \tanh \beta_2\gamma. \end{aligned} \tag{3.11}$$

The parameter γ plays a fundamental role in plate mechanics. In terms of the wavelength $\mathfrak{L}$ measured along the plate it is written

$$\gamma = \frac{\pi h}{\mathfrak{L}}. \tag{3.12}$$

Hence π/γ represents the ratio of the wavelength to the thickness. In the forced oscillation this is determined by the wavelength of the sinusoidal distribution of the normal and tangential forces applied to the surface. The coefficients in equations (3.10) are

$$\begin{aligned} a_{11} &= \Omega(\beta_2^2 - \beta_1^2)\frac{1}{\Lambda_a}, \\ a_{22} &= B_{22}(\beta_2^2 - \beta_1^2)z_1 z_2 \frac{1}{\Lambda_a}, \\ a_{12} &= [(\Omega + B_{21}\beta_2^2)z_1 - (\Omega + B_{21}\beta_1^2)z_2]\frac{1}{\Lambda_a} \end{aligned} \tag{3.13}$$

with

$$\Lambda_a = (\Omega - L\beta_1^2)z_2 - (\Omega - L\beta_2^2)z_1 . \tag{3.14}$$

In deriving these expressions drastic simplification of the algebra is obtained by using the relation

$$B_{22} = \frac{\Omega(B_{21} + L)^2}{(\Omega - L\beta_1^2)(\Omega - L\beta_2^2)}. \tag{3.15}$$

It may be verified that this relation becomes an identity by substituting the values $\beta_1^2 + \beta_2^2 = 2m$ and $\beta_1^2\beta_2^2 = k^2$ in accordance with the characteristic equation (3.6).

Forced vibration under a normal exciting force. If the exciting force is normal to the surface of the plate we put $\tau_a = 0$ in equations (3.10). Eliminating U_a in these equations we obtain for the normal exciting force q_a the relation

$$\frac{q_a}{lL} = \frac{a_{11}a_{22} - a_{12}^2}{a_{11}} V_a . \tag{3.16}$$

By substituting the values (3.13) and putting

$$\begin{aligned} R_1 &= \frac{(\Omega + B_{21}\beta_1^2)^2}{\Omega - L\beta_1^2}, \\ R_2 &= \frac{(\Omega + B_{21}\beta_2^2)^2}{\Omega - L\beta_2^2} \end{aligned} \tag{3.17}$$

we find

$$\frac{a_{11}a_{22} - a_{12}^2}{a_{11}} = \frac{R_2 z_1 - R_1 z_2}{\Omega(\beta_2^2 - \beta_1^2)}. \tag{3.18}$$

In order to obtain this result we must introduce the value (3.15) for B_{22} . In that case the expression is simplified by bringing out the common factor Λ_a which cancels out in numerator and denominator.

Free oscillations. By putting $q_a = 0$ in equation (3.16) we obtain the frequency equation for free oscillations of the plate. From the value (3.18) we derive

$$R_2 z_1 - R_1 z_2 = 0 . \tag{3.19}$$

Solutions of this equation yield the various branches of the plot of frequency versus wavelength corresponding to the propagation modes of antisymmetric waves in a plate under initial stress.

4. Single plate–Symmetric case. A similar analysis has been carried out for symmetric deformations of the plate as represented in figure 2b. In this case the displacements satisfy the condition

$$\begin{aligned} U\left(\frac{h}{2}\right) &= U\left(-\frac{h}{2}\right) = U_s , \\ V\left(\frac{h}{2}\right) &= -V\left(-\frac{h}{2}\right) = V_s . \end{aligned} \tag{4.1}$$

The corresponding condition for the force components are

$$\begin{aligned} \tau\left(\frac{h}{2}\right) &= -\tau\left(-\frac{h}{2}\right) = \tau_s , \\ q\left(\frac{h}{2}\right) &= q\left(-\frac{h}{2}\right) = q_s . \end{aligned} \tag{4.2}$$

We proceed exactly as in the previous section and derive the relations

$$\begin{aligned} \frac{\tau_s}{lL} &= b_{11} U_s + b_{12} V_s , \\ \frac{q_s}{lL} &= b_{12} U_s + b_{22} V_s . \end{aligned} \tag{4.3}$$

The coefficients are

$$\begin{aligned} b_{11} &= \Omega(\beta_2^2 - \beta_1^2) z_1' z_2' \frac{1}{\Lambda_s}, \\ b_{22} &= B_{22}(\beta_2^2 - \beta_1^2) \frac{1}{\Lambda_s}, \\ b_{12} &= [(\Omega + B_{21}\beta_2^2) z_2' - (\Omega + B_{21}\beta_1^2) z_1'] \frac{1}{\Lambda_s}. \end{aligned} \tag{4.4}$$

We have put

$$\begin{aligned} z_1' &= z_1/\beta_1^2 = \frac{1}{\beta_1} \tanh \beta_1 \gamma, \\ z_2' &= z_2/\beta_2^2 = \frac{1}{\beta_2} \tanh \beta_2 \gamma \end{aligned} \tag{4.5}$$

and

$$\Lambda_s = (\Omega - L\beta_1^2) z_1' - (\Omega - L\beta_2^2) z_2' . \tag{4.6}$$

Free oscillations. This case is obtained by putting $\tau_s = q_s = 0$ in equations (4.3) hence

$$b_{11}b_{22} - b_{12}^2 = 0 \,. \tag{4.7}$$

The same simplification is obtained as in the case of equation (3.18). By introducing the value (3.15) for B_{22} we find that equation (4.7) contains Λ_s as a factor which may be cancelled out. Therefore we may write equation (4.7) as

$$R_2 z_2' - R_1 z_1' = 0. \tag{4.9}$$

This is the frequency equation for the propagation of symmetric waves in the plate under initial stress.

5. Single plate–General case. We consider the case where the applied forces are still distributed sinusoidally according to equation (3.8) but without any additional condition of symmetry or antisymmetry. The displacements and forces at the top of the plate are denoted by (Fig. 1)

$$U_1 \,,\; V_1 \,,\; \tau_1 \,,\; q_1 \tag{5.1}$$

and at the bottom by

$$U_2 \,,\; V_2 \,,\; \tau_2 \,,\; q_2 \,. \tag{5.2}$$

This case can be obtained by superposition of the symmetric and antisymmetric cases which have been solved in the two preceding sections. The procedure is exactly the same as used in the writer's previous work on stability of multilayered media [3]. By superposition we obtain for the displacements

$$\begin{aligned} U_1 &= U_a + U_s \,, & V_1 &= V_a + V_s \,, \\ U_2 &= -U_a + U_s \,, & V_2 &= V_a - V_s \end{aligned} \tag{5.3}$$

and for the applied forces

$$\begin{aligned} \tau_1 &= \tau_a + \tau_s \,, & q_1 &= q_a + q_s \,, \\ \tau_2 &= \tau_a - \tau_s \,, & q_2 &= -q_a + q_s \,. \end{aligned} \tag{5.4}$$

We substitute the values (3.10) and (4.3) for τ_a, q_a, τ_s, q_s in these last four equations. We then express the result in terms of the total displacement at top and bottom by substituting values $U_a = \frac{1}{2}(U_1 - U_2)$ *etc.* obtained from equations (5.3). The result written in matrix form is

$$\begin{Bmatrix} \tau_1 \\ q_1 \\ \tau_2 \\ q_2 \end{Bmatrix} = lL \begin{Bmatrix} A & B & -D & E \\ B & C & -E & F \\ D & E & -A & B \\ -E & -F & B & -C \end{Bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{Bmatrix}. \tag{5.5}$$

The matrix coefficients are

$$\begin{aligned} A &= \tfrac{1}{2}(a_{11} + b_{11}), & D &= \tfrac{1}{2}(a_{11} - b_{11}), \\ B &= \tfrac{1}{2}(a_{12} + b_{12}), & E &= \tfrac{1}{2}(a_{12} - b_{12}), \\ C &= \tfrac{1}{2}(a_{22} + b_{22}), & F &= \tfrac{1}{2}(a_{22} - b_{22}). \end{aligned} \tag{5.6}$$

An equivalent way of writing the equations in more compact notation is obtained by introducing the quadratic form

$$\begin{aligned} I = {} & \tfrac{1}{2}A(U_1^2 + U_2^2) - DU_1U_2 \\ & + \tfrac{1}{2}C(V_1^2 + V_2^2) + FV_1V_2 \\ & + B(U_1V_1 - U_2V_2) + E(U_1V_2 - U_2V_1). \end{aligned} \tag{5.7}$$

Using this expression, equations (5.5) may be written

$$\begin{aligned} \tau_1 &= lL\frac{\partial I}{\partial U_1}, & \tau_2 &= -lL\frac{\partial I}{\partial U_2}, \\ q_1 &= lL\frac{\partial I}{\partial V_1}, & q_2 &= -lL\frac{\partial I}{\partial V_2}. \end{aligned} \tag{5.8}$$

6. Multilayered media. We consider now a system of n superposed adhering layers. In the general case this system may be embedded between two semi-infinite media (Fig. 3). The layers are numbered from 1 to n starting at the top. The top and bottom semi-infinite media if they are present are numbered 0 and $n + 1$ respectively and may be considered as layers of infinite thickness.

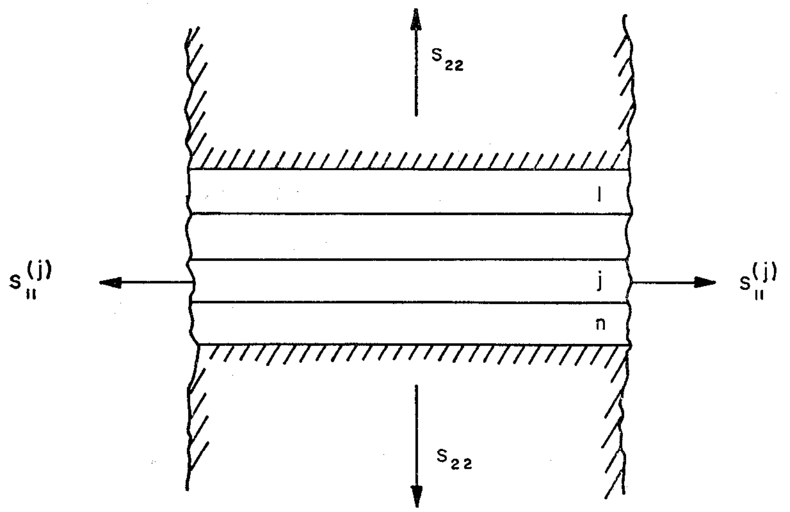

FIGURE 3. Multilayered medium under initial stress.

The state of initial stress may be different in each layer. In the j^{th} layer there is a principal stress component $S_{11}^{(j)}$ parallel with the layer and a component S_{22} normal to it. This normal component is the same in all layers.

In the analysis of the single plate it was assumed that there was no normal component S_{22} for the initial stress. However it was shown in a previous paper [3] that all the results for this case are applicable when S_{22} is not zero provided we replace P by

$$P = S_{22} - S_{11} . \tag{6.1}$$

The forces τ and q at the boundary then represent the tangential and normal stress increment on the deformed surface.

The interfaces of the medium are numbered from 1 to $n + 1$, and corresponding indices are attached to the displacement at these interfaces. The value (5.7) of I for the j^{th} layer is written

$$\begin{aligned} I_j = {} & \tfrac{1}{2}A_j(U_j^2 + U_{j+1}^2) - D_jU_jU_{j+1} \\ & + \tfrac{1}{2}C_j(V_j^2 + V_{j+1}^2) + F_jV_jV_{j+1} \\ & + B_j(U_jV_j - U_{j+1}V_{j+1}) + E_j(U_jV_{j+1} - U_{j+1}V_j). \end{aligned} \tag{6.2}$$

We now express the condition that the stresses τ and q must be continuous at the j^{th} interface. Applying equations (5.8) we derive

$$\begin{aligned} \frac{\partial}{\partial U_{j+1}} (L_jI_j + L_{j+1}I_{j+1}) &= 0, \\ \frac{\partial}{\partial V_{j+1}} (L_jI_j + L_{j+1}I_{j+1}) &= 0. \end{aligned} \tag{6.3}$$

These two equations are recurrence equations for the six displacements at three consecutive interfaces. They constitute a system of $2(n + 1)$ equations for the $2(n + 1)$ interfacial displacements. They are homogeneous and provide the frequency versus wavelength characterisitic equation by evaluation of the determinant.

These equations may also be expressed in still more compact notation by introducing the total quadratic form

$$\mathcal{I} = \sum_{j=0}^{n+1} L_jI_j . \tag{6.4}$$

Equations (6.3) are then written

$$\frac{\partial \mathcal{I}}{\partial U_j} = 0, \qquad \frac{\partial \mathcal{I}}{\partial V_j} = 0. \tag{6.5}$$

They are equivalent to the variational principle

$$\delta \mathcal{I} = 0. \tag{6.6}$$

These equations are completely general and include the case where the top or bottom are free. For example if the top and bottom surfaces of the multilayered systems are free we simply put $L_0 = L_{n+1} = 0$. The values of I_0 and I_{i+1} corresponding to layers of infinite thickness are also simplified because we have assumed unattenuated modes of propagation. This requires the roots β_1 and β_2 to have a real part different from zero. It may be chosen positive. In that case for a layer of infinite thickness

$$\tanh \beta_1\gamma = \tanh \beta_2\gamma = 1. \tag{6.7}$$

Hence

$$z_1 = \beta_1\,, \qquad z_2 = \beta_2\,, \tag{6.8}$$
$$z_1' = \frac{1}{\beta_1}, \qquad z_2' = \frac{1}{\beta_2}.$$

Substituting these values in expressions (3.13), (4.4) and (5.6) we derive

$$\begin{aligned} a_{11} &= b_{11} = A, \\ a_{12} &= b_{12} = B, \\ a_{22} &= b_{22} = C, \\ D &= E = F = 0. \end{aligned} \tag{6.9}$$

This results in considerable simplification of the quadratic forms I_0 and I_{n+1} associated with the top and bottom half space.

Computational scheme by matrix multiplication. The frequency equation may be solved numerically by using the computational scheme suggested by Thomson [4] and developed by Haskell [5] for the propagation of modes in layered media. It has been extended to anisotropic media by Harkrider and Anderson [9]. The method has been further developed by the writer for stability problems [3] using different matrix elements such that they reveal the mathematical structure and are immediately applicable to a large class of problems of the same type.

The matrix equation (5.5) is written in the form

$$\begin{bmatrix} \tau_1 \\ q_1 \\ lU_1 \\ lV_1 \end{bmatrix} = \mathfrak{M} \begin{bmatrix} \tau_2 \\ q_2 \\ lU_2 \\ lV_2 \end{bmatrix}. \tag{6.10}$$

This relates the values at the top face of a layer to the values at the bottom. The matrix $\mathfrak{M}$ is

$$\mathfrak{M} = \begin{bmatrix} B_1 & B_2 & LB_5 & LB_6 \\ B_3 & B_4 & -LB_6 & LB_7 \\ \frac{1}{L}B_8 & \frac{1}{L}B_9 & B_1 & -B_3 \\ -\frac{1}{L}B_9 & \frac{1}{L}B_{10} & -B_2 & B_4 \end{bmatrix}. \tag{6.11}$$

The ten elements B_i are functions of a_{ij} and b_{ij} and are taken from the earlier paper [3]. Their values are listed in the Appendix. The procedure of matrix multiplication and the resulting frequency equation is the same as already discussed in several papers ([3], [5], [9]).

Surface waves and surface instability. For a plate of infinite thickness, equation (3.19) yields the characteristic equation for the propagation of surface waves. As already pointed out z_1 and z_2 in this case take the limiting values (6.8) and equation (3.19) becomes

$$R_2\beta_1 - R_1\beta_2 = 0. \tag{6.12}$$

The values of β_1 and β_2 must be chosen so that their real part is positive.

Velocity curves for surface wave propagation under initial stress have been derived by Buckens [10] as an application of the general equations (2.2) and (2.7). They correspond to solutions of equation (6.12).

By putting $\alpha = 0$ equation (6.12) becomes the general condition for surface instability of an elastic half space. This problem has been discussed in detail by the writer in previous work and solved numerically for the particular case of an incompressible material (see for example [11]).

7. Discussion for the incompressible medium. The case of an incompressible medium leads to considerable simplification in the algebra. This limiting case is obtained as follows: We write the elastic coefficients in the form

$$\begin{aligned} B_{11} &= K + N + P, \\ B_{12} &= K - N + P, \\ B_{21} &= K - N, \\ B_{22} &= K + N. \end{aligned} \tag{7.1}$$

We substitute these values in the stress-strain relations (2.2). They become

$$\begin{aligned} s_{11} &= Ke + (P - N)e + 2Ne_{xx} , \\ s_{22} &= Ke - Ne + 2Ne_{yy} , \\ s_{12} &= 2Qe_{xy} \end{aligned} \tag{7.2}$$

with

$$e = e_{xx} + e_{yy} . \tag{7.3}$$

We introduce the limiting values

$$\begin{aligned} K &\rightarrow \infty , \\ Ke &\rightarrow s, \\ e &= 0. \end{aligned} \tag{7.4}$$

With these values equations (7.2) become

$$\begin{aligned} s_{11} - s &= 2Ne_{xx} , \\ s_{22} - s &= 2Ne_{yy} , \\ s_{12} &= 2Qe_{xy} . \end{aligned} \tag{7.5}$$

These are the incremental stress-strain relations for an incompressible medium in plane strain.

All expressions derived for the general case are therefore immediately applicable to the incompressible medium by introducing the values (7.1) for the coefficients and putting $K = \infty$. For example for the antisymmetric case we find

$$\begin{aligned} a_{11} &= \frac{\beta_1^2 - \beta_2^2}{z_1 - z_2}, \\ a_{22} &= a_{11} z_1 z_2 , \\ a_{12} &= \frac{(\beta_1^2 + 1)z_2 - (\beta_2^2 + 1)z_1}{z_1 - z_2}. \end{aligned} \tag{7.6}$$

The coefficients (3.7) of the biquadratic in the limiting case $K = \infty$ become

$$\begin{aligned} 2m &= \frac{1}{L}\left(4M - 2L - \frac{\alpha^2\rho}{l^2}\right), \\ k^2 &= \frac{1}{L}\left(L - P - \frac{\alpha^2\rho}{l^2}\right), \end{aligned} \tag{7.7}$$

where

$$M = N + \tfrac{1}{4}P \tag{7.8}$$

is a coefficient whose physical significance was already discussed in detail in a previous paper [8].

We note that expressions (7.6) and (7.7) are identical with the result derived for the case of static stability of an incompressible plate [3] by putting $\alpha = 0$.

8. Dynamic stability. Problems of static and dynamic instability are included in the present theory. Under conservative boundary forces the elastic medium must obey the classical theory of small motions of a conservative system in the vicinity of an equilibrium state.

This property is embodied in the variational principle for the elastic medium under initial stress [1]. The variational formulation was illustrated in detail in a recent paper on acoustic-gravity waves [12] and will also be discussed more extensively in a forthcoming book by the writer.

As a consequence the characterisitic exponents α^2 of the solutions are always real. Positive values of α^2 corresponds to oscillations proportional to exp $(i\alpha t)$. A negative value $\alpha^2 = -p^2$ yields a solution proportional to an increasing exponential exp (pt) and corresponds to a dynamic instability.

If we examine for example the expressions (3.13) we notice that the only place where the frequency appears is in

$$\Omega = B_{11} - \alpha^2 \rho / l^2 \tag{8.1}$$

and in the combination

$$P' = P + \alpha^2 \rho / l^2 \tag{8.2}$$

in the values (3.7). A solution for which

$$P' < P \tag{8.3}$$

corresponds to a dynamic instability.

Analogy between buckling and free oscillations. These results lead to an interesting conclusion. Let us assume that a buckling instability exists for a small value of P. Consider for example condition (3.19) for antisymmetric oscillations of the free plate. Let us put equal to zero the frequency α. Assume the elastic coefficients B_{ij} and L to be approximately independent of the initial stress P. If the frequency α is also small the value of Ω will be approximately constant, and the solution of the characteristic equation contains only the unknown P'. Therefore according to equation (8.2) the same solution represents either a buckling under a compression P or an oscillation of an initially stress ree plate provided

$$P = \alpha^2 \rho / l^2. \tag{8.4}$$

This points to a fundamental similarity between dispersion curves for wave propagation and buckling stress as a function of the wavelength.

This similarity extends to body waves, Stoneley and Rayleigh waves. It was pointed out by the writer that these waves are analogous to the phenomena of internal [8], surface [11] interfacial instability [13]. If the frequency varies and goes through a zero value there will be a continuous transition from an oscillation to an instability.

Internal instability of first and second kind. This phenomenon which was analyzed in detail for the incompressible medium [3] [8] may be derived in the same way for the more general type of elastic material governed by the stress-strain relations (2.2). Internal instability will appear if at least one of the roots β_1 and β_2 of equation (3.6) is a pure imaginary. There are two physically distinct cases depending on whether one or two roots are imaginary. They may be referred to respectively as internal instabilities of the first and second kind. The conditions under which this occurs is easily found in terms of the coefficients m and k^2 along exactly the same lines as for the case of incompressibility [8]. By taking into account equation (2.9) the same important conclusion is derived that *internal instability of the first kind is not possible in a medium which is isotropic for finite strain* ([3], [8]).

Appendix: Coefficients in the matrix (6.11).

$$\Delta = (a_{12} - b_{12})^2 - (a_{11} - b_{11})(a_{22} - b_{22})$$

$$B_1 = \frac{1}{\Delta}[(a_{12}^2 - b_{12}^2) - (a_{11} + b_{11})(a_{22} - b_{22})]$$

$$B_2 = \frac{2}{\Delta}(a_{11}b_{12} - a_{12}b_{11})$$

$$B_3 = \frac{2}{\Delta}(a_{12}b_{22} - a_{22}b_{12})$$

$$B_4 = \frac{1}{\Delta}[(a_{22} + b_{22})(a_{11} - b_{11}) - (a_{12}^2 - b_{12}^2)]$$

$$B_5 = \frac{2}{\Delta}[a_{12}^2b_{11} - a_{11}b_{12}^2 - a_{11}b_{11}(a_{22} - b_{22})]$$

$$B_6 = \frac{2}{\Delta}[-a_{12}b_{12}(a_{12} - b_{12}) + a_{11}a_{22}b_{12} - a_{12}b_{11}b_{22}]$$

$$B_7 = \frac{2}{\Delta}[a_{22}b_{22}(a_{11} - b_{11}) + a_{22}b_{12}^2 - a_{12}^2b_{22}]$$

$$B_8 = -\frac{2}{\Delta}(a_{22} - b_{22})$$

$$B_9 = -\frac{2}{\Delta}(a_{12} - b_{12})$$

$$B_{10} = \frac{2}{\Delta}(a_{11} - b_{11})$$

References

[1] M. A. Biot, Nonlinear theory of elasticity and the linearized case for a body under initial stress, *Phil. Mag.*, **27** (1939) 468–489.
[2] M. A. Biot, The influence of initial stress on elastic waves, *Jour. Appl. Phys.*, **11** (1940) 522–529.
[3] M. A. Biot, Theory of stability of multilayered continua in finite anisotropic elasticity, *Jour. Franklin Inst.*, **276** (1963) 128–153.

[4] W. T. Thomson, Transmission of elastic waves through a stratified solid medium, *Jour. Appl. Phys.*, **21** (1950) 89–93.
[5] N. A. Haskell, The dispersion of waves in multilayed media, *Bull. Seism. Soc. Amer.*, **43** (1953) 17–34.
[6] M. A. Biot, Stability of multilayered continua including the influence of gravity and viscoelasticity, *Jour. Franklin Inst.*, **276** (1963) 231–252.
[7] M. A. Biot, Incremental elastic coefficients of an isotropic medium in finite strain, *Applied Scient. Res. Ser. A*, 1963. (in the press)
[8] M. A. Biot, Internal buckling under initial stress in finite elasticity, *Proc. Roy. Soc.*, **A273** (1963) 306–328.
[9] D. G. Harkrider & D. L. Anderson, Computation of surface wave dispersion for multilayered anisotropic media, *Bull. Seism. Soc. Amer.*, **52** (1962) 321–332.
[10] F. Buckens, The velocity of Rayleigh waves along a prestressed semi-infinite medium assuming a two-dimensional anisotropy, *Annali di Geofisica*, **XI** (1958) 99–112.
[11] M. A. Biot, Surface instability in finite anisotropic elasticity under initial stress, *Proc. Roy. Soc.*, **A273** (1963) 329–339.
[12] M. A. Biot, Variational principles for acoustic-gravity waves, *The Physics of Fluids*, **6** (1963) 772–778.
[13] M. A. Biot, Interfacial instability in finite elasticity under initial stress, *Proc. Roy. Soc.*, **A273** (1963) 340–344.

Shell Development Company