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Validity of Thin-Plate Theory in Dynamic Viscoelasticity*

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Resonance damping for a vibrating plate is investigated both according to the exact equations of dynamical viscoelasticity and the classical thin-plate equations derived in mechanics of materials. The plate is assumed as isotropic and homogeneous and no shear- or rotatory-inertia corrections have been included in the thin-plate approximations. Two types of materials are investigated that correspond to real and complex values of the bulk modulus. For each case, the complex shear modulus is $\mu(1+ig)$ and values of g up to 0.10 were used in the calculations. The two theories are in excellent agreement in a range of wavelengths as low as about ten times the thickness. It is found that thin-plate theory evaluates the damping more accurately than it does the static rigidity.

LIST OF MAIN SYMBOLS

a_{ij}	coefficients in Eq. (3)
c_1	$(\mu/\rho)^{\frac{1}{2}}$
c_2	$[(\lambda+2\mu)/\rho]^{\frac{1}{2}}$
e	$=e_{xx}+e_{yy}$
e_{xx}	$=\partial u/\partial x$
e_{xy}	$=(1/2)(\partial u/\partial y+\partial v/\partial x)$
e_{yy}	$=\partial v/\partial y$
E	Young's modulus
$E/(1-\nu^2)$	$=4\mu(\lambda+\mu)/(\lambda+2\mu)$
g	damping constant in shear modulus
G	measure of the damping, exact theory
G_p	measure of the damping, plate theory
h	thickness of the plate
i	$\sqrt{-1}$
K	bulk modulus $=\lambda+(2/3)\mu$
l	wavenumber, Eq. (8)
$\mathcal{L}$	wavelength ($\mathcal{L}/h=\pi/\gamma$)
$\bar{q}_r/q_s$	$\epsilon+iG$ (exact theory)
$\bar{q}_{rp}/q_{sp}$	ϵ_p+iG_p (plate theory)
R	ratio of static load to static displacement, exact theory
$\bar{R}$	complex ratio of load to dynamic displacement, exact theory
R_p	ratio of static load to static displacement, plate theory
$\bar{R}_p$	complex ratio of load to dynamic displacement, plate theory
$R/\bar{R}$	$=1/(\epsilon+iG)$, amplification factor at resonance, exact theory
$R_p/\bar{R}_p$	$=1/(\epsilon_p+iG_p)$, amplification factor at resonance, plate theory
t	time
u	$=\partial\varphi/\partial x-\partial\psi/\partial y$, horizontal displacement, Fig. (1).
v	$=\partial\varphi/\partial y+\partial\psi/\partial x$, vertical displacement, Fig. (1).
U, V	displacement amplitudes of u, v, respectively
x, y	horizontal and vertical coordinates, Fig. (1).
β_1^2	$1-(\omega^2/c_1^2l^2)$
β_2^2	$1-(\omega^2/c_2^2l^2)$
ϵ	Eq. (24)
γ	$=lh/2$
θ	$=ly$
ω	frequency
Ω	$=\omega/c_1l$

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σ_{xy}	$=2\mu e_{xy}$
σ_{yy}	$=2\mu e_{yy}+\lambda e$
ρ	mass density
φ, ψ	solutions of wave equations, Eqs. (5)
μ	$=E/[2(1+\nu)]$, shear modulus
λ	$=\nu E/[(1+\nu)(1-2\nu)]$, Lamé's constant
$\bar{\mu}$	$=\mu(1+ig)$, complex shear modulus
$\bar{\lambda}$	$=(2\mu/3)[3\nu/(1-2\nu)-ig]$, complex Lamé's constant for material 1.
ν	Poisson's ratio
∇^2	Laplace's operator, in rectangular coordinates
$\exp(x)$	$=e^x$

INTRODUCTION

VIBRATING structural elements may reach dangerously large amplitudes at resonant frequencies. In the past, the resonant conditions have frequently been avoided in the design stages, but in many present-day problems this is no longer possible unless heavy weight penalties are acceptable. It is therefore necessary to introduce damping into the design of the structural components to reduce the peak stresses and amplitudes and to minimize the problem of fatigue.

Among the many sources of structural damping, that due to viscoelastic dissipation (material damping) is considered in this report. The presence of even a small amount of damping can have a decisive effect on the resonant amplitude and stress levels, and the actual reduction is very sensitive to the precise amount of damping that is present.

General studies of three-dimensional viscoelastic media have been developed in recent years based upon new methods and concepts in linear irreversible thermodynamics.[1–4] In the context of these general theories, a correspondence principle and an associated variational principle were developed[1,5,6] that are of extreme generality and are applicable to problems including dynamics and anisotropy. They provide a complete generalization of the so-called viscoelastic analogy, which had been developed earlier by Alfrey[7] and was restricted to the static stress analysis of isotropic incompressible media.

The generalization to compressible isotropic and anisotropic media was derived in a paper by Biot in 1954.[1] In two companion papers by the same author in 1955,[5,6] the term *correspondence principle* (or correspondence rule) was introduced in order to emphasize the far-reaching consequences of this development. It was pointed out that by this principle a vast body of results in the theory of elasticity becomes immediately applicable to viscoelasticity, including problems of vibrations and wave propagation in materials with isotropic or anisotropic properties.

Applications to the dynamic theory of viscoelastic plates were also developed in some detail in the quoted papers,[5,6] along with some general theorems for viscoelastic continua derived by variational procedures. This general formulation and its extension and generalizations make possible the solution of a large number of technically important problems involving plates and shells. We consider here an application to problems of *viscoelastic dissipation in plates.*

Practical engineering needs require the use of simplified theories such as the well-known thin-plate approximation that is derived in texts on the mechanics of materials. It is not clear how accurately theories of this type can predict the energy dissipation and resonant amplitude due to the viscoelastic nature of the material. The elementary theories of the type indicated usually predict displacements quite accurately, but stresses must be interpreted carefully for design purposes; the dissipative effects are an even more sensitive reflection of the displacement patterns and, for a proper evaluation of the effectiveness of simplified theory, it is necessary to work out fully a problem by the exact equations of dynamic viscoelasticity. The correspondence principle[1,5,6] can then be used to determine the damping according to the exact theory and the results can be compared with those derived from the thin-plate theory. In the present application, no shear or rotatory inertia corrections have been made to the thin-plate equations.

To carry out this program, a simple but typical example is taken that is capable of an exact solution by the three-dimensional equations of elasticity; the elastic moduli can then be replaced by their complex equivalents for the viscoelastic material and the dissipation can be calculated. An approximate numerical procedure has been devised for the evaluation of resonance amplitudes, which eliminates handling of complex functions. The thin-plate theory can also be applied to the same problem, and a precise comparison can then be made to determine the validity of the thin-plate approximations. The procedure to be used

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[1] M. A. Biot, "Theory of Stress-Strain Relations in Anisotropic Viscoelasticity and Relaxation Phenomena," J. Appl. Phys. **25**, 1385–1391 (1954).

[2] M. A. Biot, "Variational Principles in Irreversible Thermodynamics with Applications to Viscoelasticity," Phys. Rev. **97**, 1463–1469 (1955).

[3] M. A. Biot, "Deformation of Viscoelastic Plates Derived from Thermodynamics," Phys. Rev. **98**, 1869–1870 (1955).

[4] M. A. Biot, "Linear Thermodynamics and the Mechanics of Solids," Cornell Aeron. Lab. Rept. No. SA-987-S-6 (June 1958); presented as a general lecture at the 3rd U.S. National Congress of Applied Mechanics, Brown University, June 1958, and published in Proc. Natl. Congr. Appl. Mech., 3rd U.S. (1958).

[5] M. A. Biot, "Dynamics of Viscoelastic Anisotropic Media," in Proc. Midwestern Conf. Solid Mech., 4th, Purdue Univ., Purdue Engr. Expt. Sta. Publ. No. 129 (Sept. 1955).

[6] M. A. Biot, "Variational and Lagrangian Methods in Viscoelasticity," in *Proceedings of the Madrid Colloquium on Deformation and Flow of Solids, September 1955* (Springer-Verlag, Berlin, 1956).

[7] T. Alfrey, "Non-Homogeneous Stress in Viscoelasticity," Quart. Appl. Math. **2**, 113–119 (1944).

subsequently in this paper begins with the exact elasticity equations and shows how the thin-plate approximation arises as the result of geometric assumptions as to the "thinness" of the plate, and dynamical approximations that involve characteristic times of the applied sinusoidal loads and the velocities of the elastic waves in the plate.

The object of such comparisons is to arrive at the simplest technologically useful governing equations for use in more realistic situations.

The general problem of energy-dissipation mechanisms in structures has been considered by Lazan,[8,9] with a discussion of various engineering measures of the dissipation. The measurement of the damping has been considered by Plunkett[10]; vibrating-reed tests have been described by Bland and Lee[11] and by Horio and Onogi[12] that determine the complex moduli of given materials.

Demer[13] has presented an extensive bibliography in the field of material damping up to 1955; a selected bibliography[14] to 1959 is also available.

I. SOLUTION BY EXACT EQUATIONS OF DYNAMIC VISCOELASTICITY

The plate, Fig. (1), is assumed to be isotropic and homogeneous; $-\infty<x<\infty$, $-h/2\leq y\leq +h/2$. The motion is assumed to be independent of the coordinates z, and the horizontal and vertical coordinates are (x,y), as shown in the Figure.

The stresses in Fig. (1) are denoted by σ_{xy}, σ_{yy}, and displacements at any point (x,y,t) by (u,v); the stress amplitudes at the surface are denoted by (q,τ). The surfaces of the plate are at $y=\pm(h/2)$; additional information can be found in the List of Main Symbols.

Solutions of the dynamical equations of elasticity are required that at $y=\pm(h/2)$ reduce to

$$\begin{aligned}\sigma_{yy}&=\pm q\cos(lx)\exp(i\omega t),\\ \sigma_{xy}&=\tau\sin(lx)\exp(i\omega t).\end{aligned}\tag{1}$$

It is therefore assumed that surface tractions have been applied that are periodic along x and also periodic

[8] B. J. Lazan, "Review of Structural Damping Mechanism," in *WADC–University of Minnesota Conference on Acoustical Fatigue*, W. J. Trapp and D. M. Forney, Eds., WADC TR 59-676, pp. 168–184 (Mar. 1961).

[9] B. J. Lazan, "Energy Dissipation Mechanism in Structures, with Particular Reference to Material Damping," in *Structural Damping*, J. E. Ruzicks, Ed. (American Society of Mechanical Engineers, New York, 1959), pp. 1–34.

[10] R. Plunkett, "Measurement of Damping," in *Structural Damping*, J. E. Ruzicka, Ed. (American Society of Mechanical Engineers, New York, 1959), pp. 133–158.

[11] D. R. Bland and E. H. Lee, "Calculation of the Complex Modulus of Linear Viscoelastic Materials from Vibrating Reed Measurements," J. Appl. Phys. **26**, 1497-1503 (1955).

[12] M. Horio and S. Onogi, "Forced Vibration of a Reed as a Method of Determining Viscoelasticity," J. Appl. Phys. **22**, 977-981 (1951).

[13] L. J. Demer, "Bibliography of the Material Damping Field," (with abstracts and punched-card codings) Univ. Minn. WADC TR-56-180 (June 1956).

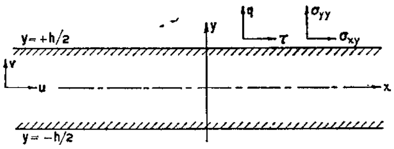

FIG. 1. Notations for the plate.

in time. The deformation is antisymmetric with respect to the x axis and represents a bending of the plate. The displacements are functions of (x,y,t), and at $y=\pm(h/2)$ it is required that

$$\begin{aligned}u&=\pm U\sin(lx)\exp(i\omega t),\\ v&=V\cos(lx)\exp(i\omega t).\end{aligned}\tag{2}$$

The surface stresses (q,τ) and the surface amplitudes (U,V) can be related linearly as

$$\begin{aligned}(\tau/l\mu)&=a_{11}U+a_{12}V,\\ (q/l\mu)&=a_{21}U+a_{22}V,\end{aligned}\quad (a_{12}=a_{21})\tag{3}$$

where μ is the shear modulus. The first problem is the determination of the a_{ij} coefficients in Eqs. (3).

It is well-known that the displacements can be written in the form

$$u=(\partial\varphi/\partial x)-(\partial\psi/\partial y)\quad\text{and}\quad v=(\partial\varphi/\partial y)+(\partial\psi/\partial x),\tag{4}$$

and, if u and v satisfy the dynamical equations of elasticity, then the conditions on φ and ψ are that these functions satisfy the scalar wave equations

$$c_1^2\nabla^2\varphi=\partial^2\varphi/\partial t^2,\quad c_2^2\nabla^2\psi=\partial^2\psi/\partial t^2,\tag{5}$$

where ∇^2 is Laplace's operator in rectangular coordinates and $c_1=(\mu/\rho)^{\frac{1}{2}}$ is the speed of the shear or S wave and $c_2=[(\lambda+2\mu)/(\rho)]^{\frac{1}{2}}$ is the speed of the P wave; λ is Lamé's constant, ρ is the mass density, and μ is the shear modulus. In later work, we shall also make use of the known relations

$$\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}=\frac{E}{(1-\nu^2)};\ \mu=\frac{E}{2(1+\nu)};\ \lambda=\frac{\nu E}{(1+\nu)(1-2\nu)},\tag{6}$$

where ν is Poisson's ratio and E is Young's modulus.

The solutions of the wave equation will be taken in the form

$$\begin{aligned}l^2\psi&=f_1(\theta)\sin(lx)\exp(i\omega t),\\ l^2\varphi&=f_2(\theta)\cos(lx)\exp(i\omega t),\end{aligned}\tag{7}$$

where

$$ly=\theta\quad\text{and}\quad\gamma=lh/2.\tag{8}$$

The parameter γ has a fundamental physical significance. It is inversely proportional to the ratio of the wavelength $\mathcal{L}$ to the plate thickness. We write

$$\mathcal{L}/h=\pi/\gamma,$$

where $\mathcal{L}$ is the wavelength of the deformation. Note that in the fundamental mode of a freely supported plate the span is represented by $\mathcal{L}/2$. A value of 0.3 for γ corresponds to a span of about 5 times the thickness; see Fig. 2. We show that this particular value plays an important part as a limiting case for the validity of the thin-plate theory.

The functions f_1 and f_2 satisfy the ordinary differential equations

$$f_1''-\beta_1^2 f_1=0,\quad ('=d/d\theta)\quad f_2''-\beta_2^2 f_2=0, \tag{9}$$

where $\beta_1^2=1-(\omega^2/c_1^2l^2)$ and $\beta_2^2=1-(\omega^2/c_2^2l^2)$. The appropriate solutions that have the correct evenness and oddness conditions for the vertical coordinates are

$$f_1=A_1\cosh(\beta_1\theta)\quad\text{and}\quad f_2=A_2\sinh(\beta_2\theta). \tag{10}$$

Thus, the functions (φ,ψ) are now known in terms of (θ,x,t) to within arbitrary constants (A_1,A_2). If the results, Eqs. (7), and (10), are used in Eq. (4), u and v are known and by (2) U and V can be written as

$$\begin{aligned} U&=A_1\beta_1\sinh(\beta_1\gamma)-A_2\sinh(\beta_2\gamma),\\ V&=A_2\cosh(\beta_1\gamma)+A_2\beta_2\cosh(\beta_2\gamma). \end{aligned} \tag{11}$$

The stresses are given by

$$(1/\mu)\sigma_{yy}=2e_{yy}+(\lambda/\mu)e,\quad (1/\mu)\sigma_{xy}=2e_{xy}, \tag{12}$$

where

$$e=e_{xx}+e_{yy}=(\partial u/\partial x)+(\partial v/\partial y)$$

and

$$e_{xy}=\tfrac{1}{2}[(\partial u/\partial y)+(\partial v/\partial x)].$$

Substitution of Eqs. (11) into Eq. (3) leads to the load-displacement relations in the form

$$\begin{aligned} (q/l\mu)&=2A_1\beta_1\sinh(\beta_1\gamma)+A_2(1+\beta_1^2)\sinh(\beta_2\gamma),\\ (\tau/l\mu)&=-A_1(1+\beta_1^2)\cosh(\beta_1\gamma)-2A_2\beta_2\cosh(\beta_2\gamma). \end{aligned} \tag{13}$$

The coefficients a_{ij} in Eqs. (3) can now be found by elimination of A_1, A_2 from among Eqs. (11), (13). The results are

$$\begin{aligned} Da_{11}&=\beta_2(1-\beta_1^2),\\ Da_{12}=Da_{21}&=2\beta_1\beta_2\tanh(\beta_1\gamma)-(1+\beta_1^2)\tanh(\beta_2\gamma),\\ Da_{22}&=\beta_1(1-\beta_1^2)\tanh(\beta_1\gamma)\tanh(\beta_2\gamma),\\ D&=\tanh(\beta_2\gamma)-\beta_1\beta_2\tanh(\beta_1\gamma). \end{aligned} \tag{14}$$

If only normal surface forces are present on the plate, then $\tau=0$, $q\neq 0$, and

$$(q/l\mu)=\left[\frac{a_{11}a_{22}-a_{12}^2}{a_{11}}\right]V. \tag{15}$$

Direct substitutions of the a_{ij} from Eq. (14) into (15) show that

$$\frac{a_{11}a_{22}-a_{12}^2}{a_{11}}=\frac{4\beta_1\beta_2\tanh(\beta_1\gamma)-(1+\beta_1^2)^2\tanh(\beta_2\gamma)}{\beta_2(1-\beta_1^2)}.$$

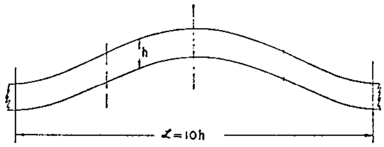

FIG. 2. Deformations of the plate for $\mathcal{L}=10h$ ($\gamma=0.3$). Drawn to scale.

Thus, the relationship between the normal-load component q and the displacement component V is known, Eq. (15).

For a purely elastic plate, resonance takes place if $q=0$, $\tau=0$. The characteristic equation is derived by introducing this condition in Eqs. (13), or, equivalently, by putting $q=0$ in (15). This yields the well-known form

$$\tanh(\beta_2\gamma)/\tanh(\beta_1\gamma)=4\beta_1\beta_2/(1+\beta_1^2)^2. \tag{16}$$

This characteristic equation is a relationship between frequency and wavelength that may be represented on a plot using the nondimensional variables $\Omega=\omega/c_1l$ and γ. The physical significance of Ω becomes evident when it is recalled that ω/l is the phase velocity along x.

In the present analysis, we assume that if the medium is elastic—that is if μ and λ are real—Eq. (16) is satisfied. In other words, the frequency is chosen as a function of the wavelength in such a way that the plate is at resonance. Moreover, the resonant condition is assumed to be defined by the lowest-frequency branch of Eq. (16). This branch corresponds to bending vibrations that degenerate into Rayleigh waves for wavelengths smaller than the plate thickness.

A. Complex Moduli

To compute the damping at resonance, the moduli μ and λ are replaced by complex quantities representing the viscoelastic properties of the medium. The moduli are functions of the imaginary frequency $p=i\omega$. The values adopted here are those corresponding to the resonant frequency. We consider two types of materials: one with zero bulk damping, the other with equal values of bulk and shear damping. Two values $\nu=\frac{1}{4}$ and $\nu=\frac{1}{2}$ of Poisson's ratio will be assumed. The examples treated below are intended as extreme cases in order to bracket a range of properties of common materials.

1. *Material I*

This material is purely elastic for volume changes and viscoelastic for shear deformation. The shear modulus μ is replaced by $\bar{\mu}=\mu(1+ig)$. For a material with bulk elasticity, it was shown[1] that λ is replaced by

$$\bar{\lambda}=K-(2/3)\bar{\mu} \tag{17}$$

where K is a real quantity representing the bulk modulus. This may also be written as

$$\bar{\lambda}=(2/3)\mu[(3\nu)/(1-2\nu)-ig], \tag{18}$$

where ν is Poisson's ratio for the purely elastic medium.

2. *Material II*

For a material of the second type, we choose a particular case where $\bar{\mu}$ and $\bar{\lambda}$ contain the same complex factor; that is,

$$\bar{\mu}=\mu(1+ig), \quad \bar{\lambda}=\lambda(1+ig). \tag{19}$$

The bulk viscoelasticity is then represented by the complex modulus

$$K=(\lambda+\tfrac{2}{3}\mu)(1+ig). \tag{20}$$

General properties of materials of this type were discussed in an earlier paper.[6] The term "homogeneous spectrum" was used to indicate this property. The term *uniform spectrum* seems preferable and is used in this paper.

Representative values of g for structural materials are in the range 0.01 to 0.10, with values 0.25 being typical of materials used in solid propellant grains.

3. *Amplification Factor at Resonance*

This factor is the ratio of the dynamic deflection at resonance to the static deflection under the same load at zero frequency.

We first evaluate the plate amplitude under dynamic conditions. By the correspondence principle, we now replace μ and λ by $\bar{\mu}$ and $\bar{\lambda}$ in Eq. (15). The result is

$$\bar{q}/l\bar{\mu}=(\bar{a}_{11}\bar{a}_{22}-\bar{a}_{12}^2)V/\bar{a}_{11}, \tag{21}$$

or

$$\bar{q}=\bar{R}V \quad \text{with} \quad \bar{R}=(l\mu/\bar{a}_{11})(\bar{a}_{11}\bar{a}_{22}-\bar{a}_{12}^2). \tag{22}$$

The load q is now a complex quantity $\bar{q}$. The complex quantities $\bar{\beta}_1$, $\bar{\beta}_2$ and the $\bar{a}_{ij}$ also replace β_1, β_2 and a_{ij}, respectively.

Under static conditions, $\omega=0$ and the static load q_s is given by the limiting value of Eq. (15) as β_1 and β_2 approach unity. Hence,

$$q_s=RV \quad \text{with} \quad R=\frac{2l\mu(\lambda+\mu)}{\lambda+2\mu}(\tanh\gamma-\operatorname{sech}^2\gamma). \tag{23}$$

What we are interested in here is the ratio $\bar{q}/q_s$ at resonance. We write this ratio as

$$\bar{q}_r/q_s=\bar{R}/R=\epsilon+iG. \tag{24}$$

The complex load $\bar{q}_r$ is that given by Eq. (21) at resonance; that is, assuming that if we put $g=0$ Eq. (16) is satisfied. As pointed out above, this relates the frequency to the wavelength by a curve that is chosen to be the lowest-frequency branch of the characteristic equation (16).

The physical significance of the ratio $\epsilon+iG$ is obtained by comparing the static deflection V_s to the dynamical deflection $\bar{V}_r$ under the same load q at resonance. We write

$$\begin{aligned} q&=RV_s,\\ q&=\bar{R}\bar{V}_r. \end{aligned} \tag{25}$$

Hence,

$$\bar{V}_r/V_s=R/\bar{R}=1/(\epsilon+iG) \tag{26}$$

represents the *amplification factor at resonance.*

At resonance, $\bar{R}$ vanishes for a purely elastic medium. The quantity G is a measure of the damping and is of the order g for small g. The real part ϵ is of the order g^2. Since g is sufficiently small, only the linear terms are retained in the analysis and G/g is then independent of g. The validity of this approximation is established in the numerical analysis of the problem.

II. THIN-PLATE APPROXIMATION

The equation of motion of a thin plate is

$$\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4 V}{\partial x^4}+\rho h\frac{\partial^2 V}{\partial t^2}=\text{applied load.} \tag{27}$$

If $V(x,t)$ in Eq. (27) is assumed as $V\cos(lx)\exp(i\omega t)$ and the applied load as $2q_p\cos(lx)\exp(\omega t)$, then Eq. (27) becomes

$$2q_p=-\rho h\omega^2 V+[Eh^3/12(1-\nu^2)]l^4V, \tag{28}$$

or, with $\gamma=lh/2$ and $E/(1-\nu^2)=4\mu(\lambda+\mu)/(\lambda+2\mu)$, as

$$(q_p/l\mu)=\left[\frac{4(\lambda+\mu)}{3(\lambda+2\mu)}\gamma^3-\frac{\rho\omega^2}{\mu l^2}\right]V. \tag{29}$$

In the static case ($\omega=0$), the result is

$$(q_{sp}/l\mu)=(4/3)\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)\gamma^3 V. \tag{30}$$

With $\bar{q}_{rp}$ denoting the complex load at resonance for the viscoelastic thin-plate theory, we set $\bar{q}_{rp}/q_{sp}=\epsilon_p+iG_p$, which is the analog of the expression $(\epsilon+iG)$ from the exact theory. By the same reasoning used in obtaining Eq. (26), we find the amplification factor at resonance for the thin plate to be

$$\bar{V}_{rp}/V_{sp}=1/(\epsilon_p+iG_p). \tag{31}$$

In the present case, we find that

$$G_p/g=\frac{2}{3}\left(\frac{1-\nu+\nu^2}{1-\nu}\right)\times\left[\frac{1+[(1-2\nu)^2/3(1-\nu+\nu^2)]g^2}{1+(4/9)[(1-2\nu)/(1-\nu)]^2g^2}\right] \tag{32}$$

or

$$G_p/g=(2/3)\left(\frac{1-\nu+\nu^2}{1-\nu}\right) \tag{33}$$

for $g^2 \ll 1$; however, the exact dependence of G_p/g can be computed in this case. If $\nu=\frac{1}{2}$, G_p/g is independent of g; if $\nu=\frac{1}{4}$, for example, the multiplying factor is $1-0.095g^2/(1+0.197g^2)$. If $g=\frac{1}{4}$, this factor is 0.9944; at $g=\frac{1}{2}$, the factor is 0.96. Hence, the linearized value of G_p/g is sufficiently accurate in the case of the thin-plate equation.

A. Thin-Plate Equation as a Limiting Case of the Exact Theory

To show how the exact theory may be used to determine the thin-plate equation as a limiting case, we return to Eq. (16) and expand the hyperbolic tangents to two terms to write the result as

$$(q_p/l\mu)=\left\{-\gamma(1-\beta_1^2) + \frac{1}{3}\frac{\gamma^3}{1-\beta_1^2}[\beta_2^2(1+\beta_1^2)^2-4\beta_1^4]\right\}V. \quad (34)$$

Since $\beta_1^2=1-(\omega^2/c_1^2l^2)$, $\beta_2^2=1-(\omega^2/c_2^2l^2)$, and $c_1^2=(\mu/\rho)$, $c_2^2=(\lambda+2\mu)/\rho$, we may write Eq. (34) as

$$2q_p=-\rho h\omega^2V+\frac{l^4h^3\mu}{12}\times\left[\beta_2^2(1-\beta_1^2)+\frac{4\beta_1^2(\beta_2^2-\beta_1^2)}{1-\beta_1^2}\right]V. \quad (35)$$

If $(\omega^2/c_1^2l)\ll 1$ and $(\omega^2/c_2^2l^2)\ll 1$, since $1-(c_1/c_2)^2=(\lambda+\mu)/(\lambda+2\mu)$, Eq. (35) immediately reduces to Eq. (28). The equation of motion of a thin plate can therefore be obtained as a limiting form of the exact equations, under conditions that require assumptions both of a geometric nature as well as of a dynamic nature.

von Kármán[15] has also treated the problem of applying the equations of static elasticity theory to obtain a simplified beam equation; the final results are the same when proper comparison is made of the differences between plane stress and plane strain.

In the derivation of the thin-plate/load-deflection relationship from the exact solution, two terms were neglected in comparison with the terms retained. If the neglected terms are included in the analysis, the results are more complex, but in a typical case ($\nu=\frac{1}{4}$) the previous linearized result of $G_p/g=13/18$ becomes $G_p/g=(13/18)+(1/18)(5\Omega^2+\Omega^4)$, where now $\gamma^2=(9\Omega^2)/(8-5\Omega^2-\Omega^4)$ and the G_p/g ratio is now a function of γ. If we anticipate the numerical results and take $\Omega^2=0.302$ ($\gamma=0.763$), then the improved value of G_p/g is 0.810, as compared with $13/18=0.722$. The exact result for G/g in this case is 0.813, so that good agreement is possible at least up to values of $\gamma=\frac{3}{4}$, or well beyond the range of the conventional beam theory, which would not be expected to hold much beyond $\gamma=0.30$.

III. NUMERICAL PROCEDURE

The quantity $\bar{q}_r/(l\mu)=f(ig)$, a function of the argument (ig), where $f(0)=0$ from the resonance condition. Hence, the expansion begins with the term linear in g and, to first-order terms, $\bar{q}_r/(l\mu)=f'(0)ig$. Only real quantities are needed to evaluate $f'(0)$. However, the frequency equation (16) must be solved to determine the relationship between $\Omega=(\omega c_1/l)$ and $\gamma=lh/2$. The outline of the procedure may be illustrated as follows.

A. Material I

Material I is purely elastic for volume changes and viscoelastic for shear deformation ($K=\bar{K}$). We have the following data:

$$\nu=\tfrac{1}{4}, \qquad 0<\gamma<2$$
$$\Omega=\omega/c_1l,$$
$$\beta_1^2=1-\Omega^2;\quad \beta_2^2=1-(1/3)\Omega^2,$$
$$\lambda=\mu;\quad \bar{\mu}=\mu(1+ig),\quad \bar{\lambda}=\mu(1-\tfrac{2}{3}ig).$$

To obtain an approximate solution of the frequency equation (16), we write this equation in the form

$$\sinh(\beta_2\gamma)\cosh(\beta_1\gamma)(1+\beta_1^2)^2=4\beta_1\beta_2\sinh(\beta_1\gamma)\cosh(\beta_2\gamma),$$

and approximate each hyperbolic function to three terms to obtain the approximate equation

$$\Omega^2+(\gamma^2/9)(-8+14\Omega^2-5\Omega^4)+(\gamma^4/270)(-48+116\Omega^2-86\Omega^4+19\Omega^6)=0. \quad (36)$$

If the first two terms of Eq. (36) are used and γ is assigned, Ω^2 can be computed from a quadratic equation and resubstituted into (36) to obtain γ^2 from a quadratic equation. This easily determines a set of initial $(\Omega-\gamma)$ values that can be used in the transcendental equation (16) to determine the final values of $(\Omega-\gamma)$, which are used in the subsequent calculations. In the thin-plate approximation, $G_p/g=13/18$, a constant independent of γ.

To simplify the calculation of G/g to terms that are linear in g ($g\ll 1$), we write

$$\bar{q}/(l\mu)=[(1+ig)F(ig)V]/[\bar{\beta}_2(1-\bar{\beta}_1^2)]=-f(ig)=f'(0)ig+\cdots,$$

[14] "Selected Bibliography on Structural Damping" in *Structural Damping*, J. E. Ruzicka, Ed. (American Society of Mechanical Engineers, New York, 1959), pp. 159–165.

[15] T. von Kármán, "Über die Grundlagen der Balkentheorie," in *Collected Works II* (Butterworth's Scientific Publications Ltd., London, 1956), pp. 153–163.

where

$$F(ig)=4\bar{\beta}_1\bar{\beta}_2 \tanh(\bar{\beta}_1\gamma)-(1+\bar{\beta}_1^2)^2 \tanh(\bar{\beta}_2\gamma). \quad (37)$$

Since $\bar{\beta}_1^2=1-\Omega^2/(1+ig)$, $(d\bar{\beta}_1/dg)_{g=0}=(\Omega^2/2\beta_1)i$ and β_1 may be expanded in terms of g; the same procedure can be used for $\bar{\beta}_2$ and for all combinations that appear in $f(ig)$. The quantities β_1 and β_2 are, of course, real and known since Ω is known. Consistent application of this expansion procedure leads to the value of $f'(0)$, which will involve only the real quantities β_1, β_2, γ and hyperbolic functions of real arguments.

B. Material II

For a material of the second kind, which exhibits bulk viscosity $\bar{K}=(\lambda+\frac{2}{3}\mu)(1+ig)$ and $\bar{\lambda}=\lambda(1+ig)$, $\bar{\mu}=\mu(1+ig)$; this material has a uniform spectrum. For a given value of ν, the resonant $(\Omega-\gamma)$ relationship is the same as for the first material. However, $\bar{\beta}_1^2=1-\Omega^2/(1+ig)$ and $\bar{\beta}_2^2=1-\Omega^2/3(1+ig)$ if $\nu=\frac{1}{4}$, and this change requires only slight modifications in the previous work; in fact, only two numerical coefficients of F in Eq. (37) must be changed and all other numerical work is the same.

When $\nu=\frac{1}{2}$, $\lambda=\infty$ and $\beta_1^2=1-\Omega^2$, $\beta_2^2=1$; $\bar{\beta}_1^2=1-\Omega^2/(1+ig)$, $\bar{\beta}_2^2=1$ and these results are the same as before in the incompressible case. Thus, the results for both material are the same if $\nu=\frac{1}{2}$.

An exact evaluation for the elasticity solution is involved but straightforward; a linearized calculation is again a simple matter. A complete (nonlinear) calculation in this case showed precisely the same behavior upon g as in the thin-plate approximation, which justifies the use of the linearized calculations throughout the analysis.

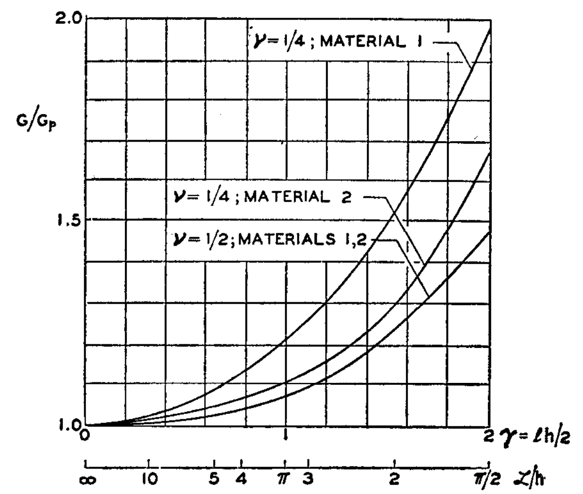

FIG. 3. Ratio of damping factor G/G_p as a function of $\gamma=lh/2$. *Material I*: Elastic for volume changes and viscoelastic for shear deformation. *Material II*: Uniform spectrum (exhibits bulk viscosity). Corresponding values of wavelength-to-thickness ratio $\mathcal{L}/h$ are also shown.

IV. PRESENTATION AND DISCUSSION OF RESULTS

The ratio G/G_p gives a comparison of the amplification factor at resonance for the exact and thin-plate theories. This ratio as a function of $\gamma=lh/2$ is shown in Fig. (3) for materials 1 and 2 ($\nu=\frac{1}{2}$ and $\frac{1}{4}$). In the calculations made to obtain the curves, the quantities G/g and G_p/g were computed for small g. Since the technological range of interest is for values of g to 0.10, the results are accurate for purposes of comparison and for direct use in applications.

Figure (3) shows that in the range $0<\gamma<0.3$ there is almost no detectable error in the case $\nu=\frac{1}{2}$; for $\nu=\frac{1}{4}$, the error is about 4% at $\gamma=0.30$. At values of $\gamma=0.60$, the errors are ($\nu=\frac{1}{2}$) 1% and ($\nu=\frac{1}{4}$) 9% for Material I and 3% for Material II. In each case, the thin-plate theory under-estimates slightly the damping that is present.

For completeness, the resonance amplitudes $\bar{V}_r$ and $\bar{V}_{rp}$ themselves should be compared. This requires an evaluation of the static deflection. We write

$$\bar{V}_r/V_s=1/(\epsilon+iG), \quad \bar{V}_{rp}/V_{sp}=1/(\epsilon_p+iG_p),$$

$$\bar{V}_r/\bar{V}_{rp}=\frac{\epsilon_p+iG_p}{\epsilon+iG}(V_s/V_{sp})\cong(G_p/G)(V_s/V_{sp}).$$

From Eqs. (23), we find the static-deflection ratio

$$(V_{sp}/V_s)=3/(2\gamma^3)(\tanh\gamma-\gamma\,\mathrm{sech}^2\gamma)$$
$$=1-(4/5)\gamma^2+(51/105)\gamma^4+.$$

This ratio is plotted in Fig. (4).

For the static deflection (at zero frequency), the thin-plate theory underestimates the deflection; for $\gamma=0.30$, the error is about 7%. The resonant amplitude results from the combined effect of the damping and the static-deflection corrections, which act in opposite directions. The latter correction is appreciably larger and the validity of the thin-plate theory is appreciably better for the evaluation of the damping than for the static rigidity. For wavelengths larger than ten times the thickness, the error of the thin-plate theory decreases rapidly.

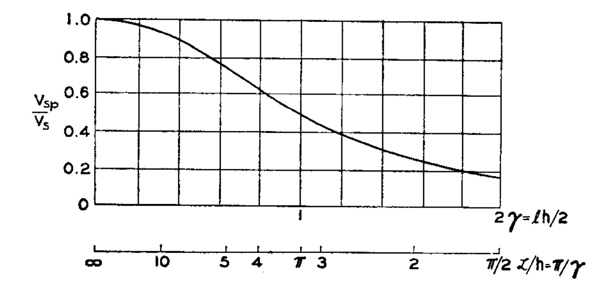

FIG. 4. Comparison of static deflections by exact (V_s) and thin-plate (V_{sp}) theories.

V. CONCLUDING REMARKS

In evaluating the resonance amplitude of a homogeneous isotropic viscoelastic plate, the thin-plate approximation is found to hold in the range $0<\gamma<0.30$, or for wavelengths larger than about ten times the thickness. The present results, Fig. (3), show that G/G_p is close to unity in this range, for both cases $\nu=\frac{1}{4}$ and $\frac{1}{2}$. The incompressible case ($\nu=\frac{1}{2}$) provides better agreement than $\nu=\frac{1}{4}$, although the latter is in error by less than 4% in the stated range. In each case, the thin-plate theory slightly underestimates the damping; the thin-plate theory will therefore slightly overestimate the resonant-amplitude factor. Comparison of the thin-plate and exact theories for static deflections shows an opposite behavior; i.e., the thin-plate theory underestimates the deflections. The error of the thin-plate theory for the damping is smaller than for the static deflection.

ACKNOWLEDGMENT

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