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By M. A. BIOT

(Shell Development Company, New York)

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CONTINUUM THEORY OF STABILITY OF AN EMBEDDED LAYER IN FINITE ELASTICITY UNDER INITIAL STRESS

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(*Shell Development Company, New York*)

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SUMMARY

The writer's theory of elasticity under initial stress is applied to the problem of buckling of a thick elastic slab embedded in an elastic medium of infinite extent. The initial stressed state of the system is one of homogeneous finite strain, and perfect adherence is assumed at the interface of the slab and the embedding medium. The characteristic equation is solved numerically and the relation between the stability and wave length parameters is plotted for various values of the ratio of the rigidities of the two media. It is shown that for vanishing wave length the buckling degenerates into an interfacial instability in analogy with Stoneley waves. By viscoelastic correspondence the present result is also an exact solution for two viscoelastic media whose operators and initial effective compressive stresses differ only by the same constant factor.

1. General results

Consider a layer of thickness h (Fig. 1) of an isotropic elastic medium whose strain energy per unit volume is

$$W = \tfrac{1}{2}\mu_0(\lambda_1^2+\lambda_2^2+\lambda_3^2-3) \tag{1}$$

The finite principal extension ratios are denoted by $\lambda_1\lambda_2\lambda_3$. The medium is assumed incompressible, i.e.

$$\lambda_1\lambda_2\lambda_3 = 1 \tag{2}$$

The layer is embedded in a similar medium of strain energy

$$W = \tfrac{1}{2}\mu_{01}(\lambda_1^2+\lambda_2^2+\lambda_3^2-3) \tag{3}$$

Perfect adherence is assumed at the interface.

Consider now a state of homogeneous finite strain defined by the three extension ratios λ_i. Assume the direction λ_1 to be parallel to the layer, while λ_3 is perpendicular to the plane of Fig. 1.

Choose coordinates x, y in the plane of the figure, x being parallel to the layer. The differences of the principal initial stresses in the x, y plane are, for the layer,

$$P = S_{22}-S_{11} = \mu_0(\lambda_2^2-\lambda_1^2) \tag{4}$$

and for the embedding material,

$$P_1 = S_{22}-S_{11}^{(1)} = \mu_{01}(\lambda_2^2-\lambda_1^2) \tag{5}$$

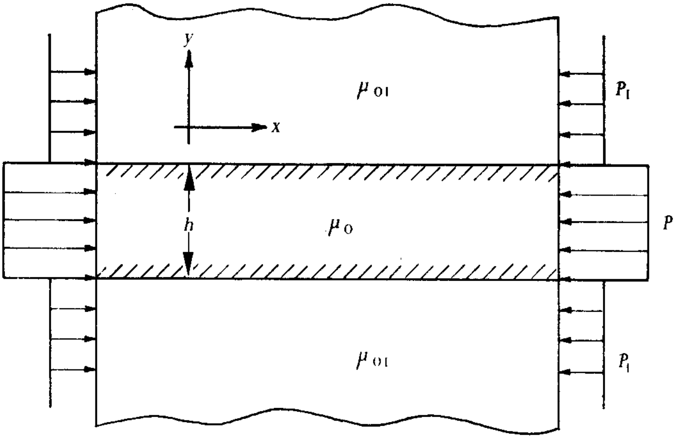

FIG. 1. Embedded layer and state of initial stress. (The initial stress component S_{22} is not represented.)

We superpose small deformations upon this initial state of stress and assume that the incremental deformation is a plane strain in the x, y plane. The incremental stress in the layer is then given by

$$\left.\begin{aligned} s_{11}-s &= 2\mu e_{xx} \\ s_{22}-s &= 2\mu e_{yy}. \\ s_{12} &= 2\mu e_{xy} \end{aligned}\right\} \tag{6}$$

These stress components are referred to axes which rotate locally with the material. Because of incompressibility we must satisfy the condition

$$e_{xx}+e_{yy} = 0. \tag{7}$$

It is easy to show (**1**, **2**, **3**) that for a medium defined by expression (1) the incremental elastic coefficient μ is given by

$$\mu = \tfrac{1}{2}\mu_0(\lambda_1^2+\lambda_2^2) \tag{8}$$

The coefficient μ_0 is the shear modulus in the unstressed state ($\lambda_1 = \lambda_2 = 1$). Similarly, in the embedding medium the incremental stresses are expressed by equations (6) where μ is replaced by

$$\mu_1 = \tfrac{1}{2}\mu_{01}(\lambda_1^2+\lambda_2^2) \tag{9}$$

The stability equations of this system are obtained by applying the results derived in some earlier work (**4**, **5**) inserting for the elastic coefficients used in these papers the values

$$\bar{Q} = \mu, \qquad \bar{Q}_1 = \mu_1, \qquad R = \infty \tag{10}$$

The last condition ($R = \infty$) corresponds to incompressibility.

Under conditions of instability, the deformation of the layer is a sinusoidal bending which is represented by an antisymmetric solution. The displacement components of the upper interface are written

$$\begin{aligned} u &= U \sin lx \\ v &= V \cos lx \end{aligned} \tag{11}$$

Equating normal and tangential stresses on both sides of the interface yields the equations

$$\begin{aligned} \mu_{01}[-a'_{11}U+a'_{12}V] &= \mu_0[a_{11}U+a_{12}V] \\ \mu_{01}[a'_{12}U-a'_{22}V] &= \mu_0[a_{12}U+a_{22}V] \end{aligned} \tag{12}$$

The coefficients are

$$\begin{aligned} a'_{11} &= 1+k & a_{11} &= \frac{1-k^2}{z_1-z_2} \\ a'_{12} &= k-1 & a_{12} &= \frac{2z_2-(1+k^2)z_1}{z_1-z_2} \\ a'_{22} &= k(1+k) & a_{22} &= a_{11}z_1z_2 \end{aligned} \tag{13}$$

with

$$z_1 = \tanh\gamma, \qquad z_2 = k\tanh k\gamma, \qquad \gamma = \tfrac{1}{2}lh \tag{14}$$

The values of k and ζ are the same in both media, i.e.

$$\left.\begin{aligned} k &= \sqrt{\frac{1-\zeta}{1+\zeta}} \\ \zeta &= \frac{P}{2Q} = \frac{P_1}{2Q_1} = \frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2+\lambda_1^2} \end{aligned}\right\} \tag{15}$$

We shall denote the ratio of rigidities of the two media by

$$n = \frac{\bar{Q}_1}{\bar{Q}} = \frac{\mu_{01}}{\mu_0} \tag{16}$$

The characteristic equation corresponding to instability is found by equating to zero the determinant of equations (12). We find

$$(a_{11}a_{22}-a_{12}^2)+n(a_{22}a'_{11}+a_{11}a'_{22}+2a_{12}a'_{12})+n^2(a'_{11}a'_{22}-a'^2_{12}) = 0 \tag{17}$$

The bracketed expressions in this equation may be written

$$\left.\begin{aligned} a_{11}a_{22}-a_{12}^2 &= \frac{1}{z_1-z_2}[4z_2-(1+k^2)^2z_1], \\ a'_{11}a_{22}+a_{11}a'_{22}+2a_{12}a'_{12} &= \frac{1-k}{z_1-z_2}[(1+k)^2z_1z_2-4z_2+2(1+k^2)z_1+k(1+k)^2], \\ a'_{11}a'_{22}-a'^2_{12} &= k(1+k)^2-(1-k)^2 \end{aligned}\right\} \tag{18}$$

2. Limiting cases

(a) The case $n = 0$ corresponds to the free layer, and equation (17) becomes

$$4z_2-(1+k^2)^2z_1 = 0 \tag{19}$$

This equation for the buckling of a thick plate was already derived by the writer in (**6**) and discussed in more detail in later publications (**4**, **7**).

(b) The case $n = \infty$ is that of a half space with a free surface. The characteristic equation becomes

$$k(1+k)^2-(1-k)^2 = 0 \tag{20}$$

It is the buckling condition of a free surface. Multiplying equation (20) by $(1-k)$ it becomes

$$4k-(1+k^2)^2 = 0 \tag{21}$$

In this form it is readily seen to be identical with the limiting case of equation (19) for the buckling of a free plate of infinite thickness. This can be shown by putting $\gamma = \infty$ into the values of z_1 and z_2. They become $z_1 = 1$ and $z_2 = k$. For these values equations (19) and (21) are the same. It can be seen that surface instability is already implicit in equation (19) of the writer's paper (**6**).

Other equivalent forms of equation (21) for surface instability are

$$k(1+\zeta)^2-1 = 0 \tag{22}$$

or

$$\zeta^3+2\zeta^2-2 = 0 \tag{23}$$

These forms of the equations were discussed in detail in the context of elasticity and viscoelasticity in several previous papers (**3**, **4**). The root of equation (23) is

$$\zeta = 0{\cdot}839 \tag{24}$$

(c) Another particular case of interest is to consider the limiting case of equation (17) for a layer of infinite thickness. In this case $(\gamma = \infty)$

$$\begin{aligned} a'_{11} &= a_{11} = 1+k \\ a'_{12} &= a_{12} = k-1 \\ a'_{22} &= a_{22} = k(1+k) \end{aligned} \tag{25}$$

Equation (17) then reduces to

$$k\left(\frac{1+k}{1-k}\right)^2 = \left(\frac{1-n}{1+n}\right)^2 \tag{26}$$

This is the condition for interfacial instability of two adherent half spaces of different rigidities. It coincides with the results derived in previous work (**8**) where it was solved numerically for the root ζ as a function of n.

3. Numerical solution for the general case

For the general case we may solve the characteristic equation (17) for ζ as a function of γ for given values of the rigidity ratio n. This yields a one-parameter family of curves plotted in Fig. 2.

The value of ζ attains a minimum. The corresponding value of γ yields the buckling wave length which suddenly appears for a given

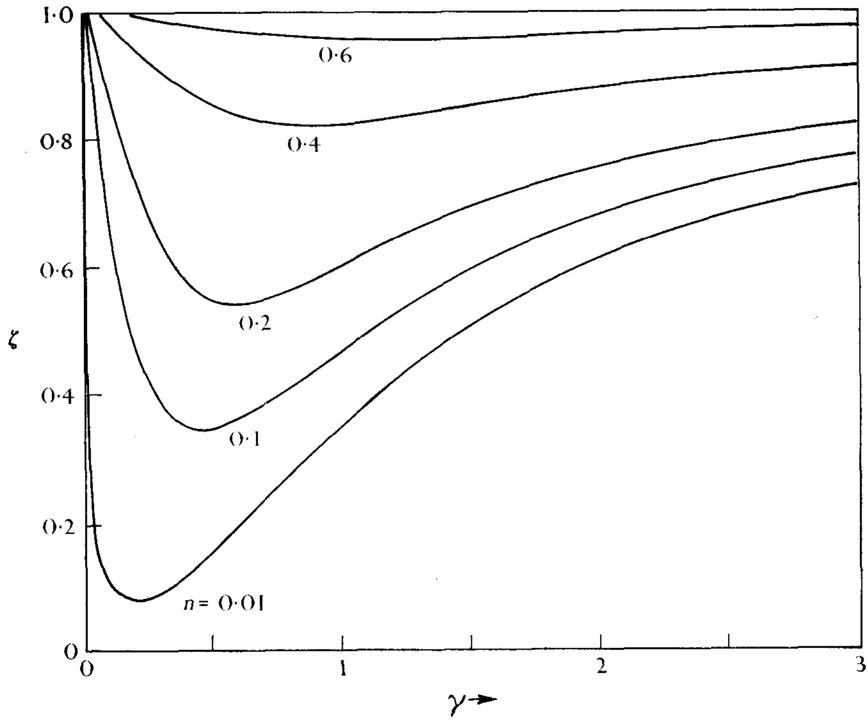

FIG. 2. Stability parameter ζ as a function of the wave length parameter γ for five values of the rigidity ratio $n = \mu_{01}/\mu_0$. Asymptotes for $\gamma = \infty$ represent interfacial instability.

value of n when the compressive strain is gradually increased. Other wave lengths can be induced by introducing a constraint in the system. For vanishing wave lengths ($\gamma = \infty$) the value of ζ tends to a horizontal asymptote corresponding to interfacial instability. The asymptotic value of ζ is the root of equation (26).

This phenomenon is entirely analogous to the dynamical case where bending waves in the layer degenerate into Stonely waves at the interface for increasing frequency and vanishing wave length.

For the materials assumed here the value (15) of ζ is always smaller than unity. Hence values are plotted only in that range. However, as already shown in the similar case discussed earlier (**5**), there are an infinite number of branches for the solution of equation (17) in the range $|\zeta| > 1$. Such branches correspond to the phenomenon of internal buckling, which has been discussed in detail in another paper (**2**).

4. Viscoelastic correspondence

The solution obtained here for the elastic medium is directly applicable to viscoelasticity. It constitutes also an exact solution for the case where the operators $\bar{Q}$ and $\bar{Q}_1$ which represent the two media and the initial stresses P and P_1 differ only by the same constant factor n. This value of n is the same as in the numerical solution discussed here for the elastic case. If the medium is assumed to obey thermodynamic principles $\bar{Q}$ is an increasing function of the time differential operator p. Hence, the dominant wave length corresponds to the minimum of ζ and is the same as in the elastic case. Strictly the medium should be at rest under the initial stress but for all practical purposes the theory is applicable to viscous fluids.

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