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THEORY OF VISCOUS BUCKLING AND GRAVITY INSTABILITY OF MULTILAYERS WITH LARGE DEFORMATION

Abstract: An exact treatment of the stability of multilayered viscous fluids in slow motion with large deformations leads to an analog model which includes the effect of gravity. A distinction can be made between true mechanical instability and an apparent instability of purely kinematic nature. A mechanism for concentric folding is derived, leading to predictions in good agreement with experimental results (De Sitter, 1939). Exact equations are obtained for the time history of viscous buckling of multilayered fluids for small deflections superimposed upon a large variable compressible strain. A numerical application to the single embedded layer checks the practical validity of the simplified thin-layer theory with interfacial slip. For gravity instability, the analog model provides a method of analysis applicable to very large deformations in salt structures.

Introduction

Continuum theories of folding instability have been established which are rigorously applicable if the medium is initially at rest under the initial stress, and for small incremental deformations superimposed on this initial state. An exact treatment of viscous buckling of multilayered fluids has been developed (Biot, 1964a) which is not subject to the aforementioned restrictions and is applicable to large deformations. The theory makes a distinction between an apparent instability which is of purely kinematic origin and a true mechanical instability. The latter may be represented by an analog model where the fluid is initially at rest and stress-free. The model is extended to include gravity instability in addition to pure buckling. By using such a model one may be able to predict a type of folding with finite strain, referred to by De Sitter (1939) as concentric folding.

In the case of small deformations superimposed on a state of flow, the analog concept leads to exact differential equations for the time history of viscous buckling and simultaneous gravity instability of multilayered fluids undergoing a variable finite strain. We will discuss the exact solution numerically for a single viscous layer in a viscous medium and demonstrate the practical validity of the simplified formulas for large compressive strain.

The present results constitute the counterpart for viscous fluids of the exact theories derived for purely elastic materials in the author's earlier papers. The two theories correspond to extreme cases and provide reliable predictions for the behavior of actual materials with intermediate properties.

The physical results of the two theories merge in the range of significant viscous instability, and in this range the principle of viscoelastic correspondence is applicable for incremental deformations in a state of initial flow.

Kinematic Amplification

We consider a viscous, incompressible fluid subject to a uniform and constant compressive stress P (Fig. 1). The viscosity is assumed to be high so that inertia effects are negligible. The fluid is deformed at a constant uniform strain-rate. For simplicity we also assume that the deformation is two dimensional. However, as the author (1964a; 1965) has shown in the more general treatment, this assumption is not essential. The finite deformation is represented as a function of the time t, by the two extension ratios

$$\lambda_1 = e^{-p_0 t}$$
$$\lambda_2 = e^{p_0 t}. \tag{1}$$

The extension ratio λ_1 represents the distance between two particles of fluid initially ($t = 0$) at a unit distance along the direction of compression, whereas λ_2 is defined similarly in the perpendicular direction. The relationship $\lambda_1\lambda_2 = 1$ expresses the incompressibility of the fluid. The constant strain-rate is

$$-\frac{1}{\lambda_1}\frac{d\lambda_1}{dt} = p_0 = \frac{P}{4\eta}, \tag{2}$$

where η is the viscosity coefficient. These re-

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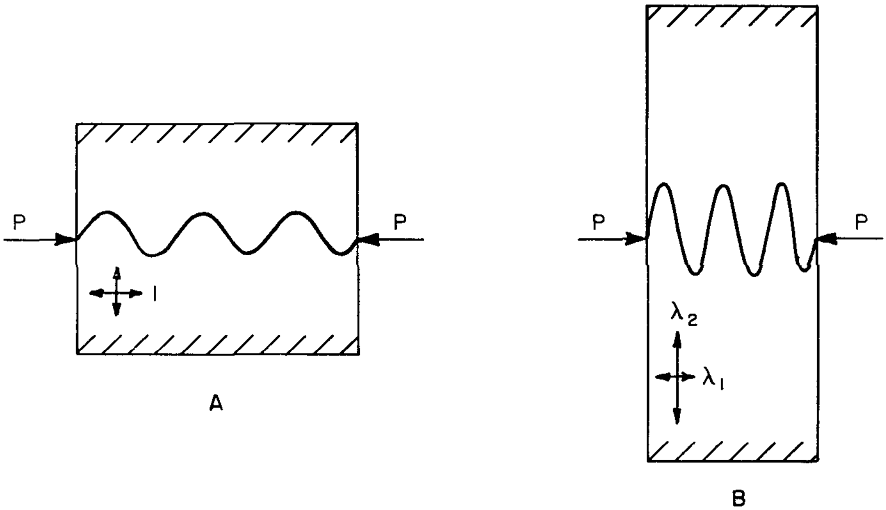

Figure 1. Kinematic amplification associated with a large uniform strain

sults are easily derived (Biot, 1964a; 1965) from the classical equations of fluid mechanics.

A sinusoidal line drawn in the fluid at the instant $t = 0$ (Fig. 1A) will be compressed accordionwise in the direction of the compression and stretched in the normal direction as shown in Figure 1B. This stretching is proportional to an amplification factor $\lambda_2 = \exp(p_0 t)$. Such an increasing exponential represents a purely kinematic effect which mathematically looks like an instability. It is not, however, a true instability in the mechanical sense.

Analog Model for Viscous Buckling with Large Deformation

Let us imagine that in Figure 1A the portion of the fluid lying above the sinusoidal line is removed. We then have a free boundary of sinusoidal shape, and the homogeneous deformation will be disturbed. However the original deformation will remain undisturbed if we apply surface forces which restore the initial stress field. These surface forces are $P \sin \alpha$ per unit area of the sinusoidal surface, where α is the slope angle of the free surface with the direction of compression, as indicated in Figure 2A. With these surface forces the deformation is uniform and remains the same as if the boundary did not exist. The boundary itself is deformed as in Figure 1, and the depth of the corrugations increases in proportion to the factor $\exp(p_0 t)$.

The effect of the free surface is now obtained by superposing surface stresses which cancel the previously applied stresses. They are equal to

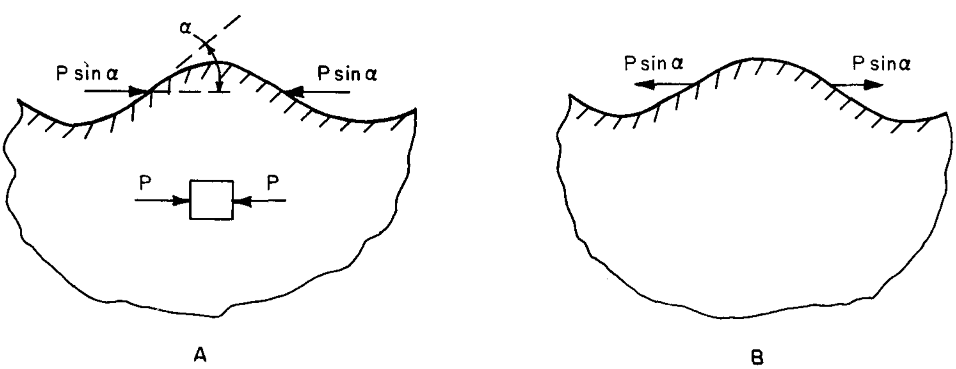

Figure 2. Derivation of the analog model. A, Boundary stresses in the undisturbed motion; B, analog model

$P \sin \alpha$ per unit area and act in the opposite direction, as shown in Figure 2B. These stresses induce a velocity field in the fluid which disturbs the uniform strain-rate. The important property which is applied here is a principle of superposition based on the fact that the equations of Navier-Stokes for the mechanics of a viscous fluid are linear when inertia forces are neglected. Hence the effect of the free surface is obtained by adding the velocity fields of Figures 2A and B. Note that in Figure 2B there is no initial stress P, and the velocity field is entirely a result of the boundary stresses acting on a fluid initially at rest. The second velocity field may be considered an *analog model* representing the influence of the boundary.

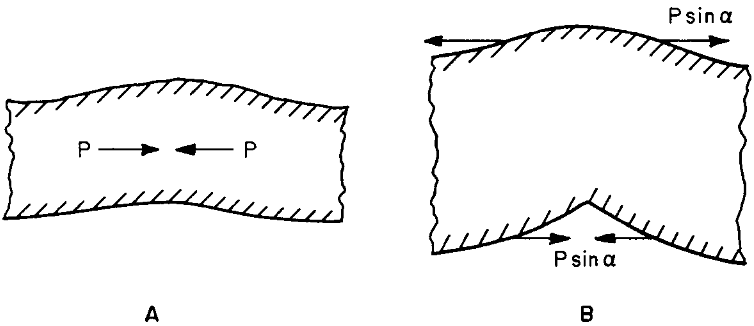

Figure 3. Application of analog model to viscous buckling with large deformation, showing the tendency to "concentric folding"

By using this model one may be able to predict the viscous buckling of a plate for large deformations. A slight kink in a fluid plate, as shown in Figure 3A, will develop into a shape obtained by applying the fictitious boundary forces $P \sin \alpha$. On the bottom face these forces must be reversed. As the deformation proceeds the convex side will tend to remain *smooth* whereas the concave side will be *pinched*, as shown in Figure 3B.

It is interesting to note that this result provides a mechanism for the pattern of "concentric folding" proposed and verified experimentally by De Sitter (1939).

In this discussion we have assumed that there is no stress in the direction normal to the compression. Such a stress component may be added by superposing an over-all hydrostatic stress on the whole medium. This, however, would not modify the phenomenon in any way since the fluid is incompressible.

The analog model is applicable to multiple and embedded layers by applying interfacial forces and leads to similar conclusions regarding the generation of concentric folding in viscous media.

Analog Model for Gravity Instability with Large Deformation

Consider two incompressible viscous fluids: one of density ρ_0 lying below the other of density $\rho_1 = \rho_0 + \Delta\rho_1$ in a gravity field of potential $U = gz +$ Const, where z equals altitude. Assume that the transition from ρ_0 to ρ_1 is continuous and occurs through a very thin surface S of thickness ϵ. The density $\rho(x, y, z)$ is then a continuous function of the coordinates. If the surface S is not a horizontal plane the fluid may be maintained in equilibrium by applying to it a body force field $-\mathbf{F}$ per unit volume. We must satisfy the equilibrium condition

$$-\operatorname{grad} p - \rho \operatorname{grad} U - \mathbf{F} = 0\,, \tag{3}$$

where p is the fluid pressure. This equation may be written

$$-\operatorname{grad}(pU + p) + U \operatorname{grad} \rho - \mathbf{F} = 0\,. \tag{4}$$

We choose a force $\mathbf{F}$, given by

$$\mathbf{F} = U \operatorname{grad} \rho = g(z - z_0) \operatorname{grad} \rho\,. \tag{5}$$

The pressure is therefore $p = -\rho U +$ const. The altitude z_0 is an arbitrary constant. In particular we may choose it to represent the altitude of the horizontal surface of separation of the two fluids in the initial equilibrium state when $\mathbf{F} = 0$.

The force $\mathbf{F}$ is zero except in the thin surface S. By integrating this force across the thickness

ϵ we obtain a force $\mathbf{F}\epsilon$. Its magnitude per unit area is

$$\mathbf{F}\epsilon = g(z - z_0)\frac{\Delta\rho_1}{\epsilon}\epsilon = g(z - z_0)\Delta\rho_1 . \quad (6)$$

This force is normal to the surface of separation of the two fluids, and expression (5) remains valid for the limiting case ($\epsilon = 0$) of an actual discontinuity.

representing the various densities as ρ_0, $\rho_1 = \rho_0 + \Delta\rho_1$, $\rho_2 = \rho_0 + \Delta\rho_1 + \Delta\rho_2$, *etc.* and treating each term $\Delta\rho_1$, $\Delta\rho_2$, *etc.* as we have done for the single discontinuity.[1]

Note that this analog model is valid for any incompressible material for which the deformation is independent of the hydrostatic component, since this is the only property invoked in the derivation. The interfacial forces are

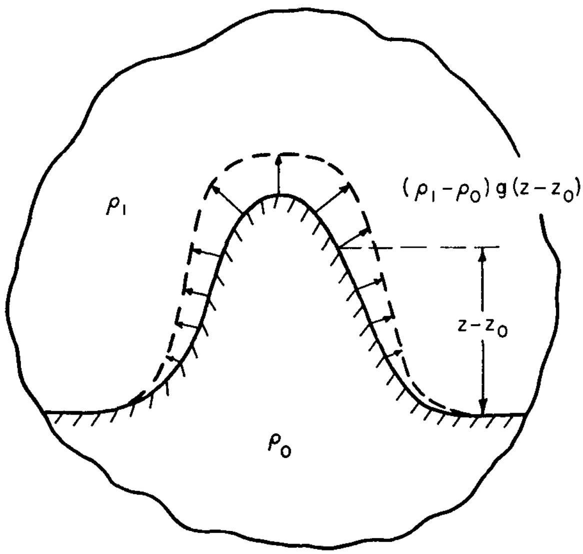

Figure 4. Analog model for a dome-shaped intrusion caused by a gravity instability of two fluids of different densities ρ_0 and ρ_1

These results provide an analog model as follows: The gravity field may be written $-\rho$ grad U $-$ $\mathbf{F}$ + $\mathbf{F}$. However as we have seen, the first two terms generate a hydrostatic pressure, hence no motion, since the medium is incompressible. Therefore these forces may be cancelled, leaving only F to cause the motion. Thus we obtain an analog model. As an illustration, consider a dome-shaped intrusion caused by a gravity instability of two fluids of densities ρ_0 and ρ_1 (Fig. 4). In the analog model gravity has been replaced by forces distributed normally to the surface of separation and of magnitude $g(z - z_0)(\rho_1 - \rho_0)$ per unit area. They are oriented toward the medium of higher density. The analog model is valid for an arbitrary number of fluid layers if one applies to each discontinuity surface the procedure just described. This is easily shown by

identical with buoyancy forces on a submerged body in various fluids of densities $\Delta\rho_1$, $\Delta\rho_2$, *etc.* The total potential energy of a vertical column of fluid is the surface integral

$$V = -\tfrac{1}{2} g \sum^{i} \iint_A z_i^2 \Delta\rho_i dA , \quad (7)$$

where A is the horizontal base area of the fluid, and z_i is the altitude of the interface of discontinuity $\Delta\rho_i$, each measured from an arbitrary origin. Equation (7) expresses the work done against the interfacial forces in the model, as can be verified by using the property $dn\, dS = dz\, dA$, where dn is the normal displacement of an element dS of the surface of discontinuity. These results generalize the analog model derived and used by the author

[1] *See* also the author's book (1965).

for small deformations in a series of earlier papers.

Variational Methods Applied to Viscous Buckling and Gravity Instability

Application of the principle of minimum dissipation suggests itself naturally for the approximate evaluation of large deformations in viscous media. Of particular interest is its application to the determination in the analog model of the additional velocity field super-

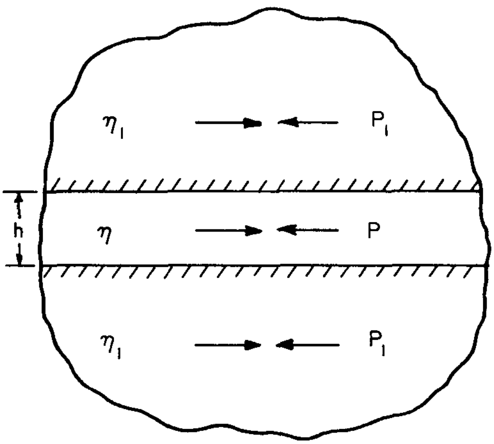

Figure 5. Fluid layer of viscosity η embedded in a fluid of viscosity η_1

posed upon the kinematic amplification. This velocity field is obtained by applying boundary and interfacial forces to a medium initially at rest and stress-free, with the configuration corresponding to the actual deformation at the instant considered. We then evaluate the velocity field for which the dissipation is minimum for a given power input (Biot, 1955). The finite deformation may thus be determined step by step at successive instants.

The method is of course applicable to gravity instability, including the case in which viscous buckling occurs in combination with it.

The analog model, when used as an intuitive guide in the choice of approximations, is helpful in the application of variational procedures.

Viscous Buckling of an Embedded Layer Undergoing Finite Strain

A fluid layer of viscosity η is embedded in a fluid of viscosity η_1. Under uniform compressive strain-rate the compressive stresses in the layer and the embedding medium are P and P_1, respectively. (Fig. 5.) In this case the analog model is obtained by applying fictitious forces $(P - P_1) \sin \alpha$ at the interfaces instead of $P \sin \alpha$ as in Figure 3B. If this analog model is used, it is possible to formulate in exact mathematical terms the time history of buckling of the layer for large strain, provided we assume that the superimposed flexural deformation remains small. The thickness of the layer increases with time. Its initial value is h_0 at $t = 0$; at the time t the thickness has become $h = h_0 \exp(p_0 t)$. An initial flexural deflection $v_0 = V_0 \cos l_0 x$ of wave length $\mathcal{L}_0 = 2\pi/l_0$ becomes $v = V \cos lx$ after a time t. The wave length at this time has been shortened owing to the compressive strain and has become $\mathcal{L} = 2\pi/l = \mathcal{L}_0 \exp(-p_0 t)$. The author (1964a; 1965) has shown that the amplification of the deflection is

$$\frac{V}{V_0} = A(t) e^{p_0 t}, \tag{8}$$

with

$$\log A = \int_0^t p dt, \tag{9}$$

where $p = P/2\eta\zeta$ and

$$\zeta = \frac{1}{2(1-c)\gamma}[(1+c^2)\sinh 2\gamma + 2c\cosh 2\gamma + 2(c^2-1)\gamma]. \tag{10}$$

The parameter $c = \eta_1/\eta$ represents the ratio of viscosities of the two materials. The variable $\gamma = \pi h/\mathcal{L} = (\pi h_0/\mathcal{L}_0) \exp(2p_0 t)$ is a function of time proportional to the ratio of the instantaneous value of the thickness h and the wave length $\mathcal{L}$.

In equation (8) the factor $A(t)$ represents the amplification owing to the *intrinsic* instability. The intrinsic instability is thus separated from the factor $\exp(p_0 t)$ which represents the pure kinematic amplification. For large viscosity contrast and large wave length ($c \ll 1$, $\gamma \ll 1$), expression (10) reduces to the approximate value

$$\zeta = \frac{2}{3}\gamma^2 + \frac{c}{\gamma}. \tag{11}$$

This value is the same as that obtained from thin-plate theory neglecting interfacial adherence (Biot, 1957). The exact value (10) of ζ and the approximate value (11) have been plotted in Figure 6 as functions of γ for two values of the viscosity ratio ($c = 1/50$ and $c = 1/100$).

The value γ_d of γ for which ζ is a minimum determines the instantaneous dominant wave length, *i.e.* that for which the rate of amplification is maximum at a given instant. It is obtained by equating to zero the derivative of expression (10) with respect to γ. This yields the equation

$$\frac{2\gamma_d - \tanh 2\gamma_d}{1 - 2\gamma_d \tanh 2\gamma_d} = \frac{2c}{1 + c^2}. \tag{12}$$

The solution γ_d of this equation is plotted as a function of $\sqrt[3]{c}$ in Figure 7. The value $\gamma_d = \sqrt[3]{(3/4)c}$ derived from the approximate equation (11) is also plotted as a straight line in Figure 7. One can see that this approximate value is satisfactory for $c < 1/50$. This corresponds to $\gamma_d < 0.3$, hence to wave lengths larger than about 10 times the thickness.

A similar comparison of the approximate equation (11) has been made with results obtained from the exact theory of stability of a continuum initially at rest under the initial stress with and without interfacial adherence (Biot, 1959; 1964b; Biot and Odé, 1962). Similar conclusions were derived regarding the validity of the approximate equation (11).

As already shown (Biot, 1959) the range of validity of equation (11) also coincides with the range of significant instability for viscous materials. Hence for such materials equation (11) will generally be sufficient in practical applications.

Compensating Effects of Shortening and Thickening of the Layer

The amplification A after a time t is obtained by integrating expression (9). The integrand p may be replaced by its average value p_{av} over the time interval t. Expression (9) becomes

$$\log A = p_{av} t. \tag{13}$$

The average value p_{av} is obtained by substituting into ζ the average value of $\gamma = (\pi h_0/\mathcal{L}_0) \exp (2p_0 t)$. We may write approximately

$$\gamma_{av} = \frac{\pi h_0}{\mathcal{L}_0} \exp (p_0 t) = \frac{\pi h_0}{\mathcal{L}}. \tag{14}$$

Hence the dominant wave length $\mathcal{L}$ *measured after deformation* is approximately the same as if there were no change of thickness ($h = h_0$) and no shortening. This is the result of a

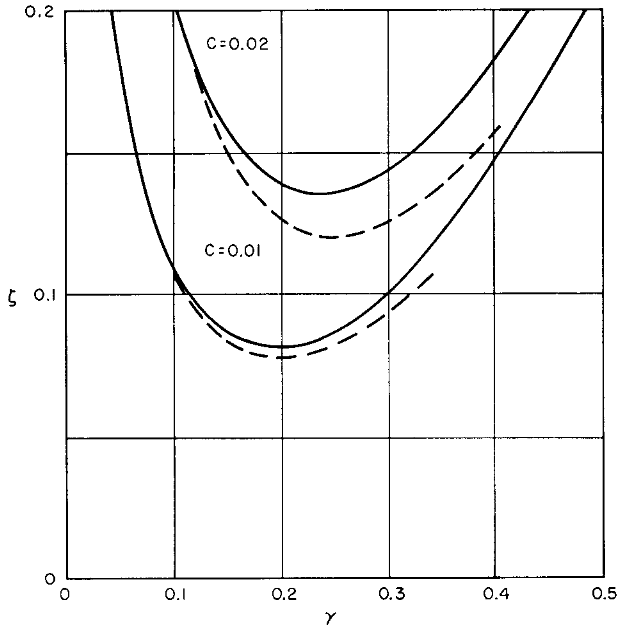

Figure 6. Stability diagram for the embedded layer of Figure 5. Solid lines represent the exact equation (10) and dotted lines, the approximate equation (11).

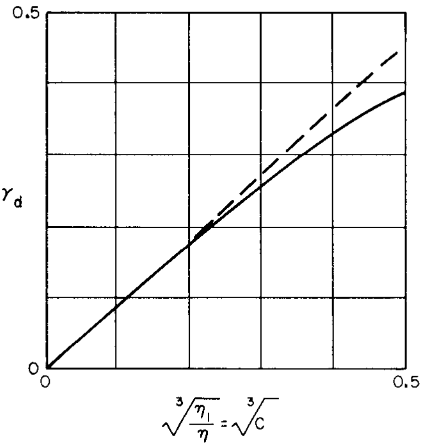

Figure 7. Wave length parameter γ_d for the dominant wave length. Solid line represents the exact value given by equation (12) and dotted line, the approximate value.

mutual compensation of these two factors which act in opposite directions.

Theory of Viscous Buckling of Multilayers Undergoing Finite Strain

One may use the same analog model to derive general differential equations for the time history of buckling of an arbitrary number of horizontal layers of viscous fluids undergoing a large compressive deformation (Fig. 8). The equations were derived in another paper (Biot, 1964a).[2] They are expressed by means of six coefficients:

$$\begin{aligned} A &= \tfrac{1}{2}(a_{11} + b_{11}) & D &= \tfrac{1}{2}(a_{11} - b_{11}) \\ B &= \tfrac{1}{2}(a_{12} + b_{12}) & E &= \tfrac{1}{2}(a_{12} - b_{12}) \\ C &= \tfrac{1}{2}(a_{22} + b_{22}) & F &= \tfrac{1}{2}(a_{22} - b_{22}), \end{aligned} \tag{15}$$

with

$$\begin{aligned} a_{11} &= \frac{4\cosh^2\gamma}{\sinh 2\gamma + 2\gamma} & b_{11} &= \frac{4\sinh^2\gamma}{\sinh 2\gamma - 2\gamma} \\ a_{12} &= -\frac{4\gamma}{\sinh 2\gamma + 2\gamma} & b_{12} &= \frac{4\gamma}{\sinh 2\gamma - 2\gamma} \\ a_{22} &= \frac{4\sinh^2\gamma}{\sinh 2\gamma + 2\gamma} & b_{22} &= \frac{4\cosh^2\gamma}{\sinh 2\gamma - 2\gamma}. \end{aligned} \tag{16}$$

The following differential equations for the time history of folding were established:

$$\begin{aligned} (D_{j-1}\dot{\mathfrak{u}}_{j-1} + E_{j-1}\dot{\mathfrak{v}}_{j-1} - A_{j-1}\dot{\mathfrak{u}}_j + B_{j-1}\dot{\mathfrak{v}}_j)\eta_{j-1} \\ - P_{j-1}\mathfrak{v}_j = (A_j\dot{\mathfrak{v}}_j + B_j\dot{\mathfrak{v}}_j - D_j\mathfrak{u}_{j+1} \\ + E_j\dot{\mathfrak{v}}_{j+1})\eta_j - P_j\mathfrak{v}_j \\ (-E_{j-1}\dot{\mathfrak{u}}_{j-1} - F_{j-1}\dot{\mathfrak{v}}_{j-1} + B_{j-1}\dot{\mathfrak{u}}_j - C_{j-1}\dot{\mathfrak{v}}_j)\eta_{j-1} \\ + (1/l)\rho_{j-1}g\mathfrak{v}_j = (B_j\dot{\mathfrak{u}}_j + C_j\dot{\mathfrak{v}}_j \\ - E_j\dot{\mathfrak{u}}_{j+1} + F_j\dot{\mathfrak{v}}_{j+1})\eta_j + (1/l)\rho_j g\mathfrak{v}_j . \end{aligned} \tag{17}$$

The dots represent time derivatives; each layer is characterized by the subscript j. The viscosity, density, compressive stress, and instantaneous thickness of each layer are respectively η_j, ρ_j, P_j, and h_j. Gravity is taken into account, and g is the acceleration of gravity. The coefficient A_j is obtained from equations (15) and (16), replacing γ by $\gamma_j = \frac{1}{2}lh_j = \kappa_j \exp(2p_0t)$, where κ_j equals the initial value of $\frac{1}{2}lh_j$. Other coefficients are obtained in the same way. The vertical deflection V_j of the top interface of layer j is $V_j = \mathfrak{v}_j \exp(p_0t)$. Hence $\mathfrak{v}_j$ is the interface deflection after elimination of the kinematic amplification factor. The variables $\mathfrak{u}_j$ are related to tangential interfacial displacements.

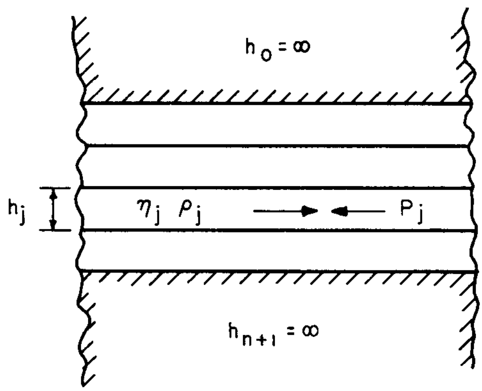

Figure 8. Multilayered viscous fluid. In the jth layer of thickness h_j the viscosity is η_j, the density ρ_j, and the compression P_j.

[2] Note the erratum in equations (6.6), (6.9), (6.13), and (6.17) of that paper, where V_j and $\mathfrak{v}_j$ must be replaced by V_{j+1} and $\mathfrak{v}_{j+1}$; *see* also the author's book (1965).

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