Short Notes

M. A. BIOT

THEORY OF SIMILAR FOLDING OF THE FIRST AND SECOND KIND

Abstract: It is shown that in the initial phase there are two basic types of similar folding of multilayered structures, referred to here as folding of the first and second kind, depending on the prevalence of over-all flexure or shear. A simple theory brings to light the underlying mechanics and the controlling parameters which govern these two types of folding and the transition from one to the other. The dominant wave length obtained in folding of the second kind is the same as derived previously (Biot, 1961) assuming the incompetent layers to act as lubricants. In folding of the first kind the system behaves approximately as a single layer (Biot, 1957; 1961). The same theory solves at the same time the problem of folding of a single anisotropic layer.

Introduction

A structure composed of alternating competent and incompetent layers exhibits a collective behavior when undergoing folding. In the present analysis we shall consider similar folding defined by the property that approximately the folded layers differ only by a vertical translation (Van Hise, 1896). In the initial phase of the deformation two types of similar folding may be distinguished. In one type where the viscosity contrast between layers is small the structure behaves like a single anisotropic plate in bending; in the other type the incompetent layers tend to act as lubricant and the collective deformation resembles a shear buckling. These two types are referred to here as similar folding of the first and second kind, respectively. As indicated in the last paragraph they are not restricted to small deformations.

We have considered the folding of a multilayered structure embedded in a soft medium. The questions which naturally arise are: what are the controlling parameters that determine the type of folding? How does the transition occur from one type to the other, and what are the significant parameters which govern the dominant wave length in each case? Our purpose is to answer these questions quantitatively and develop a simple theory which brings to light in intuitive form the fundamental mechanics underlying the characteristic behavior.

Similar folding of the second kind of multilayered structure was already analyzed in a previous paper (Biot, 1961), and a very simple expression for the dominant wave length was obtained which is in complete agreement with the present more sophisticated analysis.

The results presented here developed in the context of multilayered structure are equally applicable to the folding of a single anisotropic layer. The folding of such a layer is governed by the approximate equation (11). It is in excellent agreement with the exact theory developed in a previous paper (Biot, 1963b).

Equilibrium Equations for the Collective Buckling of a Multilayered Structure as a Single Plate

Consider a plate of thickness h under axial compressive stress P and a normal load q per unit length. The normal deflection is denoted by v (Fig. 1). The x axis coincides with the axis of the plate. Equilibrium of a deformed cross-sectional element of thickness dx implies the following equations:

$$\frac{d\mathfrak{M}}{dx} = \mathfrak{N}$$

$$\frac{d\mathfrak{N}}{dx} + q = Ph\frac{d^2v}{dx^2}, \qquad (1)$$

where $\mathfrak{M}$ denotes the bending moment and $\mathfrak{N}$ the total shear over the cross section. Equations (1) are immediately derived by inspection of Figure 1. They have also been derived as a rigorous consequence of the general mechanics of continuous media under initial stress (Biot,

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1965). The first of equations (1) expresses the equilibrium of moments about an axis normal to the figure. The second of equations (1) expresses the equilibrium of forces in the vertical direction.

Average Anisotropic Stress-Strain Relationships in a Multilayered Structure

We shall consider a purely viscous incompressible medium composed of a superposition of alternating competent and incompetent layers. They are respectively of viscosities η_1 and η_2, and occupy fractions α_1 and α_2 of the total thickness (Fig. 2). By averaging stresses and strains such a structure behaves approximately as a continuous viscous plate with anisotropic viscosity. The normal stress σ_{xx} along the axis and the shear σ_{xy} satisfy the following stress-strain relationships:

$$\begin{aligned}\sigma_{xx} &= 4\eta_n \dot{e}_{xx}\\ \sigma_{xy} &= 2\eta_t \dot{e}_{xy}\ ,\end{aligned}\tag{2}$$

where $\dot{e}_{xx}$ and $\dot{e}_{xy}$ are the time derivatives of the two strain components indicated in Figure 2. The two viscosity coefficients defining the anisotropic viscous properties are derived by viscoelastic correspondence from results obtained earlier (Biot 1963a, p. 318; 1963b, p. 244). They are

$$\begin{aligned}\eta_n &= \alpha_1\eta_1 + \alpha_2\eta_2\\ \eta_t &= \frac{\eta_1\eta_2}{\alpha_1\eta_2 + \alpha_2\eta_1}\,.\end{aligned}\tag{3}$$

We shall assume a deformation such that the vertical displacement is the same over the cross section whereas the horizontal displacement is a linear function across the thickness (Fig. 2). We write

$$v = v(x)\,, \qquad u = yu_1(x)\,. \tag{4}$$

Note that the horizontal displacement is the average value after smoothing out the fine structure due to the heterogeneity of the layering. With the displacements (4) the stresses (2) become:

$$\begin{aligned}\sigma_{xx} &= 4\eta_n\dot{e}_{xx} = 4\eta_n\frac{\partial \dot{u}}{\partial x} = 4\eta_n y\frac{d\dot{u}_1}{dx}\\ \sigma_{xy} &= 2\eta_t\dot{e}_{xy} = \eta_t\left(\frac{\partial \dot{v}}{\partial x} + \frac{\partial \dot{u}}{\partial y}\right)\\ &= \eta_t\left(\frac{d\dot{v}}{dx} + \dot{u}_1\right)\,.\end{aligned}\tag{5}$$

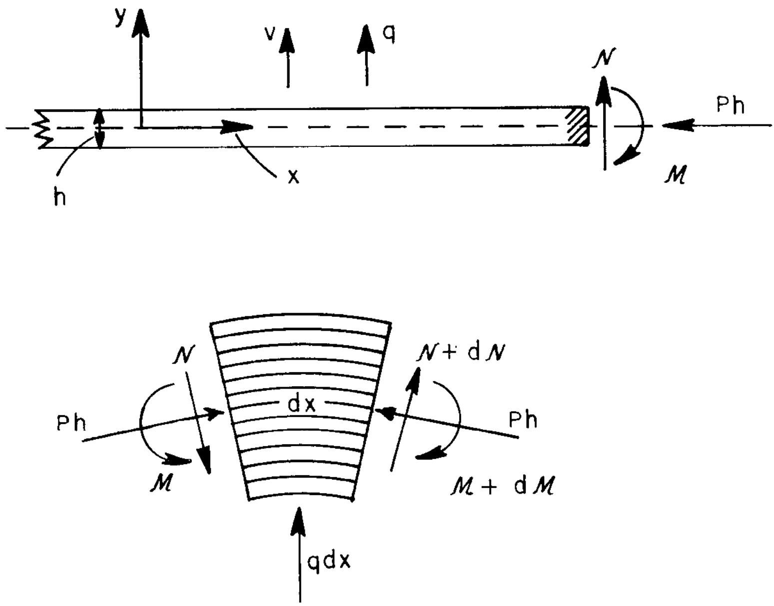

Figure 1. Equilibrium diagram for a vertical slice of thickness dx of the multilayered structure

For the bending moment $\mathfrak{M}$ and total shear $\mathfrak{N}$ we derive

$$\mathfrak{M} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} y\, dy = \frac{1}{3} \eta_n h^3 \frac{d\dot{u}_1}{dx}$$

$$\mathfrak{N} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xy}\, dy = \eta_t h \left(\frac{d\dot{v}}{dx} + \dot{u}_1\right). \quad (6)$$

with

$$A = \frac{1}{3} \frac{\eta_n}{\eta_t} h^2. \quad (8)$$

By assuming a sinusoidal deflection along x we put

$$v = \bar{v} \cos lx\,, \qquad q = \bar{q} \cos lx\,. \quad (9)$$

The wave length is $\mathfrak{L} = 2\pi/l$. Equation (7) becomes[1]

$$Ph(Al^2 + 1)l^2 v + (Al^2 + 1)q = Ahl^4 \eta_t \dot{v}\,. \quad (10)$$

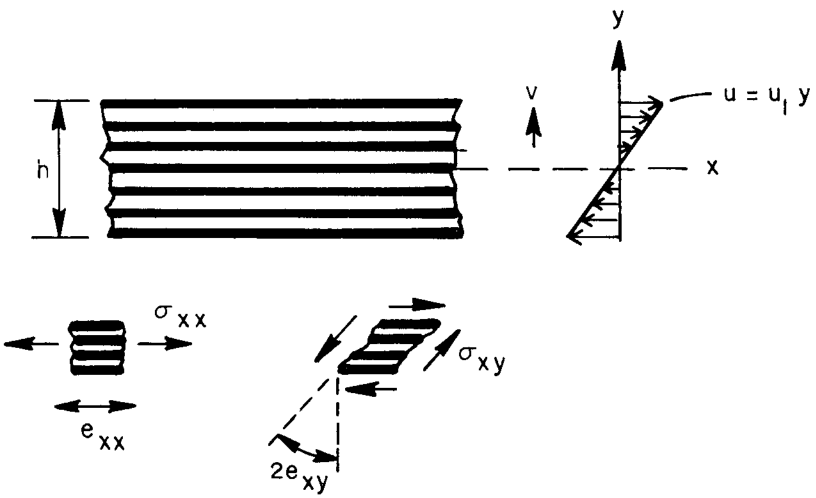

Figure 2. Deformation and stresses in a multilayered structure

We have neglected the tangential stress acting at the top and bottom of the multilayered structure since it was shown previously that its influence is not significant (Biot, 1959; Biot and Odé, 1962). Whereas the analysis was carried out for an isotropic layer the conclusion is even more valid for a multilayered structure, since such a structure will tend to fold with less horizontal displacement at the top and bottom faces.

Buckling Equation for an Embedded Multilayered Structure

Substituting the values (6) for $\mathfrak{M}$ and $\mathfrak{N}$ into the equilibrium equations (1) and eliminating $\dot{u}_1$ we derive

$$Ph\frac{d^2}{dx^2}\left(A\frac{d^2 v}{dx^2} - v\right) - A\frac{d^2 q}{dx^2} + q = Ah\eta_t \frac{d^4 \dot{v}}{dx^4} \quad (7)$$

Solving for P we find

$$P = -\frac{q}{hl^2 v} + \frac{Al^2}{Al^2 + 1} \eta_t \frac{\dot{v}}{v}\,. \quad (11)$$

The normal load q may be considered to represent the reaction of a large variety of lateral constraints. For example consider the case where the multilayer is embedded in an isotropic medium of infinite extent. (Fig. 3A). It was shown (Biot, 1959; 1961) that for a sinusoidal deflection of wave length $2\pi/l$ the value of q is

$$q = -4\eta l \dot{v}\,. \quad (12)$$

By substituting this value into equation (11) and putting $p = \dot{v}/v$

[1] Note that in this case we may substitute $d^2v/dx^2 = -l^2 v$, $d^2q/dx^2 = -l^2 q$, *etc.* The common factor $\cos lx$ drops out.

we find

$$\zeta = \frac{P}{\eta_t p} = \frac{4\eta}{\eta_t}\frac{1}{lh} + \frac{Al^2}{Al^2+1}. \qquad (13)$$

This may be considered a differential equation for v. If the right side is independent of time the solution v is proportional to exp (pt) where p is a constant (*see* last paragraph). The accuracy of the approximate equation (13) has been verified by comparing it with the exact theory of buckling of the anisotropic plate derived in another paper (Biot, 1963b). The two results are in excellent agreement.

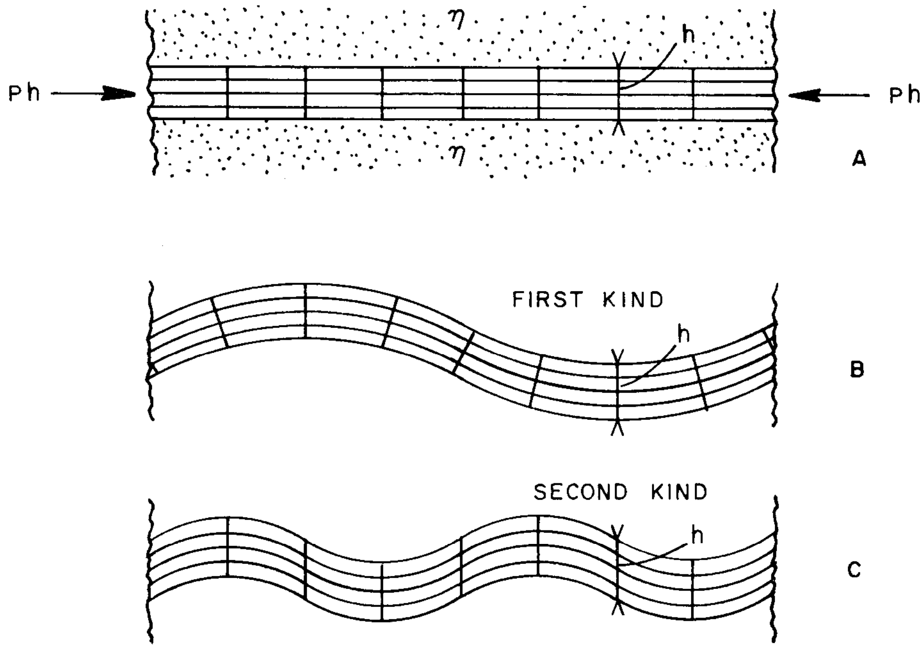

Figure 3. (A) Multilayered structure embedded in an infinite medium of viscosity η; (B) Similar folding of the *first kind;* (C) Similar folding of the *second kind*

Similar Folding of the First and Second Kind

A discussion of equation (13) leads to the establishment of two fundamental types of folding which will be referred to as similar folding of the first and second kind.

Folding of the first kind occurs at the larger wave lengths and is characterized by a deformation of the multilayered structure which resembles pure bending of a homogeneous plate as illustrated in Figure 3B. In such a case cross-sectional planes which are vertical in the initial unfolded state tend to remain normal to the layers after folding has occurred.

Folding of the second kind occurs at the smaller wave lengths, and the over-all deformation of the multilayered structure resembles a pure shear buckling (Biot, 1963b, p. 235). In this case the average cross section tends to remain vertical, as illustrated in Figure 3C. An adequate treatment of this case requires the introduction of an additional term into equation (11) in order to account for the bending stiffness of the individual competent layers which enters into play at the shorter wave lengths.

In the analysis which follows we shall determine the parameters which determine the preponderance of one or the other of these two types of folding and the transition from one to the other.

Similar Folding of the First Kind

For this kind of folding to occur the value (13) of ζ plotted as a function of l must have a minimum which is appreciably smaller than unity, as illustrated in Figure 4, and is located near the origin. Hence $Al^2 \ll 1$ and equation (13) may be simplified to

$$\frac{P}{p} = \frac{4\eta}{lh} + \frac{1}{3}\eta_n l^2 h^2 . \qquad (14)$$

This equation is identical with that obtained for the folding of an embedded layer of isotropic viscosity η_n whose deformation is a pure bending. Hence this case corresponds to a folding of the first kind. Equation (14) has been discussed extensively in earlier papers (Biot, 1957; 1961). The dominant wave length was found to be

$$\mathcal{L} = \frac{2\pi}{l} = 2\pi h \sqrt[3]{\frac{\eta_n}{6\eta}} . \qquad (15)$$

This wave length for which P/p is a minimum corresponds to the maximum rate of growth of the amplitude of folding.

Folding of the first kind will take place if the first term in equation (13) is appreciably smaller than 1/2 at the point where $Al^2 = 1$. This is expressed by the condition

$$\frac{\eta}{\eta_t} \ll 0.2 \sqrt{\frac{\eta_t}{\eta_n}} . \qquad (16)$$

As shown earlier significant folding requires that η_n be at least 50 to 100 times larger than η. This condition should also be kept in mind when applying the inequality (16).

Similar Folding of the Second Kind

By reversing the inequality (16) we obtain

$$\frac{\eta}{\eta_t} \gg 0.2 \sqrt{\frac{\eta_t}{\eta_n}} . \qquad (17)$$

This condition also implies $Al^2 \gg 1$. The folding in this case is of a different type, described heretofore as a folding of the second kind. The plot of expression (13) for ζ in this case is represented by curve 3 in Figure 5. The value of ζ is greater than unity, and the system of multilayers *tends to behave as an anisotropic plate* buckling in pure shear.

An additional limitation must be put on the ratio η/η_t which must not be large; otherwise the embedding medium acts as a strong confinement which prevents similar folding and will tend to generate internal buckling (Biot, 1963a; 1964a).

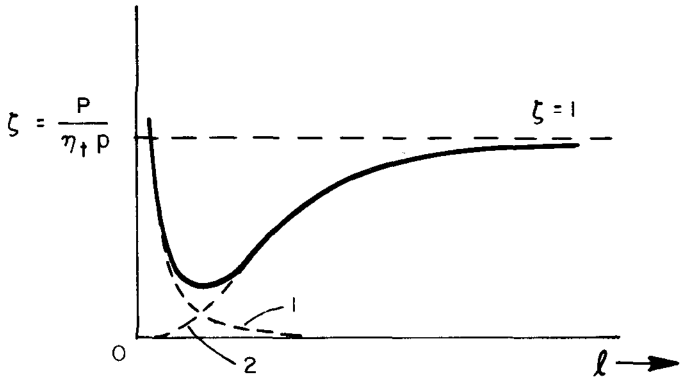

Figure 4. Stability diagram for similar folding of the *first kind* as given by equation (13). Curves 1 and 2 represent respectively the first and second terms of equation (13).

As can be seen, equation (13) yields a plot (Fig. 5, curve 3) from which the minimum associated with the existence of a dominant wave length has disappeared. This is because we have treated the layered structure as an anisotropic continuum which tends to buckle in pure shear at the shorter wave lengths. This assumption neglects the bending resistance of the competent layers which also enters into play at the shorter wave lengths. This effect may be taken into account by adding a correction term to equation (13) as follows. Owing to this bending resistance, the buckling of competent layer of thickness h_1 requires a compression[2] $P_b = \frac{1}{3}\eta_1 l^2 h_1^2 \dot{v}/v$. If there are n_c competent layers they contribute a total compressive stress $P_b n_c h_1/h = \frac{1}{3} n_c \eta_1 l^2 (h_1^3/h)(\dot{v}/v)$. The correction due to bending resistance is obtained by

[2] See equation (4.9) of an earlier paper (Biot, 1961).

adding this term to expression (11). Hence

$$P = -\frac{q}{hl^2v} + \frac{Al^2}{Al^2+1}\eta_t\frac{\dot v}{v} + \frac{1}{3}n_c\eta_1 l^2\frac{h_1^3}{h}\frac{\dot v}{v}. \tag{18}$$

For a layer embedded in an infinite medium we substitute the value (12) for q and write

$$\zeta = \frac{P}{\eta_t p} = \frac{4\eta}{\eta_t}\frac{1}{lh} + \frac{Al^2}{Al^2+1} + \frac{1}{3}n_c\frac{\eta_1}{\eta_t}l^2\frac{h_1^3}{h}. \tag{19}$$

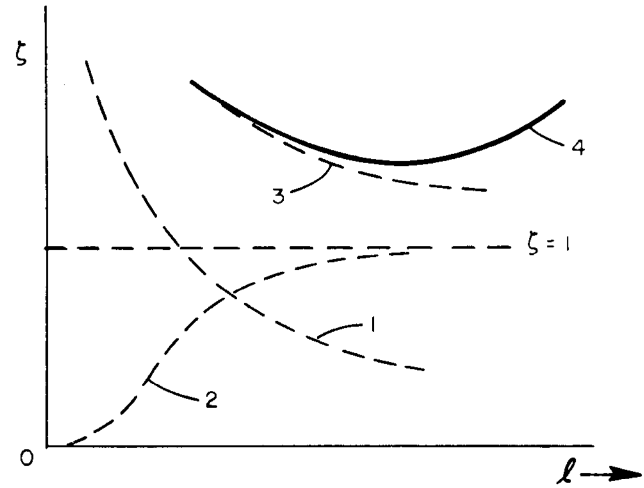

Figure 5. Stability diagram for similar folding of the *second kind.* Curve 3 represents equation (13) as the sum of two terms (curves 1 and 2). Curve 4 is obtained by adding a correction term for the bending rigidity of the competent layers as given by equation (19).

This intuitive result may also be derived more rigorously by adding corrective terms into expressions (6), *i. e.*,

$$\mathfrak{M} = \frac{1}{3}\eta_n h^3\frac{d\dot u}{dx} + \mathfrak{M}_1$$

$$\mathfrak{N} = \eta_t h\left(\frac{d\dot v}{dx} + \dot u_1\right) + \frac{d\mathfrak{M}_1}{dx}, \tag{20}$$

where

$$\mathfrak{M}_1 = -\frac{1}{3}n_c h_1^3\eta_1\frac{d^2v}{dx^2} \tag{21}$$

represents the additional moment due to the bending resistance of the individual competent layers. Note that it is important to include a term $d\mathfrak{M}_1/dx$ in the second of equations (20) because of the shear which is required in order to equilibrate the moment $\mathfrak{M}_1$. By introducing the values (20) into the equilibrium equations (1) and eliminating u_1 as before we derive equation (18).

In folding of the second kind we have assumed $Al^2 \gg 1$. Hence equation (19) is simplified to

$$\frac{Ph}{n_c h_1 p} - \frac{\eta_t h}{n_c h_1} = \frac{4\eta}{n_c l h_1} + \frac{1}{3}\eta_1 l^2 h_1^2. \tag{22}$$

The expression on the right side is the same as for a single layer of thickness h_1 of viscosity η_1 embedded in a medium of viscosity η/n_c. The dominant wave length is

$$\mathfrak{L} = 2\pi h_1\sqrt[3]{\frac{n_c\eta_1}{6\eta}}. \tag{23}$$

This result was already obtained in an earlier paper (Biot, 1961)[3] by assuming that the incompetent layers act primarily as *lubricant.* Equation (22) shows that this assumption is essentially correct for evaluating the dominant wave length in similar folding of the second kind. The effect of the sliding friction between layers is represented by the constant term $\eta_t h/n_c h_1$ in equation (22) which corresponds to

[3] *Note* that the notations η and η_1 have been interchanged in the 1961 paper.

the shear threshold (*see* discussion, further on). It is independent of l and therefore does not affect the dominant wave length but reduces the over-all rate of folding.

Elastic Materials

By viscoelastic correspondence the present solution is also immediately applicable to the buckling of elastic multilayers. We put $\mu_1 = \eta_1 p$, $\mu_2 = \eta_2 p$, $\mu = \eta p$ respectively for the elastic moduli of the two layered materials and the embedding medium. This defines anisotropic elastic moduli $M = \eta_n p$ and $L = \eta_t p$ by equations (3). Similar buckling of the first and second kind will be governed by the same equations, and the dominant wave length becomes the buckling wave length.

Shear Threshold and Incipient Internal Buckling

It is interesting to bring out the significance of the constant term in equation (22). Without bending stiffness and without the restraint owing to embedding the right side of the equation vanishes. Hence the equation for P becomes

$$P = \eta_t p = L. \tag{24}$$

This value of the compressive stress has been discussed previously and referred to as the shear threshold (Biot, 1964a). Physically it represents a pure shear buckling. Hence as already mentioned the constant term $\eta_t h/n_c h_1$ in equation (22) is a result of the sliding resistance between layers.

Note that the shear threshold corresponds to $\zeta = P/\eta_t p = 1$. As seen from Figure 5 similar folding of the second kind occurs above the shear threshold. According to previous results (Biot, 1963a; 1964a) a value $\zeta > 1$ implies incipient internal buckling. The amount of internal buckling superimposed on the similar folding and the magnitude of the correction required may be evaluated by applying the previous analysis (Biot, 1964a).

Similar Folding of the Second Kind with Competent Layers of Unequal Thickness

Similar folding of the second kind for competent layers of unequal thickness was already analyzed in a previous paper (Biot, 1961). It was shown that the dominant wave length is the same as for a single equivalent layer. This result was obtained by considering the equation of folding of a particular competent layer designated by the subscript i. It is written here with an additional term including the sliding friction between competent layers:

$$\frac{1}{3}\eta_i h_i^3 \frac{d^4 \dot{v}}{dx^4} + P_i h_i \frac{d^2 v}{dx^2} = q_i + \frac{dm_i}{dx}. \tag{25}$$

The thickness, viscosity, and compressive stress of this layer are respectively h_i, η_i, and P_i. The total vertical force per unit length acting on the layer is q_i. Owing to the sliding friction in the incompetent layers there is a clockwise moment per unit length which may be written

$$m_i = K_i \frac{d\dot{v}}{dx}. \tag{26}$$

We add equations (25) for all competent layers and assume a sinusoidal deflection along x of wave length $\mathcal{L} = 2\pi/l$. We also substitute $\Sigma q_i = -4\eta l \dot{v}$ for the total reaction of the embedding medium according to equation (12). We obtain

$$\frac{v}{\dot{v}} \overset{i}{\Sigma} P_i h_i - \overset{i}{\Sigma} K_i = \frac{1}{3} l^2 \overset{i}{\Sigma} \eta_i h_i^3 + \frac{4\eta}{l}. \tag{27}$$

This is basically the same type of equation as (22). By minimizing the right side with respect to l we derive the dominant wave length

$$\mathcal{L} = 2\pi \sqrt[3]{\frac{\overset{i}{\Sigma} \eta_i h_i^3}{6\eta}}. \tag{28}$$

This result is the same as obtained earlier (Biot, 1961). For equal thickness it reduces to equation (23). The additional term ΣK_i in equation (27) represents the sliding resistance acting tangentially on the layers. This term is wave-length-independent and corresponds to what we have called the shear threshold. It only affects the over-all rate of folding.

Values of the coefficients K_i are easily obtained. Denote by η_i' and h_i' the viscosity and thickness respectively of the incompetent layer lying on top of the competent layer i. We find

$$K_i = \frac{1}{4} h_i \left[\eta'_i \left(\frac{h_{i-1} + h_i}{h'_i} \right) + \eta'_{i+1} \left(\frac{h_i + h_{i+1}}{h'_{i+1}} \right) \right]. \tag{29}$$

Numerical Discussion

The transition of similar folding of the first to the second kind is controlled by the relative magnitudes of the three viscosity coefficients η_1, η_2, and η, respectively, of the competent layer, the incompetent layer, and the embedding medium. In order to obtain some estimate of the orders of magnitude involved consider the case of layers of equal thickness ($\alpha_1 = \alpha_2 = \frac{1}{2}$). Assume $\eta_2/\eta_1 = 1/10$ and $\eta/\eta_2 = 1/100$. In that case we find $\eta_t/\eta_n \cong 2/5$ and $\eta/\eta_t \cong 1/200$. Hence the inequality (16) is verified. The folding will be of the first kind. Note that the condition $\eta/\eta_n < 1/100$ required for significant folding (Biot, 1961, p. 1611) is also fulfilled. On the other hand let us assume a large viscosity contrast between competent and incompetent layers, say $\eta_2/\eta_1 = 1/1000$, and an embedding medium of same viscosity as the incompetent layer $\eta = \eta_2$. In this case $\eta_t/\eta_n \cong 1/250$ and $\eta/\eta_t \cong \frac{1}{2}$, and the inequality (17) is verified. Hence the folding is now of the second kind. For intermediate values of the viscosities the folding will be of a mixed type and may be analyzed by using the complete equation (19).

Characteristic Transition Wave length

It follows from the foregoing analysis that the relationship $Al^2 = 1$ corresponds to a characteristic wave length

$$\mathfrak{L}_c = 2\pi h \sqrt{\frac{1}{3}\frac{\eta_n}{\eta_t}} \tag{30}$$

which determines the region of transition from one type of folding to the other. Note that $\mathfrak{L}_c/h$ depends only on the ratio η_n/η_t which is a measure of the average anisotropy.

Application to Large Deformations

The present results are applicable to large compressive strains. In this case the thickness and the wave length are functions of time. The coefficients in the differential equation (13) become time-dependent (Biot, 1964b).

Beam-type equations as applied here are not restricted to small deformations. They provide the foundation for an analysis of the later phase where steep slopes and the influence of plasticity must be considered (Biot, 1961). Whether the character of similar folding is retained in this later phase will be discussed in subsequent papers.

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