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Theory of Buckling of a Porous Slab and Its Thermoelastic Analogy

A theory is developed for the buckling of a fluid-saturated porous slab under axial compression. The problem is discussed in the context of the thermodynamics of irreversible processes. It is shown that there is a range of compressive loads, between a lower and upper critical value, for which the slab exhibits creep buckling. The problem of folding instability of a porous layer embedded in a viscous or viscoelastic medium is also analyzed and the dominant wavelength is evaluated. Identical behavior is derived by analogy for a thermoelastic slab with a critical range between isothermal and adiabatic buckling. The theory is applicable to a large class of two-phase materials obeying the same thermodynamics. It also provides a simple analysis of thermoelastic damping of plates.

In recent years some basic problems of stability of nonhomogeneous viscoelastic media have been treated by the author. The analysis was based on a general theory of stability of continua under initial stress. One of the main interests of these problems lies in their application to geophysics and structural geology. In this connection it is important to consider the behavior of fluid-saturated porous media. A general theory for the stability and consolidation of porous media under initial stress has been developed by the author [1].[1] However, it is important to bring out the characteristic features for the buckling of porous media by using a more simplified treatment. This is the purpose of the present paper.

The existence of a lower and upper buckling load for a porous slab is derived and discussed. The analogy with the viscoelastic behavior of nonporous media is shown to be a consequence of thermodynamic principles.

This analogy is applied to the analysis of folding instability of an embedded layer.

It is of interest to point out that the physical problem is quite different from that of a viscoelastic continuum since the stresses depend not only on the local strain, but also on the fluid pressure whose value is determined by solving the complete field problem. The simplification which leads to the viscoelastic analogy in the present case is due to the particular nature of the approximation associated with the concept of bending moment.

Attention also should be called to the overall perspective, of the problem of buckling of a layered porous medium, which is provided by the simple case analyzed in this paper. A solution is derived for a porous slab embedded in an impervious medium. However, the other extreme case of infinite permeability of the embedding medium is immediately obtained by applying a factor to the relaxation constants in the operator $B(p)$ representing the bending properties of the slab. Hence an estimate can be made for the more complex intermediate case where the finite permeability of the embedding medium enters into play.

Another analogy based on thermodynamics leads to the theory of thermoelastic buckling of a purely elastic homogeneous slab. In general the theory is applicable to the case of any two-component material where one component diffuses through the other provided the same thermodynamic principles are valid.

The results may also be used to derive a theory of thermoelastic damping of plates as shown briefly in a later section of the paper.

Operational Relation Between Bending Moment and Curvature

In the bending of a porous elastic slab the response to the sudden application of a bending moment involves an instantaneous elastic deformation and a delayed after effect. The time history of the after effect depends on the fluid flow throughout the pores.

In order to evaluate this effect we shall consider a plane-strain deformation of the slab. In particular let us analyze the behavior of a slice of unit thickness cut along a cross section, Fig. 1. With the x and y-axes oriented as shown in the figure, we shall impose a deformation of the slab such that the longitudinal strain e_{xx} is a linear function of the distance y to the neutral axis. Hence we put

$$e_{xx} = \kappa y \tag{1}$$

The bending curvature is represented by κ. The faces of the slab are located at

$$y = \pm h/2 \tag{2}$$

where h is the slab thickness. The stress-strain relations in a porous medium are [2]

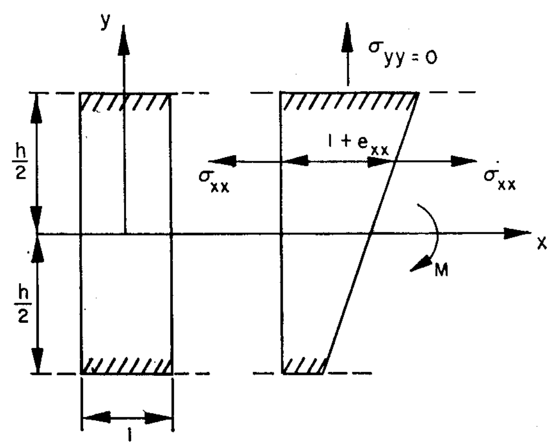

Fig. 1 Deformation of a cross-sectional slice of porous slab

[1] Numbers in brackets designate References at end of paper.

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Table 1 Comparisons of properties and susceptibility to catastrophic-shear as function of strain rate

Material	Hard-ness Bhn	ρ pcf	C, Btu/lb deg F	k, Btu/deg F hr ft	$\frac{\partial\tau/\partial\epsilon}{\partial\tau/\partial\theta}$, deg F	τ_y, psi	Ratio of critical strain rate to that of titanium[a] $\frac{\dot{\epsilon}_c}{\dot{\epsilon}_c T}$
Mild steel SAE 1020.........	120	490	0.12	29	4200	20000	1400
Stainless SAE 310..........	160	490	0.12	8	6050	27000	450
Stainless SAE 446..........	160	490	0.14	12	2200	30000	80
Vascojet 1000...	...	490	0.15	20	2900	125000	16
Titanium RC-70	...	275	0.12	8	550	40500	1
Titanium alloy RC-130-A....	...	290	0.12	8	1250	75000	1.3
Aluminum 2SH14.......	...	170	0.22	120	700	8000	700

[a] Average critical shearing strain rate $\dot{x}/L$, Fig. 1.

machining geometry at machining speeds of the order of 1 sfm; mild steel behaves in a similar fashion when speeds approach 1300 sfm. Titanium is being considered seriously for a number of applications subject to dynamic loading. Its high strength-to-weight ratio is very attractive. Dynamic loads, which might simply induce local stress-relieving plastic flow in aluminum or steel structures, might induce catastrophic shear failure in an improperly designed titanium structure. Titanium will be very useful for space-vehicle structures if proper design criteria are established.

Criteria governing the onset of catastrophic adiabatic slip have been presented. Accurate evaluation requires isothermal static stress-strain relationships which describe those deformation characteristics of the material which are independent of geometrical and necking conditions. A series of properly designed tests should be performed to obtain true stress-true strain relationships which accurately represent materials under consideration.

The machining geometry is very useful for studying shear deformation. Results of ultrahigh-speed machining tests imply that dynamic shear strength tends to become insensitive to strain rate when catastrophic shear is well developed. This implication is very useful during analyses of ballistic impact.

The thinness of adiabatic slip zones is helpful for heat-transfer considerations. Reasonably accurate computations of shear-zone temperatures can be made using a model based upon a plane which uniformily generates heat at a constant rate within an infinite medium.

Acknowledgments

Machining experiments were sponsored by the Shell Oil Company through a grant-in-aid of research. The author's interest in the dynamic behavior of materials was generated during the performance of an Air Force Subcontract for Lockheed Aircraft Corporation (USAF 33(600)36232) which concerned the feasibility of machining at ultrahigh speeds. Mr. Bruce Hanna directed the technical research on this subcontract for the Denver Research Institute, a department of the University of Denver. Mr. Larry Mordock, a high-school senior participant in a summer NSF program at the University, performed much of the dynamometer data reduction which led to determinations of shearing stress and temperature within catastrophic slip zones. Mr. Ron Kilgore performed the photomicrography. The author is especially indebted to Mr. Jack Siekmann who supplied Mr. Hanna with the machining chips removed at ultrahigh speeds during experiments at the Carboloy Division of the General Electric Company.

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$$
\begin{aligned}
\sigma_{xx} &= 2\mu e_{xx} + \lambda e - \alpha p_f \\
\sigma_{yy} &= 2\mu e_{yy} + \lambda e - \alpha p_f \\
p_f &= -\alpha M e + M\zeta
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
\sigma_{xx}, \sigma_{yy} &= \text{total stress} \\
e_{xx}, e_{yy} &= \text{strain components} \\
e &= e_{xx} + e_{yy} = \text{dilatation} \\
p_f &= \text{pressure of fluid in pores} \\
\zeta &= \text{fluid content.}
\end{aligned}
\tag{4}
$$

Other quantities are material constants.[2] The relative fluid volume displacement w in the solid is assumed to be normal to the x-axis. Hence by definition

$$
\zeta = -\frac{\partial w}{\partial y} \tag{5}
$$

The fluid displacement and pressure are related by Darcy's law

$$
\frac{\partial w}{\partial t} = -\frac{k}{\eta}\frac{\partial p_f}{\partial y} \tag{6}
$$

where k is the permeability and η the fluid viscosity.

From these equations we shall derive a relation between the curvature κ and the bending moment. We shall assume $\sigma_{yy} = 0$.

Elimination of e_{yy} between the first two equations (3) yields

$$
\sigma_{xx} = Be_{xx} - \alpha\varphi p_f \tag{7}
$$

where

$$
\begin{aligned}
B &= \frac{4\mu(\mu+\lambda)}{2\mu+\lambda} = \frac{E}{1-\nu^2} \\
\varphi &= \frac{2\mu}{2\mu+\lambda} = \frac{1-2\nu}{1-\nu}
\end{aligned}
\tag{8}
$$

In these expressions, E and ν are, respectively, Young's modulus and Poisson's ratio of the porous medium for $p_f = 0$; i.e., for infinitely slow deformations.

From equations (3), (5), and (6) we also derive by elimination

$$
\frac{\partial p_f}{\partial t} = D\frac{\partial^2 p_f}{\partial y^2} - \alpha\varphi M_c \frac{\partial e_{xx}}{\partial t} \tag{9}
$$

with

$$
\begin{aligned}
D &= \frac{kM_c}{\eta} \\
M_c &= \frac{2\mu+\lambda}{2\mu+\lambda+\alpha^2 M} M
\end{aligned}
\tag{10}
$$

The constant D represents a diffusivity of the fluid in the pores. Putting

$$
p = \frac{\partial}{\partial t} \tag{11}
$$

equation (9) may be written operationally

$$
p_f - \frac{D}{p}\frac{d^2 p_f}{dy^2} = -\alpha\varphi M_c e_{xx} \tag{12}
$$

A solution of this equation is obtained by expanding e_{xx} in a Fourier series. Since e_{xx} is given by (1) we write

$$
e_{xx} = \kappa y = 4\kappa h \sum_{n=0}^{\infty} \frac{(-1)^n}{a_n^2} \sin\frac{a_n y}{h} \tag{13}
$$

with

$$
a_n = (2n+1)\pi \tag{14}
$$

The solution of equation (12) is therefore,

$$
p_f = 4\alpha\varphi M_c \kappa h \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{a_n^2}\frac{p}{p+r_n}\sin\frac{a_n y}{h} \tag{15}
$$

where

$$
r_n = \frac{a_n^2 D}{h^2} = (2n+1)^2\pi^2\frac{D}{h^2} \tag{16}
$$

The solution (15) satisfies the condition $\partial p_f/\partial y = 0$ at $y = h/2$. Hence it corresponds to a slab for which the *pores are sealed at the surface.*

The bending moment is

$$
\mathfrak{M} = \int_{-h/2}^{+h/2} \sigma_{xx} y\, dy \tag{17}
$$

From expression (7) using the values (1) and (15) for e_{xx} and p_f we derive

$$
\mathfrak{M} = B(p)\frac{h^3}{12}\kappa \tag{18}
$$

with the operator

$$
B(p) = B + \sum_{n=0}^{\infty}\frac{p}{p+r_n}B_n \tag{19}
$$

and the coefficients

$$
B_n = \frac{96}{\pi^4}\frac{\alpha^2\varphi^2}{(2n+1)^4}M_c \tag{20}
$$

We note that expression (19) is a rapidly convergent series. Hence using only the first term we may write the approximate value

$$
B(p) = B + \frac{p}{p+r_0}B_0 \tag{21}
$$

In many problems this approximation will be amply satisfactory.

Relation to Viscoelasticity and Irreversible Thermodynamics

The particular form of expression (19) for the operator is of considerable interest here. It is found to be identical with the general expression derived by the author [3] from irreversible thermodynamics for viscoelastic and relaxation properties. The fact that all terms are positive in expression (19) is a direct consequence of the thermodynamic principles.

Note that the porous slab behaves as if it were constituted by an homogeneous viscoelastic material whose stress-strain relation in uniaxial stress would be

$$
\sigma_{xx} = B(p)e_{xx} \tag{22}
$$

Actually in the present case the thermodynamic system is composed of a large number of degrees of freedom. One of these is the curvature. This may be considered as a generalized coordinate whose conjugate generalized force is the bending moment. The other generalized coordinates of the system correspond to the hidden degrees of freedom of the slab. They are theoretically infinite in number. They are represented by the normal coordinates corresponding to relaxation modes of the pore fluid over the cross section. Actually the quantities r_n appearing in the operator

[2] The Lamé constants λ and μ are defined as in classical elasticity for zero fluid pressure ($p_f = 0$). The dimensionless constant α is the ratio ζ/e for $p_f = 0$. The elastic modulus M is defined physically for constant volume ($e = 0$)

(19) are the relaxation constants of these hidden coordinates.

The operator B could have been evaluated by variational Lagrangian equations, using a potential energy and a dissipation function as developed in the original thermodynamic theory [3] and illustrated in more detail in the thermoelastic theory [4].

Another interesting consequence of the thermodynamic theory is that the particular form (19) of $B(p)$ is valid also in the more general case where the *porous matrix itself is viscoelastic.* This is quite evident since it is equivalent to increasing the total number of hidden degrees of freedom included in the overall system.

If only one hidden degree of freedom is taken into account the approximation (21) is obtained. It contains only one relaxation constant r_0.

Creep Buckling of a Porous Slab

Consider a slab of length L subject to an axial compressive stress P and pinned at both ends, Fig. 2. The total axial force is Ph. If we denote by v the deflection of the neutral axis, equilibrium requires that the bending moment $\mathfrak{M}$ satisfy the following equation

$$\mathfrak{M} = Phv \tag{23}$$

On the other hand, with the x coordinate measured along the axis, the curvature is

$$\kappa = -\frac{d^2v}{dx^2} \tag{24}$$

Combining equations (18), (23), and (24) we find the buckling equation

$$B(p)\,\frac{h^3}{12}\,\frac{d^2v}{dx^2} + Phv = 0 \tag{25}$$

This is identical with Euler's equation for the buckling of a rod with the moment of inertia $h^3/12$ of the cross section, and the elastic modulus replaced by the operator $B(p)$.

The slab buckles in a half sine wave represented by

$$v = C \sin \frac{\pi x}{L} \tag{26}$$

By substitution in equation (25) we find the characteristic equation

$$P = \frac{\pi^2}{12}\,\frac{h^2}{L^2}\,B(p) \tag{27}$$

The significance of this equation arises from the fact that any real positive root p corresponds to an instability such that the lateral deflection increases proportionally to an exponental function of time $\exp(pt)$.

An important property of $B(p)$ is that all terms in the series are positive. Hence it is an increasing function of p. The lowest value of P for which instability can occur corresponds to $p = 0$. The corresponding value of $B(p)$ is

$$B(0) = B = \frac{E}{1-\nu^2} \tag{28}$$

This yields a *lower buckling load*

$$P_l = \frac{\pi^2}{12}\,\frac{h^2}{L^2}\,B \tag{29}$$

This is the load of incipient instability. As soon as P exceeds P_l an instability appears in the form of a *creep buckling.* When P is increased further, the rate of lateral deflection increases until it becomes infinite; i.e., purely elastic. The load at which this happens is obtained by putting $p = \infty$ in the value of $B(p)$. This upper critical load is therefore

$$P_u = \frac{\pi^2}{12}\,\frac{h^2}{L^2}\left[B + \sum_{n=0}^{\infty} B_n\right] \tag{30}$$

The value of the series is known [5], i.e.

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots . = \frac{\pi^4}{96} \tag{31}$$

Hence

$$B_c = B + \sum_{n=0}^{\infty} B_n = B + \alpha^2\varphi^2 M_c \tag{32}$$

Substituting the values (8) and (10) for B, φ, and M_c it becomes

$$B_c = \frac{4\mu(\mu+\lambda_c)}{2\mu+\lambda_c} \tag{33}$$

with

$$\lambda_c = \lambda + \alpha^2 M \tag{34}$$

Hence the upper critical load is

$$P_u = \frac{\pi^2}{12}\,\frac{h^2}{L^2}\,B_c \tag{35}$$

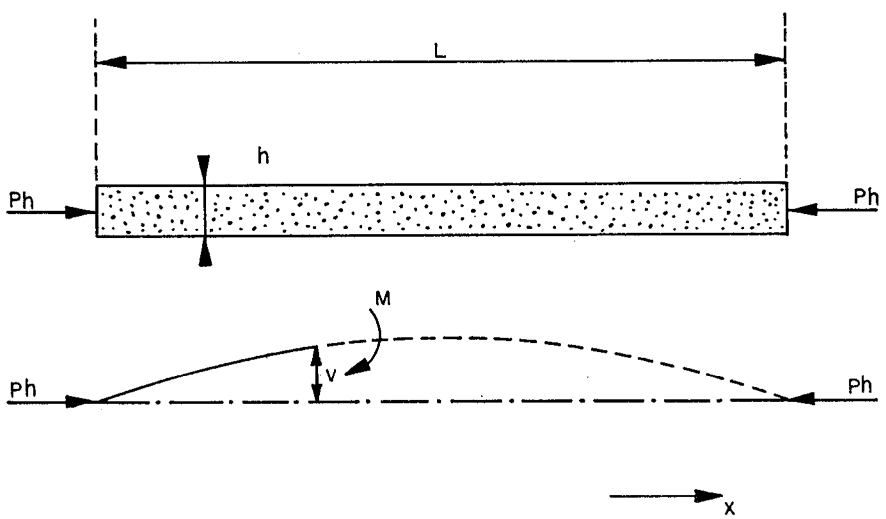

Fig. 2 Equilibrium of axial compression and bending moment during buckling

Comparing the values (8) and (33) we notice that B_c is obtained from B by substituting λ_c for λ. The physical significance of this is derived from the property of the stress-strain relations (3). For very fast deformations the fluid has no time to flow through the pores and we may put $\zeta = 0$. By then eliminating p_f we derive

$$\begin{aligned} \sigma_{xx} &= 2\mu e_{xx} + \lambda_c e \\ \sigma_{yy} &= 2\mu e_{yy} + \lambda_c e \end{aligned} \tag{36}$$

We see that the stress-strain relations for fast deformation are obtained by substituting λ_c for λ.

For a compression P lying between the lower and the upper critical value the slab will buckle gradually as a function of time. When P exceeds the upper critical load the buckling is purely elastic.

It should be pointed out that these buckling properties are due to the positive sign of the various terms in the operator $B(p)$ and this in turn is a consequence of the thermodynamics.

For the same thermodynamic reason these properties are also not dependent on the particular boundary condition assumed in the example corresponding to sealed pores at the surface. If the fluid is allowed to flow freely in and out of the surface the main effect is to change the relaxation constants r_n of the hidden degrees of freedom and multiply them by a factor 4.

Folding Instability of an Embedded Porous Layer

Consider the porous slab subject to an axial compressive stress P embedded in a viscoelastic medium, Fig. 3. We shall assume that the embedding medium is impervious. Hence we may apply the solution which has just been derived for the slab with a sealed surface.

When the layer undergoes a sinusoidal deflection

$$v = V \cos lx \tag{37}$$

the total reaction of the embedding medium on the layer per unit length is

$$-2q' = B_e(p)lv \tag{38}$$

The minus sign is used because the deflection and the reaction are in opposite directions. The operator $B_e(p)$ characterizes the embedding medium and has been previously discussed and evaluated [6, 7]. It is of the same general form as expression (19) for $B(p)$.

The theory of folding instability of the embedded porous slab is identical with the case of a layered viscoelastic medium which has been analyzed in detail in a previous paper [6]. The differential equation for the lateral deflection v of the layer is

$$\frac{1}{12} B(p) \frac{d^4 v}{dx^4} + Ph \frac{d^2 v}{dx^2} + B_e(p) lv = 0 \tag{39}$$

h^3

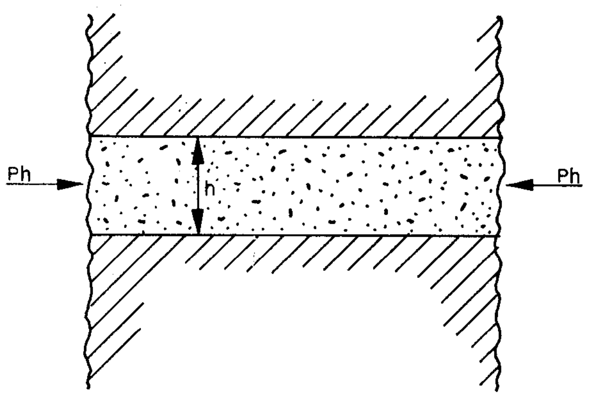

Fig. 3 **Porous layer embedded in an impervious viscous or viscoelastic medium**

For a sinusoidal deflection (37) we obtain the characteristic equation

$$P = \frac{1}{12} B(p) l^2 h^2 + \frac{B_e(p)}{lh} \tag{40}$$

Again here the thermodynamic property that $B(p)$ and $B_e(p)$ are increasing functions of p plays an important role. When l varies the value of P goes through a minimum, corresponding to the dominant wavelength. This dominant wavelength is

$$L_d = 2\pi h \left[\frac{B(p)}{6B_e(p)} \right]^{1/3} \tag{41}$$

For a given value of P this dominant wavelength is the one with the largest value of p, hence with the most rapid rate of increase of folding. The compression generating this wavelength is

$$P = \tfrac{3}{2} B_e(p) \left[\frac{B(p)}{6B_e(p)} \right]^{1/3} \tag{42}$$

As an illustration consider the case where the embedding medium is purely viscous and incompressible. In that case

$$B_e(p) = 4\eta p \tag{43}$$

where η is the coefficient of viscosity of the medium. The dominant wavelength and corresponding compression become

$$\begin{aligned} L_d &= \pi h \left[\frac{B(p)}{3\eta p} \right]^{1/3} \\ P &= 3\eta p \left[\frac{B(p)}{3\eta p} \right]^{1/3} \end{aligned} \tag{44}$$

A relation between the dominant wavelength and the compressive load P is obtained by plotting a curve of ordinate L_d and abscissa P as a parametric function of p. For a given value of P this curve yields the dominant wavelength and the rate of growth of the folding.

Thermoelastic Analogy

The author has shown that the mechanics of porous media and thermoelastic continua are isomorphic [4]. The equations of the two theories are identical and may be obtained from each other simply by a change of notation. This is more than a purely formal analogy. It reflects a deeper identity of the two phenomena when considered from the broader viewpoint of the thermodynamics of irreversible processes. The thermodynamic theory of thermoelastic continua was developed in an earlier publication [4]. It is based on new concepts, such as a generalized free energy for systems with nonuniform temperature, a dissipation function defined in terms of entropy displacement, and a generalized thermal force.

The theory of porous media applies immediately to thermoelasticity. The temperature replaces the fluid pressure, and the entropy displacement is used instead of the relative fluid displacement. Equations (3) are replaced by

$$\begin{aligned} \sigma_{xx} &= 2\mu e_{xx} + \lambda e - \beta\theta \\ \sigma_{yy} &= 2\mu e_{yy} + \lambda e - \beta\theta \\ \theta &= -\frac{\beta T_r}{c} e + \frac{T_r}{c} s \end{aligned} \tag{45}$$

where

- θ = excess temperature over equilibrium value T_r
- s = entropy density
- c = heat capacity per unit volume
- β = coefficient replacing α

The coefficients λ and μ are the isothermal elastic moduli. For adiabatic deformation we put $s = 0$ in equations (45) and derive the stress-strain relations

$$\begin{aligned}\sigma_{xx} &= 2\mu e_{xx} + \lambda_a e \\ \sigma_{yy} &= 2\mu e_{yy} + \lambda_a e\end{aligned} \tag{46}$$

with the adiabatic modulus

$$\lambda_a = \lambda + \beta^2 \frac{T_r}{c} \tag{47}$$

This modulus is the analog of λ_c defined by equation (34) in the case of a porous medium.

From these considerations we conclude that the buckling of a thermoelastic slab involves a lower and upper buckling load completely analogous to the values (29) and (35) for the case of a porous medium. The upper buckling load is

$$P_u = \frac{\pi^2}{12} \frac{h^2}{L^2} B_a \tag{48}$$

with

$$B_a = \frac{4\mu(\mu + \lambda_a)}{2\mu + \lambda_a} \tag{49}$$

The lower buckling load P_l is given by the same expression (29). The buckling of the porous and thermoelastic media is therefore exactly the same in its basic features. The only difference lies in the magnitude of the effect. Since λ_a is very close to λ the two buckling loads corresponding to *isothermal and adiabatic buckling* are so close together that they will be indistinguishable in practice.

Thermoelastic Damping of a Plate

The previous results are also applicable to the problem of propagation of thermoelastic waves in an elastic plate. The dynamical equations for a plate of thickness h and density ρ are obtained by analogy with the purely elastic case. We write

$$B(p) \frac{h^3}{12} \frac{d^4v}{dx^4} + p^2\rho h v = q(x) \tag{50}$$

A normal load $q(x) \exp (pt)$ harmonic function of time is applied. We put $p = i\omega$ where ω is the angular frequency. Equation (50) is derived from the thin-plate theory and is valid for wavelengths larger than several times the thickness. If we assume that heat loss has no time to take place at the free surface the boundary condition is the same as for the porous slab in the foregoing analysis. Hence the operator $B(p)$ is given by equation (19) where the physical constants of the porous medium are replaced by the corresponding quantities for the thermoelastic medium. We write

$$B(p) = B + \sum_{n=0}^{\infty} \frac{p}{p + r_n} B_n \tag{51}$$

with

$$\begin{aligned}B &= \frac{4\mu(\mu + \lambda)}{2\mu + \lambda} = \frac{E}{1 - \nu^2} \\ B_n &= \frac{96}{\pi^4} \frac{\beta^2\varphi^2}{(2n + 1)^4} M_a \\ \varphi &= \frac{2\mu}{2\mu + \lambda} = \frac{1 - 2\nu}{1 - \nu} \\ M_a &= \frac{2\mu + \lambda}{2\mu + \lambda_a} \cdot \frac{T_r}{c}\end{aligned} \tag{52}$$

The relaxation constants r_n are now

$$r_n = (2n + 1)^2\pi^2 \frac{KM_a}{h^2T_r} \tag{53}$$

where K is the thermal conductivity. In the analogy the ratio k/η appearing in equation (6) for the porous medium must be replaced by K/T_r [4].

Note that if the expression in equation (52) is reduced to its first two terms we obtain the same type of operator as derived in an earlier paper [4] by the variational method for a cantilever plate. The value of r_0 which was derived for that case is very close to that obtained from equation (53) by putting $n = 0$.

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