Short Notes

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THEORY OF INTERNAL BUCKLING OF A CONFINED MULTILAYERED STRUCTURE

Abstract: A simplified theory is derived for internal buckling of a multilayered structure. The layers, alternately competent and incompetent, are subject to a horizontal compression and confined between two rigid straight boundaries whose vertical separation varies with the amount of horizontal compressive strain. For purely viscous materials, the dominant folding wave length is expressed in terms of the viscosity ratio, the number of layers, and their thickness.

Introduction

The existence of internal buckling of a laminated medium with rigid confinement was brought out in a recent paper by Biot (1963a). In three other papers (Biot, 1963b; 1963c; 1964) an exact theory of buckling of multilayered continuous media has been developed that is applicable to viscous, elastic, and viscoelastic materials.

From these theories emerges the present simplified analysis which brings to light the controlling factors in buckling of multilayers. When this analysis is applied to internal viscous buckling it yields the striking result that the buckling wave length is about 20–60 times the layer thickness and is extremely insensitive to the viscosity contrast. This result provides an explanation for one of the predominant features of geological structures.

Internal Buckling

A typical buckling pattern of a confined laminated medium under a compression P parallel to the layers is shown in Figure 1. The critical value of the compression (Biot, 1963a) is

$$P = 4M\xi^2 + L(1-\xi^2)^2. \tag{1}$$

The parameter $\xi = \mathcal{L}/(2H)$ represents the ratio of the wave length $\mathcal{L}$ to twice the confinement distance H. The two elastic coefficients defining the property of the laminated medium are

$$L = \frac{\mu_1\mu_2}{\alpha_1\mu_2 + \alpha_2\mu_1} \tag{2}$$

$$M = \mu_1\alpha_1 + \mu_2\alpha_2 .$$

The laminations are composed of alternating layers of two types of incompressible materials of elastic moduli, μ_1 and μ_2, respectively occupying fractions α_1 and α_2 of the total thickness (Fig. 2). The medium behaves as an anisotropic continuous medium. An average vertical strain e_{yy} requires a vertical stress $\sigma_{yy} = 4Me_{yy}$, and an average shear strain φ requires a horizontal tangential stress $\sigma_{xy} = L\varphi$. The deformation is assumed to be one of plane strain. We have referred to L as the "slide modulus."

Shear Threshold

For $H = \infty$ the confinement disappears and equation (1) reduces to

$$P = L . \tag{3}$$

This is also the minimum value of the buckling load that occurs at a vanishing wave length for a confined medium. We shall call it the "shear threshold" since it defines a lower value of the compression below which internal buckling cannot occur in a laminated medium. For compressions above the threshold, equation (1) yields the approximate value

$$P = L + 4M\xi^2 . \tag{4}$$

The term L represents the sliding of one layer over the next.

Of considerable interest is the physical significance of the term $4M\xi^2$ which corresponds to the resistance to the vertical movement of folding. It is due to the confinement and the influence of the over-all vertical rigidity of the laminated medium represented by the modulus M. The effect of the finite thickness of the layers is represented by two major factors, bending stiffness and interstitial flow.

Bending Stiffness of the Layers

The thickness of a pair of layers is denoted by $2h$ (Fig. 2). Hence h is the average thick-

Geological Society of America Bulletin, v. 75, p. 563–568, 4 figs., June 1964

ness. We shall assume that the material of modulus μ_1 is appreciably more rigid than the other. If it were free the buckling of this layer would require a compressive stress.

$$P_1 = \tfrac{4}{3}\,\mu_1\alpha_1^{\,2}l^2h^2\,. \qquad (5)$$

This is the classical formula for buckling of a thin plate of thickness $2\alpha_1 h$ in a sinusoidal wave

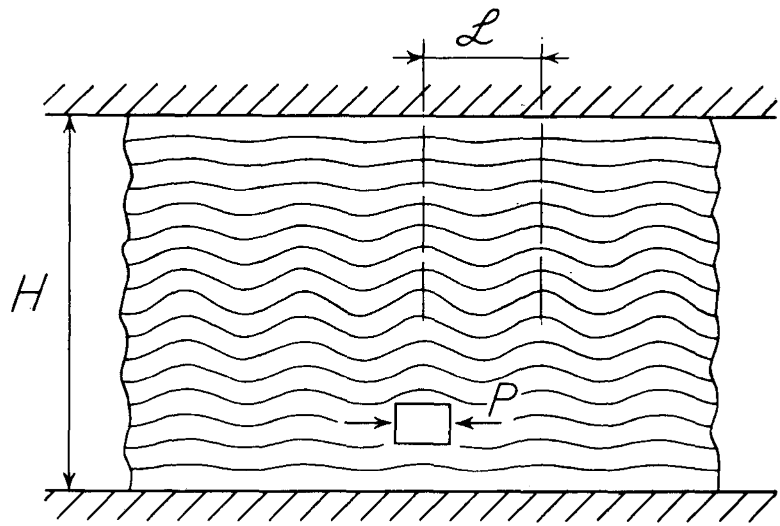

Figure 1. Internal buckling of a multilayered medium under rigid confinement

of wave length $\mathcal{L} = 2\pi/l$. The compression P_1 when averaged over the total thickness becomes

$$P = \alpha_1 P_1 = \tfrac{4}{3}\,\mu_1\alpha_1^{\,3}l^2h^2\,. \qquad (6)$$

The total buckling load of the medium is obtained by adding the values of equations (4) and (6). We obtain

$$P = L + 4M\xi^2 + \tfrac{4}{3}\mu_1\alpha_1^{\,3}l^2h^2 \qquad (7)$$

We put

$$\gamma = \tfrac{1}{2}\,lh = \frac{\pi h}{\mathcal{L}} \qquad (8)$$

$$n = \frac{H}{h}\,.$$

The parameter n is the total number of confined layers and γ/π is the ratio of the average layer thickness to the wave length. With these parameters equation (7) becomes

$$P = L + \left(\frac{\pi}{n}\right)^2 \frac{M}{\gamma^2} + \frac{16}{3}\,\mu_1\alpha_1^{\,3}\gamma^2\,. \qquad (9)$$

Because of the presence of the last term, which is due to the bending stiffness, the compression P now goes through a minimum value when plotted as a function of the wave length.

Apparent Compressibility and Interstitial Flow

When one type of layer is very soft in comparison with the other, an apparent compressibility sets in at the shorter wave lengths. In regions of higher vertical pressure the soft material tends to be squeezed out toward regions of lower pressure. This causes an interstitial flow of the soft material along the direction of the layers. (Fig. 3). The effect amounts to an apparent decrease of the vertical rigidity M by a factor which is wave length-dependent.

The effect may be evaluated as follows. The vertical stress q on the soft layer and the corresponding thickening $2V$ are distributed sinusoidally along the layering with a wave length $\mathcal{L} = 2\pi/l$ (Fig. 3). The relation between q and V was derived in a previous paper (Biot, 1963b). It was found that

$$q = l\mu_2 b_{22} V = 2\gamma\mu_2 b_{22}\frac{V}{h}\,. \qquad (10)$$

The coefficient b_{22} is given by equations (86) of the quoted paper (Biot, 1963b):

$$b_{22} = \frac{4\cosh^2\gamma_2}{\sinh 2\gamma_2 - 2\gamma_2}\,; \qquad (11)$$

where $\gamma_2 = \alpha_2 lh = 2\alpha_2\gamma$. The elastic modulus of the soft layer is μ_2. When the soft layer is thinner than about one tenth of the wave length, the value of γ_2 is small enough so that equation (10) may be written approximately as

$$2V = \frac{8}{3\mu_2}\alpha_2{}^3\gamma^2 qh\,. \tag{12}$$

The total thickening of a pair of layers is

$$2V = \left(\frac{1}{2M} + \frac{8}{3\mu_2}\alpha_2{}^3\gamma^2\right) qh\,. \tag{13}$$

The first term represents the thickening in the absence of interstitial flow, while the second

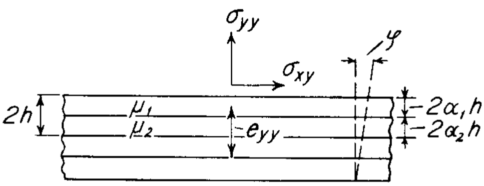

Figure 2. Details of multilayered structure and definition of the average coefficients $L = \sigma_{xy}/\varphi$ and $M = \sigma_{yy}/4e_{yy}$

term represents the additional effect due to this flow. Equation (13) shows that the apparent compressibility is measured by a modulus

$$M_a = M\delta\,, \tag{14}$$

obtained by applying a correction factor δ that is written

$$\delta = \frac{1}{1 + \kappa\gamma^2} \qquad \kappa = \frac{16}{3}\frac{M}{\mu_2}\alpha_2{}^3\,. \tag{15}$$

Replacing M by M_a the buckling load (9) becomes

$$P = L + \left(\frac{\pi}{n}\right)^2 M\frac{\delta}{\gamma^2} + \frac{16}{3}\mu_1\alpha_1{}^3\gamma^2\,. \tag{16}$$

The minimum of this expression as a function of γ determines the critical compression P and the wave length for internal buckling. The result is applicable to either viscous, elastic, or viscoelastic media by the principle of correspondence.

Internal Viscous Buckling

We shall discuss the case of purely viscous layers of viscosities, η_1 and η_2, where the ratio η_1/η_2 is large ($\eta_1/\eta_2 > 50$). We must put

$$\mu_1 = \eta_1 p \qquad \mu_2 = \eta_2 p\,. \tag{17}$$

With this coefficient p, the amplitude of the viscous buckling is proportional to the exponential function *exp* (pt) of the time t. We also assume the layers to be of equal thickness, *i.e.* $\alpha_1 = \alpha_2 = \frac{1}{2}$. Since $P = \frac{1}{2}P_1$, $L \cong 2\eta_2 p$, and $M \cong \frac{1}{2}\eta_1 p$, equation (16) becomes

$$\frac{P_1}{4\eta_1 p} = \frac{\eta_2}{\eta_1} + \left(\frac{\pi}{2n}\right)^2\frac{\delta}{\gamma^2} + \tfrac{1}{3}\gamma^2\,, \tag{18}$$

with

$$\frac{1}{\delta} = 1 + \frac{1}{3}\frac{\eta_1}{\eta_2}\gamma^2\,. \tag{19}$$

The value of γ for which expression (18) is a minimum yields the dominant wave length $\mathfrak{L}_d$. The ratio $\mathfrak{L}_d/h$ is a function of the viscosity ratio η_1/η_2 and the number n of confined layers. It is plotted in Figure 4. The striking result is that $\mathfrak{L}_d/h$ is almost independent of the viscosity ratio. The horizontal portions of the curve on the left side of the diagram are well approximated by the very simple expression

$$\mathfrak{L}_d = 1.90h\sqrt{n} = 1.90\sqrt{hH}\,. \tag{20}$$

It is obtained by neglecting the interstitial flow, putting $\delta = 1$ in equation (18). The ascending portion of the curves on the right of the diagram corresponds to large interstitial

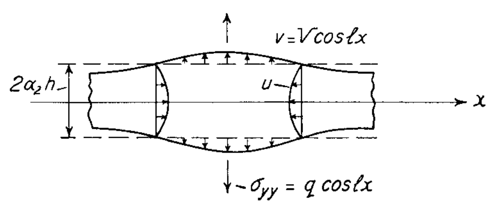

Figure 3. Deformation of the soft layer and associated interstitial flow represented by u

flow. In this case we write approximately $\delta = 3\eta_2/(\eta_1\gamma^2)$. This leads to the simplified formula

$$\frac{\mathfrak{L}_d}{h} = 1.66\, n^{1/3}\left(\frac{\eta_1}{\eta_2}\right)^{1/6}. \tag{21}$$

It is represented by the dashed curves in Figure 4.

The folding represented in Figure 1 corresponds approximately to the case of a hundred layers with a viscosity ratio $\eta_1/\eta_2 = 1000$.

The same plot yields the buckling wave length in the purely elastic case when the abscissa represents the rigidity ratio.

Discussion

Attention is called to the significance of the confinement for the case of purely viscous layers. The compression generates a constant flow rate producing a thickening and a gradual increase in the distance H between confining appear gradually with amplitudes distributed sinusoidally along the vertical in a half wave of length H vanishing at the top and bottom confining walls. The variation of H during folding is small, and the initial value may be used without much error. This is justified by a fundamental result showing that significant

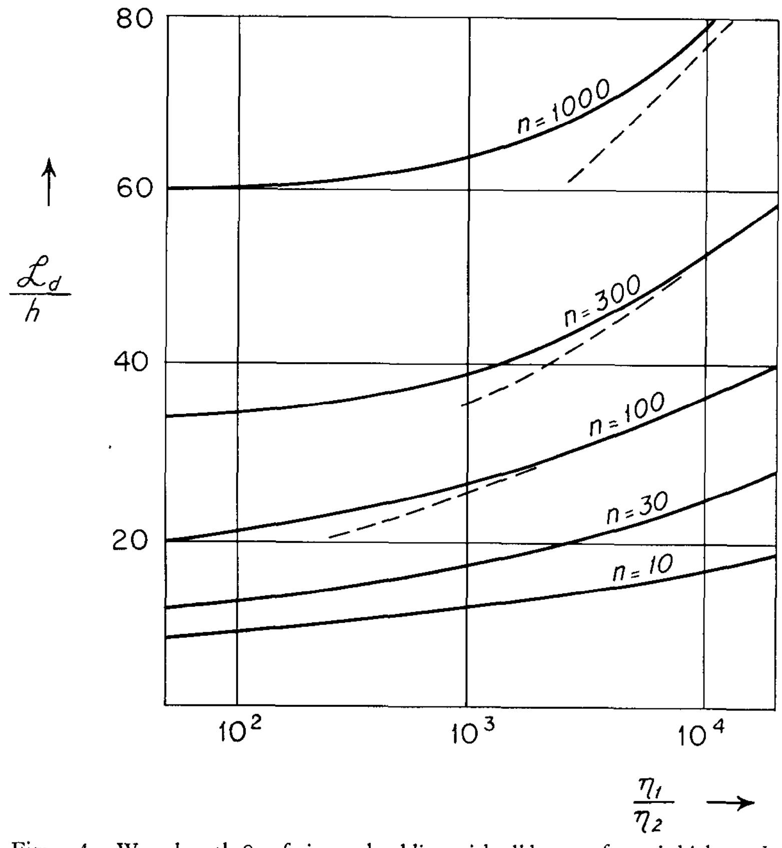

Figure 4. Wave length $\mathcal{L}_d$ of viscous buckling with all layers of equal thickness h. The number of confined layers is $n = H/h$, and the ratio of viscosities of two adjacent layers is η_1/η_2. Dashed curves represent equation 21.

walls. Geologically these confining walls correspond to the presence of thick competent layers on top and bottom of the multilayered structure that participate in the over-all compressive flow. Strictly speaking perfect slip is assumed at the rigid wall, but the presence of adherence introduces only a secondary correction. If perfect slip is assumed the compressive flow may be restricted to the multilayered structure alone.

During this process folding of the layers will folding will occur under these conditions with the emergence of the dominant wave length (Biot and others, 1961) if the viscosity contrast is sufficiently large. The conclusion is derived from a numerical evaluation of the time history of folding originating with local imperfections in the geometry of the layers.

The present results also include what might be called "self-confinement," where the multilayered medium is of infinite extent vertically and folding occurs with amplitudes distributed sinusoidally along the vertical with a wave length $2H$ as illustrated in an earlier paper (Biot, 1963a).

As pointed out, internal buckling tends to take place with the minimum wave length compatible with the microstructure. This wave length is therefore controlled primarily by the correction term for bending rigidity introduced in equation (7). The additional correction for interstitial flow is required only for cases where the value of equation (21) is larger than the value of equation (20); that is, if

$$\frac{\eta_1}{\eta_2} > 3.4\,n\,. \tag{22}$$

A theory of viscous buckling of multilayers was already developed earlier (Biot, 1961) as an extension of the single-layer problem (Biot, 1957). The result is applicable to many geological cases of similar folding with large viscosity contrast and "weak confinement." Under these conditions the incompetent layers act primarily as lubricant, and the dominant wave length is the same as if only the competent layers were present. Further justification is provided by the results obtained in the present paper that show that a similar situation prevails for internal buckling.

More elaborate theories of multilayered viscous folding may be derived by direct extension of the approximate methods used by this writer to take into account interfacial adherence (Biot, 1959). However it has been found preferable to develop the second phase of this investigation dealing with anisotropic and multilayered media by following different procedures along two distinct lines of approach.

One procedure is an analysis of collective behavior that uses a simplified formulation for the whole system that involves only the significant parameters as exemplified in the present treatment. The other, developed extensively in three separate papers (Biot, 1963b; 1963c; 1964), is an exact procedure that turns out to be relatively simple and leads to routine digital computational schemes. This opens the way to the treatment of a large number of layers of various thickness with viscous, elastic, or viscoelastic properties, isotropic or anisotropic, including the effect of gravity. No restrictions are imposed on the thickness to wave length ratio, and it is not necessary to distinguish between competent and incompetent layers.

The present theory of internal buckling is fundamentally different from a recent discussion of folding of multilayered structures (Ramberg, 1963) that assumes all deflections to be the same and requires the system of multiple layers to be separated from the rigid walls by two thick slabs of soft material and does not provide an expression for the dominant wave length. This latter problem is very close to the case of similar folding of a multilayered structure embedded in a soft medium already analyzed previously by a simple method (Biot, 1961) that brings out the controlling parameters. While Ramberg refers to "fluid dynamics" his discussion does not signify an essential departure from earlier methods that are based on approximations in the framework of viscoelastic correspondence, and lead to results applicable to either viscous or elastic materials by a simple change of language. By contrast a genuine and exact theory of viscous buckling of multilayered fluids based exclusively on fluid mechanics (Biot, 1964) leads to equations of a quite different type and provides a criterion for the range of validity of viscoelastic correspondence when applied to a fluid under initial stress undergoing initial flow with large deformations.

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Shell Development Company, Houston, Texas
Manuscript Received by the Society, December 3, 1963