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# Variational Analysis of Ablation for Variable Properties[1]

*The application of variational and Lagrangian thermodynamics is extended to problems of heat conduction with melting boundaries. The physical properties of the conducting material are considered to be temperature dependent. In particular, the material Alumina* ($Al_2O_3$) *representing a class of ceramic materials for which the effective conductivity* $k_{\text{eff}} = k_{\text{phonon}} + k_{\text{photon}}$ *is subjected to above treatment and the temperature distribution and the melting rate are found. The results are compared with the constant-conductivity case.*

## Introduction

The behavior, after the initiation of melting, of a slab insulated on one side and subjected to heat input (constant or varying with time) on the other, has been studied analytically by several workers [1, 2, 3][2] in the past. Citron [4] in 1959 has developed the method of successive approximation to study this problem under constant physical properties. He further applies Galerkin's method to study the problem when the properties of the material vary linearly with temperature. The recent work of Biot and Daughaday [5] on ablation for the material of constant properties deals with the application, to such problems, of the variational and Lagrangian thermodynamics developed earlier [6, 7]. A remarkable agreement with the exact solution of Landau [1] is obtained.

The purpose of the present work is to show that the applicability of the Lagrangian equations is not restricted to the study of ablation for constant properties. The most complex temperature dependency of the properties of the material, on the other hand, can be taken care of by this method, avoiding heavy computational work hitherto needed in analytical solutions. The method also permits one to account for the heating history prior to melting. The temperature distribution and the rate of melting are found. A numerical example for a class of ceramic materials represented by Alumina ($Al_2O_3$), whose conductivity is comprised of the phonon and photon conductivities, is solved and comparison is made with the constant conductivity case.

## Formulation of the Problem

Consider one-dimensional heat conduction in a semi-infinite cylinder of unit cross section (Fig. 1) whose thermal conductivity is a function of temperature and the heat capacity is constant. The face at $y = 0$ is heated at a constant rate $R$ per unit area, while the face at $y \to \infty$ behaves as an insulated surface. If heating continues long enough, the face at $y = 0$ reaches the melting temperature and melting commences. It is assumed that the liquid is removed immediately on formation. Let $s(t)$ denote the position at time $t$ of the face which was initially at $y = 0$, so that $\dot{s}$ is the melting rate of the solid. Let $q(t)$ be the penetration depth up to which the heat effect due to $R$ reaches. Time $t$ is measured from the start of melting. The equations describing the process are

$$\frac{\partial}{\partial y}\left[k(\theta)\frac{\partial \theta}{\partial y}\right] = c\,\frac{\partial \theta}{\partial t}, \qquad s(t) < y < \infty, \quad t > 0 \tag{1}$$

$$\left.\frac{\partial \theta}{\partial y}\right|_{y\to\infty} = 0, \qquad t > 0 \tag{2}$$

$$\dot{s}(0) = \dot{s}_0 \tag{3}$$

$$\left.\begin{array}{l} \theta(y, t) = \theta_m \qquad (4a) \\ R = \text{the rate of heat flow into the slab} + \\ \quad \text{the rate of heat removal by the} \\ \quad \text{melted material} \end{array}\right\} \text{ at } y = s(t) \qquad (4b)$$

## Variational Procedure

In order to apply the variational method to the foregoing problem, the following transformation is useful:

$$u = \int_0^{\theta} \frac{k(\theta)}{k_m}\,d\theta = \int_0^{\theta} f(\theta)d\theta \tag{5}$$

$$u_m = \int_0^{\theta_m} f(\theta)d\theta \tag{5a}$$

where $k_m$ is the conductivity at the melting temperature $\theta_m$. This represents the physical mapping of the system by a model where $u$ is the model temperature. The equations (1)–(4$a$) reduce to

$$k_m\,\frac{\partial^2 u}{\partial y^2} = \frac{c}{f(\theta)}\,\frac{\partial u}{\partial t}, \qquad s(t) < y < \infty, \quad t > 0 \tag{6}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y\to\infty} = 0, \qquad t > 0 \tag{7}$$

$$\dot{s}(0) = \dot{s}_0 \tag{8}$$

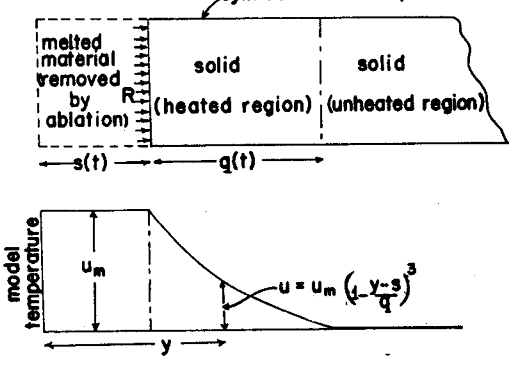

Fig. 1 Ablation of a cylinder in half space

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[2] Numbers in brackets designate References at end of paper.

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$$u(y, t) = u_m, \qquad y = s(t) \qquad (9)$$

The problem can now be solved as if $u$ were the temperature distribution, and $(c/f(\theta))$ the heat capacity per unit volume as a function of $(u/u_m)$, the constant thermal conductivity being $k_m$. The reasons for using $u$ are:

($a$) Even if the heat capacity $c$ is temperature-dependent, the parameter $c/k(\theta)$ is the only experimental function needed. It can be represented by one single curve.

($b$) Since the straight line is the exact time-independent solution for the steady-state nonablating case $\partial^2 u/\partial y^2 = 0$, it is natural to assume that a smooth-curve approximation for $u$ is also a good approximation for the transient case.

The equation (6) may be expressed as

$$k_m \frac{\partial^2 u}{\partial y^2} = cF(\tau)\frac{\partial u}{\partial t}, \qquad \frac{1}{f(\theta)} = F(\tau), \qquad \tau = u/u_m \qquad (6a)$$

Recalling some of the general results in reference [7], we define

$$h(y, u) = \int_0^u cF(\tau)du = cu_m \int_0^\tau F(\tau)d\tau \qquad (10)$$

= the total heat acquired by the unit volume. A density function is defined as

$$E(y, h) = \int_0^u cF(\tau)u\,du = cu_m^2 \int_0^\tau \tau F(\tau)d\tau \qquad (11)$$

The thermal potential is given by

$$V = \int E(y, h)dy \qquad (12)$$

Lagrangian heat-flow equations are obtained in the familiar form [6, 7]

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \qquad (13)$$

## Solution of the Problem

Assume a cubic profile for the model temperature $u$:

$$\tau = u/u_m = \left(1 - \frac{y - s}{q}\right)^3 \qquad (14)$$

or

$$\tau = \zeta^3, \qquad \zeta = \left(1 - \frac{y - s}{q}\right) \qquad (15)$$

The heat content $h$ can be represented as

$$h = cu_m\varphi(\tau), \qquad \varphi(\tau) = \int_0^\tau F(\tau)d\tau \qquad (16)$$

The density function becomes

$$E = cu_m^2\Phi(\tau), \qquad \Phi(\tau) = \int_0^\tau \tau F(\tau)d\tau \qquad (17)$$

The thermal potential is given by

$$V = \int_s^{s+q} E dy = cu_m^2 \int_s^{s+q} \Phi(\tau)dy \qquad (18)$$

$$= \frac{q}{2} cu_m^2 N \qquad (19)$$

$$N = \frac{2}{3}\int_0^1 \tau^{-2/3}\Phi(\tau)d\tau = 2\int_0^1 \tau(1 - \tau^{1/3})F(\tau)d\tau \qquad (20)$$[3]

The heat flow per unit area in the $y$ direction is denoted by $H$ and is obtained as

$$H = \int_y^{s+q} h dy = cu_m \int_y^{s+q} \varphi(\tau)dy \qquad (21)$$

$$= \frac{1}{3} cu_m q\Psi(\tau), \qquad \Psi(\tau) = \int_0^\tau \tau^{-2/3}\varphi(\tau)d\tau \qquad (22)$$

The condition $H = 0$ at $y = s + q$ has been used. The rate of heat flow is given by

$$\dot{H} = \frac{1}{3} cu_m\dot{q}\Psi(\tau) + \frac{1}{3} cu_m q\dot{\Psi}(\tau) \qquad (23)$$

$$\dot{\Psi}(\tau) = \frac{d\Psi}{d\tau}\left[\frac{\partial \tau}{\partial q}\dot{q} + \frac{\partial \tau}{\partial s}\dot{s}\right] \qquad (24)$$

$$= \frac{1}{q}\frac{d\Psi}{d\tau}[3\zeta^2(1 - \zeta)\dot{q} + 3\zeta^2\dot{s}] \qquad (25)$$

$$= \frac{1}{q}\frac{d\Psi}{d\tau}[3\tau^{2/3}(1 - \tau^{1/3})\dot{q} + 3\tau^{2/3}\dot{s}] \qquad (26)$$

Hence

$$\dot{H} = cu_m\left[\left\{\frac{1}{3}\Psi(\tau) + (1 - \tau^{1/3})\varphi(\tau)\right\}\dot{q} + \varphi(\tau)\dot{s}\right]$$
$$= cu_m[X(\tau)\dot{q} + \varphi(\tau)\dot{s}] \qquad (27)$$

$$X(\tau) = \left\{\frac{1}{3}\Psi(\tau) + (1 - \tau^{1/3})\varphi(\tau)\right\} = \int_0^\tau (1 - \tau^{1/3})F(\tau)d\tau$$

The dissipation function becomes

$$D = \frac{1}{2k_m}\int_s^{s+q} \dot{H}^2 dy = \frac{c^2u_m^2}{k_m} C \qquad (28)$$

where

$$C = \frac{1}{2}\int_s^{s+q} [X^2\dot{q}^2 + 2X\varphi\dot{q}\dot{s} + \varphi^2\dot{s}^2]dy$$
$$= \frac{q}{2}\int_0^1 \tau^{-2/3}\left[\frac{1}{3}\varphi^2\dot{s}^2 + \frac{2}{3}X\varphi\dot{q}\dot{s} + \frac{1}{3}X^2\dot{q}^2\right] \qquad (29)$$

Therefore,

$$D = \frac{c^2u_m^2}{k_m}\cdot\frac{q}{2}[L_1\dot{s}^2 + L_2\dot{s}\dot{q} + L_3\dot{q}^2] \qquad (30)$$

where the coefficients are

$$\left.\begin{aligned} L_1 &= \frac{1}{3}\int_0^1 \tau^{-2/3}\varphi^2 d\tau \\ L_2 &= \frac{2}{3}\int_0^1 \tau^{-2/3}X\varphi d\tau \\ L_3 &= \frac{1}{3}\int_0^1 \tau^{-2/3}X^2 d\tau \end{aligned}\right\} \qquad (31)$$[4]

The thermal force at the surface is

$$Q = u_m \left.\frac{\partial H}{\partial q}\right|_{y=s} = \frac{cu_m^2}{2} L_4 \qquad (32)$$

$$L_4 = \frac{2}{3}\Psi(\tau)\big|_{y\to s} = 2\int_0^1 (1 - \tau^{1/3})F(\tau)d\tau = 2X(1) \qquad (32a)$$

The Lagrangian equations to give a relationship between the velocity of movement of the melting face, i.e., $\dot{s}$ and the penetration distance $q$, is

$$\frac{\partial V}{\partial q} + \frac{\partial D}{\partial \dot{q}} = Q \qquad (33)$$

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[3] Integral is evaluated numerically.

[4] Numerical integration is performed after integrating by parts.

Substituting $V$, $D$, and $Q$ from (20), (28), and (32), we get

$$q[L_2\dot{s} + 2L_3\dot{q}] = \frac{k_m}{c}(L_4 - N) \tag{34}$$[5]

The second equation relating the two variables $\dot{s}$ and $q$ is obtained from the fourth boundary condition; viz.,
The quantity of heat supplied in time $t$

= the heat absorbed by the body to reach the melting temperature
+ that required to melt the distance $s$ in time $t$
+ that used to heat the body a distance $q$

This gives

$$Rt = (L + c\theta_m)s + \int_s^{s+q} c\theta dy \tag{35}$$

$$R = (L + c\theta_m)\dot{s} + c\theta_m\dot{q}L_5, \qquad L_5 = \int_0^1 \theta/\theta_m d\zeta \tag{36}$$

where $\theta/\theta_m$ is represented as a function of $u/u_m$ and $L$ is the latent heat. Simultaneous solution of the equations (34) and (36) gives

$$q\dot{q}\frac{2L_3}{\frac{k_m}{c}(L_4 - N)}\left[1 - \frac{L_2L_5c\theta_m}{2L_3(L + c\theta_m)}\right] + q\left[\frac{R}{\frac{k_m}{c}(L + c\theta_m)} \frac{L_2}{(L_4 - N)}\right] = 1 \tag{37}$$

$$R - c\theta_m\frac{k_m}{c}\frac{1}{q}\frac{L_5(L_4 - N)}{2L_3} = \dot{s}(L + c\theta_m)\left[1 - \frac{L_2L_5c\theta_m}{2L_3(L + c\theta_m)}\right] \tag{38}$$

Assuming that a steady state is achieved, we have $\dot{q} = 0$ at the steady state. Hence (37) yields

$$q_{st} = \frac{(L_4 - N)\frac{k_m}{c}(L + c\theta_m)}{L_2R} \tag{39}$$

In order to nondimensionalize the equations (37) and (38), the following quantities are defined:

$$\left.\begin{aligned} A &= \frac{2L_3}{(L_4 - N)}\left[1 - \frac{L_2L_5c\theta_m}{2L_3(L + c\theta_m)}\right] \\ \xi &= \frac{\frac{k_m}{c}t}{q_{st}^2} \\ Q &= \frac{q}{q_{st}} \\ S &= \frac{s}{q_{st}} \end{aligned}\right\} \tag{40}$$

The parameter $A$ can always be expressed as

$$A = \frac{2L_3}{(L_4 - N)}(1 - Q_{00}), \qquad Q_{00} = \frac{q_{00}}{q_{st}} = \frac{L_2L_5}{2L_3}\frac{c\theta_m}{(L + c\theta_m)} \tag{41}$$

where $q_{00}$ is the initial penetration depth, giving zero initial melting rate ($\dot{s}(0) = 0$) and is evaluated from the equations (34) and (36). The equations (37) and (38) in the nondimensional form become

$$AQ\frac{dQ}{d\xi} + (Q - 1) = 0 \tag{42}$$

$$\frac{dS}{d\xi} = \frac{(L_4 - N)}{L_2(1 - Q_{00})}\left(1 - \frac{Q_{00}}{Q}\right) \tag{43}$$

## Solution for Penetration Distance

The solution of equation (42) is given by

$$(Q - Q_0) + \log_e\frac{Q - 1}{Q_0 - 1} = -\frac{1}{A}\xi \tag{44}$$

where $Q = Q_0$ is the initial penetration distance at $\xi = 0$ and can be obtained by the Lagrangian analysis of the period prior to start of melting. The solution for small times is expressed as

$$Q = Q_0 + \frac{1 - Q_0}{AQ_0}\xi \tag{45}$$

$$q = q_0 + \frac{1 - q_0/q_{st}}{Aq_0}\frac{k_m}{c}t \tag{45a}$$

For the particular case when $Q_0 = Q_{00}$, equation (44) reduces to

$$(Q - Q_{00}) + \log_e\frac{Q - 1}{Q_{00} - 1} = -\frac{1}{A}\xi \tag{46}$$

Note that the steady-state solution is represented by $Q = 1$. The physical solution must tend toward this steady state when $\xi$ tends to infinity. The differential equation (42) shows that this is possible only if $A > 0$. This implies also $Q_{00} < 1$. Under these conditions there are two possible types of solutions of equation (42), depending on whether the value of $Q$ lies above or below the horizontal asymptote $Q = 1$, Fig. 2. However, if we assume that the rate of heating $R$ is the same before and after melting, the initial penetration depth $q_0$ will be smaller than the steady-state value $q_{st}$ and only the solutions for which $Q < 1$ will appear. This property is readily verified for the case of constant parameters and it seems justified to extrapolate it for variable conductivity. From equation (43) a positive melting rate also implies the condition $Q > Q_{00}$.

Curve (*a*), Fig. 2, stands for $Q < 1 (q_0 < q < q_{st})$ and the curve (*b*) for $Q > 1 (q_0 > q > q_{st})$. The dotted portion of the curve (*a*) is not an acceptable part of the solution because this holds for the negative values of $Q$, which has no physical meaning. The extended parts of the curves (*a*) and (*b*) for $\xi < 0$ are useful in the sense that the same curves may be used for different values of $Q_0$ just by an appropriate shifting of the origin. This can easily be shown analytically.

## The Melting Rate

Using equations (39) and (43) we get the nondimensional melting rate

$$\dot{S} = \frac{\dot{s}}{\dot{s}_{st}} = \left(\frac{1}{1 - Q_{00}}\right)\left(1 - \frac{Q_{00}}{Q}\right) \tag{47}$$

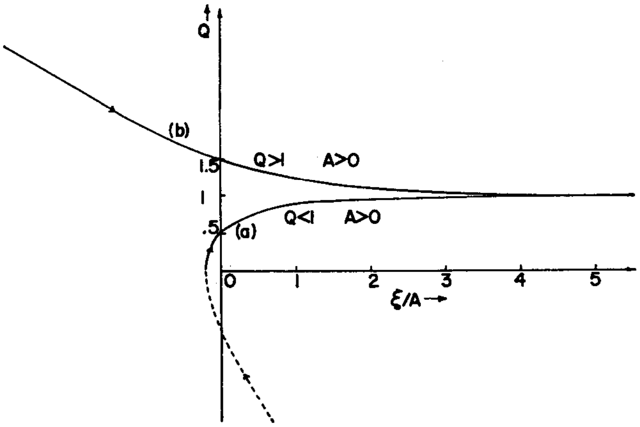

**Fig. 2** Solution of equation (44) for the penetration distance

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[5] Since $\Phi(\tau) < \varphi(\tau)$ we note that $N < L_4$.

$$= 1 + \frac{Q_{00}}{(1 - Q_{00})}\left[1 - \frac{1}{Q}\right] \qquad (48)$$

The initial melting rate is given by

$$\dot{S}(0) = 1 + \frac{Q_{00}}{(1 - Q_{00})}\left[1 - \frac{1}{Q_0}\right] \qquad (49)$$

and from equation (47)

$$\dot{S}_{\min} = 0, \qquad Q = Q_{00} \qquad (50)$$

## The Melting Rate for Small Times

Using the equation (45), the melting rate for small times is obtained as

$$\dot{S} = \frac{1}{(1 - Q_{00})}\left[1 - \frac{q_{00}/q_0}{1 + \frac{(1 - q_0/q_{st})(k_m/c)^2}{Aq_0^2}\frac{t}{t_m}\frac{\pi}{4}\frac{(c\theta_m)^2}{R^2}}\right] \qquad (51)$$

where the time $t$ has been nondimensionalized with $t_m$ $\left(t_m = \frac{\pi}{4}\frac{k_m}{c}\frac{(c\theta_m)^2}{R^2}\right.$ Landau [1]$\left.\right)$ the time required to bring the material of *constant conductivity* to its melting temperature.

If $Q_0 = Q_{00}$, the expression (51) reduces to

$$\dot{S} = \frac{1}{(1 - Q_{00})}\left\{1 - \frac{1}{1 + G\frac{t}{t_m}}\right\} \qquad (52)$$

where

$$G = \pi L_3/2L_5^2(L_4 - N)$$

## Relation Between $\xi/A$ and $t/t_m$

Equations (39), (40), and the definition of $t_m$ are used to find the relation between $\xi/A$ and $t/t_m$.

$$\frac{t}{t_m} = \frac{c}{k_m}\frac{q_{st}^2 A}{t_m}\frac{\xi}{A}$$

$$= \frac{8L_3(L_4 - N)}{\pi L_2^2}\left(1 + \frac{\sqrt{\pi}}{2m}\right)^2 (1 - Q_{00})\xi/A \qquad (53)$$

where the quantity $m$ is defined as

$$m = \frac{\sqrt{\pi}}{2}\frac{c\theta_m}{L} \qquad \text{(Landau) [1]} \qquad (54)$$

## Numerical Example

The previous analysis is applied to carry out investigations for the material Alumina ($Al_2O_3$) which represents a class of dielectric materials. The temperature dependence of phonon conductivity in dielectric solids is given by

$$k_{\text{phonon}} = \frac{A}{B + T} \qquad (55)$$

It has been noticed, however, that in addition to the vibrational energy in solids, a much smaller fraction of the energy content results from higher frequency electromagnetic radiation energy. This fraction, while usually negligible, becomes important at high temperatures because it is proportional to the fourth power of temperature. The energy per unit volume of the black body radiation at temperature $T$ is given by

$$E_T = 4\sigma n^3 T^4/c \qquad (56)$$

The volume heat capacity corresponding to the energy necessary to raise the temperature level of this radiation is

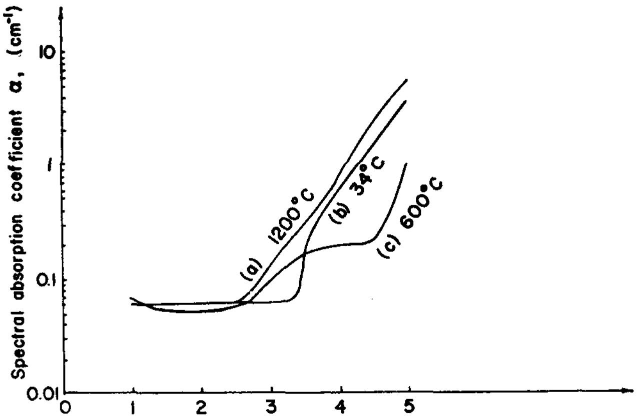

**Fig. 3 Absorption coefficient for single crystals ($Al_2O_3$) at different wavelengths and temperature levels.**

$$c_R = \frac{\partial E}{\partial T} = 16\sigma n^3 T^3/c \qquad (57)$$

where $\sigma$ is the Stefan-Boltzmann constant ($1{\cdot}37 \times 10^{-12}$ cal cm$^{-2}$ sec$^{-1}$ deg K$^{-4}$), $c$ is the velocity of light ($3 \times 10^{10}$ cm sec$^{-1}$), $n$ the refractive index, *and the velocity of this radiation is* $v = c/n$. The radiant energy conductivity or the photon conductivity is obtained as

$$k_r = \frac{1}{3}c_R v l_r = \frac{16}{3}\sigma n^2 T^3 l_r = \frac{16}{3}\sigma n^2 T^3 \cdot \frac{1}{\alpha} \qquad (58)$$ [6]

where $l_r$ is the photon mean free path and $\alpha$ the absorption coefficient, which depends on the temperature and the wavelength. The expression (58) may also be derived by using Rosseland approximation for the radiant flux vector explained in reference [9]. The combined conductivity of a dielectric material is, therefore, given by

$$k = k_{\text{phonon}} + k_{\text{photon}} = \frac{A}{B + T} + \frac{16\sigma n^2 T^3}{3\alpha} \qquad (59)$$

Lee and Kingery [10, 11], in recent years, have performed many experiments to study the behavior of conductivity with respect to temperature, of many ceramic materials. The study on thermal conductivities of $Al_2O_3$, BeO, and MgO by Mcquarrie [12] in 1954, predicts a correlation

$$k = \frac{A}{T - 125 \text{ deg K}} + 8{\cdot}5 \times 10^{-36} \times T^{10} \qquad (60)$$

The second term on the right-hand side denotes an increase in the phonon conductivity by the passage of radiant energy through the translucent specimen and differs from the theoretical expression (59) which depends only on the third power of the temperature. However, for practical purposes, the above correlation may be very useful until further, more accurate, experimental results are made available.

The validity of the concept of thermal conductivity to represent radiative energy transfer in solids depends on the magnitude of the photon mean-free-path ($l_r = 1/\alpha$) as compared with the distance over which the temperature varies appreciably. According to Wien's law the major part of the energy at 2000 deg K lies in a range of wavelength around $1\mu - 2\mu$. Fig. 3, Lee and Kingery [8], shows that for Alumina the photon mean-free-path in this range is of the order of 0.1 cm. Hence the use of the thermal conductivity coefficient for this case appears to be justified.

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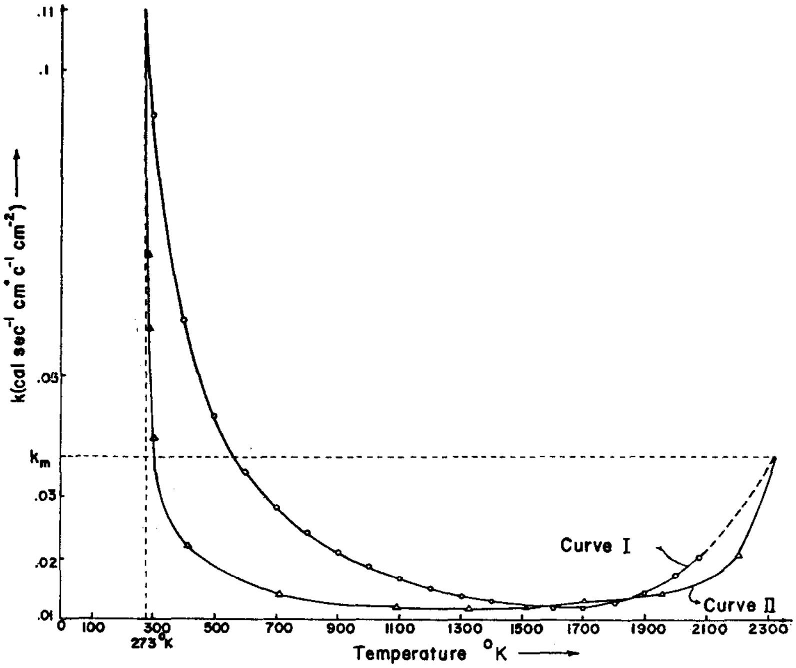
[6] For detailed description see reference [8].

**Fig. 4**



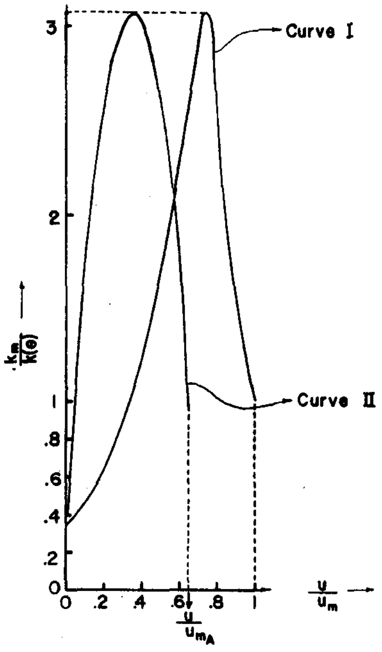


**Fig. 5**

**Table 1**

| $\theta$°c | $k$ | $F(\tau) = \frac{k_m}{k(\theta)} = 1/f(\theta)$ | $\int_0^\theta k(\theta)d\theta$ | $u/u_m = \tau$ |
|---|---|---|---|---|
| 0 | .10946 | .3334 | 0 | 0 |
| 27 | .09257 | .394 | 2.714 | .053 |
| 127 | .0589 | .6196 | 10.037 | .1968 |
| 227 | .0432 | .8449 | 15.063 | .2954 |
| 327 | .0341 | 1.07 | 18.893 | .3705 |
| 427 | .0281 | 1.295 | 21.989 | .4313 |
| 527 | .024 | 1.520 | 24.386 | .4822 |
| 627 | .0209 | 1.746 | 26.825 | .5261 |
| 727 | .0185 | 1.970 | 28.792 | .5647 |
| 827 | .0166 | 2.193 | 30.547 | .5991 |
| 927 | .0151 | 2.414 | 32.133 | .6301 |
| 1027 | .0139 | 2.626 | 33.582 | .6587 |
| 1127 | .0129 | 2.818 | 34.923 | .685 |
| 1227 | .0123 | 2.970 | 36.182 | .7097 |
| 1327 | .0119 | 3.060 | 37.388 | .7333 |
| 1427 | .0120 | 3.040 | 38.580 | .7567 |
| 1527 | .0127 | 2.870 | 39.809 | .7808 |
| 1627 | .0143 | 2.54 | 41.153 | .8072 |
| 1727 | .0173 | 2.105 | 42.721 | .8379 |
| 1800 | .02077 | 1.757 | 44.106 | .8651 |
| 1827 | .02125 | 1.717 | 44.674 | .8762 |
| 1927 | .0269 | 1.357 | 47.081 | .9234 |
| 2027 | .03475 | 1.050 | 50.164 | .9839 |
| 2050 | .0365 | 1.000 | 50.983 | 1.000 |

We make use of the expression (60) for the present study. For Alumina it is given by

$$k = \frac{16\cdot 2}{T - 125 \text{ deg K}} + 8\cdot 5 \times 10^{-36} \times T^{10}$$

$$\text{(valid up to 1800 deg C)} \quad (61)$$

Curve I (Fig. 4) shows the plot of conductivity versus temperature up to melting temperature 2050 deg C of $Al_2O_3$. The dotted line from 1800 to 2050 deg C extrapolates an approximate behavior of $k$ in this range. Curve I, Fig. 5, is the corresponding behavior of $k_m/k(\theta)$ versus $u/u_m$, which is drawn using the values calculated in Table 1.

The parabolic curve II, Fig. 5, is analogous to curve I, Fig. 5, and satisfies the conditions for initial, extremum, and final conductivities. It is introduced for the purpose of comparison[7] and is expressed by

$$\frac{k_m}{k(\theta)} = 0.3334 + 10\cdot 2(u/u_{m_A}) - 9\cdot 54(u/u_{m_A})^2 \quad (62)$$

where $u_{m_A}$ corresponds to the melting temperature $\theta_m$ and $= \int_0^{\theta_m} \frac{k(\theta)}{k_m} d\theta$. The plot of conductivity versus temperature using equation (62) is shown by the curve II, Fig. 4.

The integrals (20), (31), (32), and (36) are evaluated numerically for the actual curve I, Fig. 5, for the analogous case by using (62) and for the constant conductivity case ($k = k_m$). The values of the quantities $L_2$, $L_3$, $L_4$, $L_5$, and $N$ are obtained (Table 2).

**Table 2**

| | Actual curve I, Fig. 5 | Parabolic curve II, Fig. 5 | Constant conductivity, $k = k_m$ |
|---|---|---|---|
| $L_2$ | 0.0904 | 0.3876 | 11/112 |
| $L_3$ | 0.01096 | 0.0560 | 1/56 |
| $L_4$ | 0.4840 | 0.9886 | 1/2 |
| $L_5$ | 0.1720 | 0.2190 | 1/4 |
| $N$ | 0.2132 | 0.3610 | 1/7 |

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[7] In many cases it is possible to represent the variations of $k$ quite accurately by a parabolic approximation.

These values are used in calculating the melting rates, equation (47), and the penetration distances, equation (46), for the three different cases.

Also, the parameters $L_2L_5/2L_3$, equation (40), $Q_{00}$, equation (41); $A$, equation 41; and $G$, equation (52) which are useful in obtaining results for small times are calculated for the three cases (Table 3).

**Table 3**

| | Actual curve I, Fig. 5 | Parabolic curve II, Fig. 5 | Constant conductivity, $k = k_m$ |
|---|---|---|---|
| $L_2L_5/2L_3$ | 0.709 | 0.7569 | 11/16 |
| $Q_{00}$ | 0.1307 | 0.1397 | 0.1267 |
| $G$ | 2.15 | 2.92 | 1.257 |
| $A$ | 0.07036 | 0.1535 | 0.08733 |

## Results

Expression (39) for the penetration distance at the steady state may be rewritten as

$$q_{st} = \frac{L_4 - N}{L_2}\beta, \qquad \beta = \frac{\frac{k_m}{c}(L + c\theta_m)}{R} \tag{63}$$

Its value for the three cases is calculated and the nondimensional penetration distance $Q$ is expressed as

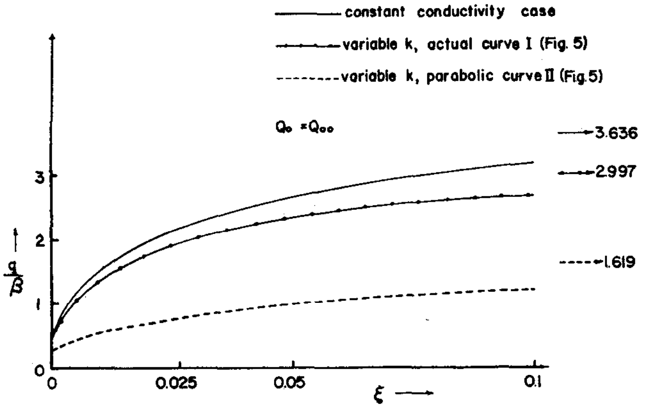

Fig. 6 Penetration distance versus time

$$Q = \frac{q}{q_{st}} = \frac{q}{2\cdot 997\beta} \qquad \text{(actual case, curve I, Fig. 5)} \tag{64}$$

$$= \frac{q}{1\cdot 619\beta} \qquad \text{(parabolic case, curve II, Fig. 5)}$$

$$= \frac{q}{3\cdot 636\beta} \qquad \text{(constant case, } k = k_m\text{)}$$

Using equation (46) and Table 3, the values of $q/\beta$ have been plotted against the nondimensional time $\xi$, Fig. 6, for $Q_0 = Q_{00}$ and $m = 0.2$. It is noticed that

$$\left(\frac{q}{\beta}\right)_{\text{constant case}} > \left(\frac{q}{\beta}\right)_{\text{actual case}} > \left(\frac{q}{\beta}\right)_{\text{parabolic case}}$$

at any finite time and also at the stationary state. On the other hand, looking at Fig. 4, we see that the effective conductivity $k_{eff}$ $\left(\right.$the effective conductivity is some kind of mean of the conductivity, for example, it may be approximated by the quadrature $\left.\frac{\int k(\theta)d\theta}{\theta_m}\right)$ in the three cases, is also arranged in the same order, i.e.,

$$(k_{eff})_{\text{constant case}} > (k_{eff})_{\text{actual case}} > (k_{eff})_{\text{parabolic case}}$$

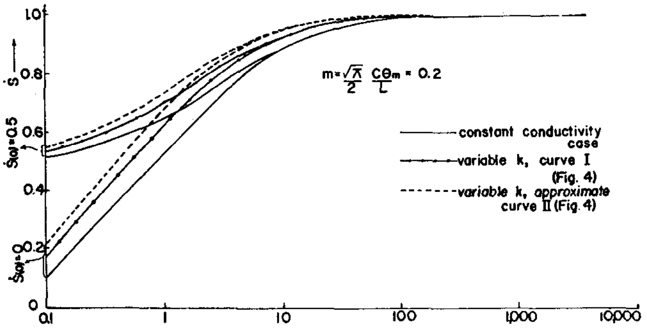

Fig. 7 Nondimensional melting rate versus nondimensional time

Hence, the smaller the effective conductivity, the smaller is the penetration distance. This is physically true, because if the effective conductivity is the smaller, heat will penetrate only to the smaller distances to warm the material and more heat is absorbed by the surface itself. It is also expected, therefore, that the melting rates at finite times should be greater under this circumstance.

The equations (48), (44), (54), and (55) and Table 2 are used to calculate the melting rates for the three cases discussed above, for two different initial values of $\dot{S}$ and for a particular value of 0.2 of the parameter $m$. These have been plotted in Fig. 7 against the nondimensional time $t/t_m$. Since the melting rate is defined as $\dot{S} = \dot{s}/\dot{s}_{st}$, all the curves approach unity as an asymptote. The melting rates at definite times are arranged as

$$\dot{S}_{\text{constant case}} < \dot{S}_{\text{actual case}} < \dot{S}_{\text{parabolic case}}$$

the reason for which has already been explained. The constant conductivity case may be considered as a first-approximation treatment. The parabolic approximation for calculating the values of $\dot{S}$ may be accepted as a good second approximation and it furnishes the order of correction to the values of $\dot{S}$ obtained in the constant-conductivity case.

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