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NONLINEAR EFFECT OF INITIAL STRESS ON CRACK PROPAGATION BETWEEN SIMILAR AND DISSIMILAR ORTHOTROPIC MEDIA*

BY

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Abstract. The theory of crack propagation in orthotropic media is developed by applying the theory of incremental deformations in the vicinity of a state of initial stress. This is carried out in the context of a new approach to analytical methods and a physical analysis which takes into account plastic deformation under prestress. The state of initial stress is triaxial along the directions of elastic symmetry, and the crack is parallel to these directions. An additional shear component for the initial stress is also taken into account and general conditions are derived for crack propagation, including the case of fluid injection into the crack. The analysis is first carried out for an homogeneous medium. The nonlinear influence of the initial stress appears in two ways: first, through a fundamental purely elastic effect related to the occurrence of surface instability, and second, through the influence of the initial stress on plastic behavior. The particular cases of an isotropic elastic medium with finite initial strain and an orthotropic incompressible medium are discussed. The analysis is extended to a crack between dissimilar orthotropic media with initial stress. The method of analysis leads to a number of simplifications and brings out new properties of the solutions for this type of problem. For incompressible media without initial stress, the typical oscillatory behavior disappears. Uniqueness of the solutions is also derived.

1. Introduction. The theory of crack propagation was first developed by Griffith [1] [2], who considered the problem of failure of a brittle elastically isotropic material. Since that time an abundant literature has become available on crack mechanics based on the classical linear theory of elasticity. The stress distribution in the vicinity of a linear and circular crack was evaluated by Sneddon [3] and the case of a linear crack with nonuniform internal pressure was discussed by Sneddon and Elliott [4]. The latter analysis is based on the solution of simultaneous integral equations obtained by Busbridge [5]. The case of a circular crack was also treated by Sack [6]. Important contributions concerning both physical and mathematical aspects of these problems were made by Barenblatt; an extensive account of these contributions is presented in a review

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paper [7]. Many original contributions are also found in a book by Sneddon and Lowengrub [8].

The problem of crack mechanics at an interface between dissimilar isotropic media was investigated by Salganik [9], Rice and Sih [10], Erdogan [11] and England [13]. The same problem for two bonded media with two-dimensional anisotropy was solved by Gofoh [13]. The particular case of an isotropic medium bonded to an anisotropic medium was treated by Clements [14], who considered the application to bonded isotropic and transverse isotropic media. Erdogan and Gupta [15] have analyzed stresses due to cracks in a medium composed of a number of adhering layers of isotropic media.

The main purpose of the present paper is to analyze crack propagation in homogeneous and bonded orthotropic media taking into account the nonlinear influence of the state of initial stress by applying the theory of incremental deformations [16, 17, 18, 19, 20] developed by the author. A physical discussion is also included which provides a novel outlook and considers the influence of initial stress on the plastic separation energy. The whole theory is developed in the context of a new analytical approach which greatly simplifies the analysis in a large variety of problems of crack mechanics.

The basic analytical procedure is outlined in Sec. 2. It starts from the solution for a halfspace with sinusoidally distributed tractions along the surface. Solutions were derived by the author for a large number of cases of orthotropic and initially stressed media. They are all analytically of the same type and differ only in values of the coefficients. By use of these coefficients it is then possible to formulate an equivalent problem by means of Laplace's equation. The same analytical procedure therefore becomes applicable to a large variety of problems. Fundamentally, the fact that this is possible is due to the similarity property of linear problems for any homogeneous halfspace, namely that the surface displacements under given sinusoidal surface tractions are proportional to the wavelengths. General expressions are obtained for the shape of a crack under any arbitrary internal loading. The case of uniform loading is obtained in a remarkably simple way. Conditions for uniqueness of the solution and its behavior at infinity are examined in detail.

The physical aspects of crack propagation are discussed in Sec. 3 in the context of linear isotropic elasticity. This provides an extension of the concepts advanced by Irwin [21], Orowan [22], Dugdale [26] and Goodier and Field [27] to include plastic properties. The influence of initial stress on the separation energy is discussed. Similarity considerations lead to a separation energy which depends linearly on crack size. The crack propagation condition is derived on the basis of energy balance for given initial stresses and given interval fluid pressure in the crack. The influence of initial stress may be quite large, since plastic properties of materials are known to increase by a large order of magnitude, for example, under high hydrostatic stress.

A preliminary analysis based on linear elasticity theory is presented in Sec. 4 for crack propagation in orthotropic media. In addition to illustrating the simplicity of the method, the result provides an insight into more complex cases. Expressions for the shape of a crack under an internal loading of arbitrary distribution are immediately derived from the results of Sec. 2.

In Sec. 5, the influence of initial stress is evaluated by applying the author's theory of incremental deformation. This theory is essentially nonlinear with respect to the influence of initial stress. However, it is linearized with respect to small incremental strains in the vicinity of the state of initial stress. The material is assumed to be orthotropic for incremental strains, with principal initial stress along the axes of symmetry. The crack is parallel to a plane of symmetry and its edges are parallel to an axis of symmetry. An additional initial shear stress is also taken into account under the assumption that it is small enough that the orthotropic symmetry is not disturbed. Fluid under pressure is also injected into the crack. The crack propagation condition is derived. The particular case of an isotropic medium whose initial condition is one of finite strain is discussed in Sec. 6. Some new features in crack propagation are also discussed which are related to the phenomenon of surface instability of an initially stressed half-space. As expected, the resistance to crack propagation is diminished when the condition of surface instability is approached. However, the effect of a shear component for the initial stress is much less pronounced than the effect of the normal stresses or the fluid pressure.

In order to complete the analysis the case of crack propagation between two dissimilar adhering materials is developed in Sec. 7. Both materials are orthotropic for incremental deformations. The axes of symmetry for incremental properties and for the state of initial stress coincide with the coordinate axes for each material. The interface and the crack are in the xz plane and the edges of the crack are parallel to the z direction. The state of initial stress in each of the media may be different except for the stress S_{22} normal to the interface. The procedures used in the foregoing section are generalized to this case and a solution is obtained for the crack deformation due to the application of internal loading with arbitrary distribution. This internal loading includes the case in which it is represented by an increment of fluid pressure in excess of the initial value. The procedure used here provides again an analysis which is remarkably simple. It leads to a classical Hilbert problem by diagonalization of a two-by-two Hermitian matrix which is shown to be positive-definite. Hence the characteristic values are positive. Furthermore, their product turns out to be unity. The solution of the Hilbert problem is obtained from Muskhelishvili [23]. Uniqueness of the solution is also discussed. Except for the values of the coefficients, the analytical solution turns out to be fundamentally the same as for the case of initially stress-free isotropic media. The same singular behavior with violent oscillations occurs near the crack tips: this singular behavior is due essentially to the presence of a coupling term in the Hermitian matrix. Under certain conditions this coupling may vanish or be negligible. In this case the oscillatory singular behavior disappears and the crack propagation condition may be derived immediately without further calculations. This is verified rigorously for incompressible materials without initial stress. Thus if the effects of initial stress on incremental deformation appear only in a change of value of the elastic coefficients, the theory of crack propagation at an interface between orthotropic incompressible materials is drastically simplified.

2. Basic procedure. The method of analysis will first be presented in the context of the classical problem in isotropic elasticity. It will be shown that the crack problem is readily solved once we have determined solutions for the elastic halfspace which are sinusoidally distributed along the surface. Since a large number of such solutions were derived earlier [16, 18, 19, 20] for the very general case of orthotropic initially stressed media, the solution of the crack problem for such cases follows immediately.

We shall consider the plane strain problem of an elastic half-space occupying the region $y < 0$, the x axis lying along the free surface. The displacements along x and y are denoted by u and v and the stress components are σ_{xx} , σ_{yy} , σ_{xy} . Surface tractions normal to the free surface are applied with a sinusoidal distribution along x. Hence at the free surface the stress is

$$\sigma_{yy} = q \cos lx \tag{2.1}$$

while the normal displacement is

$$v = V \cos lx. \tag{2.2}$$

The relation between the amplitude q of the surface traction and the amplitude V of the surface displacement is readily evaluated by the classical theory of elasticity. We find

$$q = KlV \tag{2.3}$$

with

$$K = E/2(1 - \nu^2), \tag{2.4}$$

where E is Young's modulus and ν is Poisson's ratio.

We note that by shifting the origin by a distance $\pi/2l$, i.e. by replacing x by $x - \pi/2l$, Eqs. (2.1) and (2.2) become

$$\sigma_{yy} = q_1 \sin lx, \qquad v = V_1 \sin lx, \tag{2.5}$$

where the relation between q_1 and V_1 is the same as (2.3), i.e.

$$q_1 = KlV_1 . \tag{2.6}$$

An arbitrary distribution of v may be represented by a Fourier integral

$$v = \int_0^\infty [V(l) \cos lx + V_1(l) \sin lx]\, dl. \tag{2.7}$$

Hence, according to Eqs. (2.1), (2.2), (2.3), (2.5) and (2.6), the corresponding normal surface traction is

$$\sigma_{yy} = K \int_0^\infty [lV(l) \cos lx + lV_1(l) \sin lx]\, dl. \tag{2.8}$$

Let us define a function $\phi(x, y)$ in the halfplane $y > 0$ by the relation

$$\phi(x, y) = \int_0^\infty e^{-ly}[V(l) \cos lx + V_1(l) \sin lx]\, dl. \tag{2.9}$$

The function ϕ is harmonic and satisfies Laplace's equation

$$(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = 0. \tag{2.10}$$

Moreover, it vanishes at $y = \infty$. From Eqs. (2.7) and (2.8) we derive the following basic property:

$$v = \phi(x, 0), \qquad \sigma_{yy} = -K(\partial\phi/\partial y) \quad \text{at} \quad y = 0. \tag{2.11}$$

Hence the relation between normal displacements and tractions at the surface of the halfplane is the same as between a harmonic function and its normal derivative. Note that this result does not depend on the particular nature of the elasticity problem. It is essentially due to the fact that the factor Kl in Eq. (2.3) is inversely proportional to the wavelength; this in turn is a consequence of dimensional similitude. Hence the result should be valid for any homogeneous medium in the absence of any characteristic dimension. This is of course applicable to any homogeneous halfspace whether isotropic or not.

Eqs. (2.11) may be expressed in terms of holomorphic functions in the plane $y > 0$. We denote by $Z(z)$ such a function of the complex variable $z = x + iy$. Eqs. (2.11) for $z = x$ become

$$v = Z + Z^*, \qquad i\sigma_{yy} = K(Z' - Z'^*), \tag{2.12}$$

where

$$Z' = dZ/dz \tag{2.13}$$

and * denotes the complex conjugate quantity.

A very simple solution of Eq. (2.12) is obtained for the case where $-\sigma_{yy}$ is equal to a constant pressure p on the x axis in the interval $|x| < c$ while $v = 0$ for $|x| > c$. We put

$$Z = -(p/2K)i[z - (z^2 - c^2)^{1/2}], \tag{2.14}$$

where the argument of $(z^2 - c^2)^{1/2}$ is chosen between 0 and π in the halfplane $y > 0$. Since we have

$$z - (z^2 - c^2)^{1/2} = c^2/[z + (z^2 - c^2)^{1/2}], \tag{2.15}$$

the value of Z vanishes at $|z| = \infty$ as $1/z$. That this is the required behavior and leads to a unique solution is shown below in the last paragraph of this section. Moreover, substitution of (2.14) in Eqs. (2.12) with $z = x$ yields

$$\begin{aligned} v &= 0 && \text{for} \quad |x| > c, \\ -\sigma_{yy} &= p && \text{for} \quad |x| < c, \\ -v &= (p/K)(c^2 - x^2)^{1/2} && \text{for} \quad |x| < c. \end{aligned} \tag{2.16}$$

Because of the symmetry relative to the x axis this solution corresponds to a crack of length $2c$ subject to a uniform internal fluid pressure p. The crack assumes an elliptic shape of width distribution

$$w = -2v = (2p/K)(c^2 - x^2)^{1/2}. \tag{2.17}$$

In order to solve the more general problem of an arbitrary distribution of σ_{yy} along x, we first consider the case of a concentrated point load at $x = t$ ($|t| < c$), i.e.

$$\sigma_{yy}(x) = -\delta(x - t), \tag{2.18}$$

where δ is the Dirac function. For this case the second of Eqs. (2.12) is satisfied if we put

$$dZ/dz = (1/2\pi K) \ [(c^2 - t^2)/(c^2 - z^2)]^{1/2}(z - t)^{-1}. \tag{2.19}$$

This is easily verified since we may write

$$1/(z - t) = (d/dz)[\log (z - t)] = (d/dx)[\log r + i\theta_t], \tag{2.20}$$

where

$$z - t = r \exp (i\theta_t). \tag{2.21}$$

Note that Eq. (2.19) is in agreement with the required condition that Z vanishes as $1/z$ at infinity. As shown below in the last paragraph of this section, this insures that the solution (2.19) is unique. In order to obtain Z we must now integrate (2.19). A standard procedure is to rationalize the equation. There are several ways to do this but the most suitable in this case is to use the conformal transformation

$$z = (c/2)(\zeta + 1/\zeta) \tag{2.22}$$

which transforms the unit circle on the segment $|x| < c$ on the real axis. Also consider the relation

$$t = (c/2)(\tau + 1/\tau) \tag{2.23}$$

where τ is the point on the unit circle corresponding to the point $x = t$ on the real axis. On the unit circle ζ, τ and τ^* may be written

$$\zeta = \exp(i\theta), \qquad \tau = 1/\tau^* = \exp(i\theta'). \tag{2.24}$$

They correspond to the points

$$x = c \cos \theta, \qquad t = c \cos \theta' \tag{2.25}$$

on the x axis. With the new variables the differential equation (2.19) becomes

$$dZ/d\zeta = (1/2\pi K)[1/(\zeta - \tau) - 1/(\zeta - \tau^*)]. \tag{2.26}$$

Hence

$$Z = (1/2\pi K) \log ((\zeta - \tau)/(\zeta - \tau^*)). \tag{2.27}$$

This result satisfies the condition $v = 0$ on the real axis for $|x| > c$, as required. It also embodies a wellknown solution in potential flow problems where a source and a sink are located at points τ and τ^* on a circle [24]. With this value of Z the width distribution w of the crack is obtained from the first of Eqs. (2.12) and may be written

$$w = -2v = (2/\pi K) \log [R(x, t)] \tag{2.28}$$

where

$$R(x, t) = |\zeta - \tau^*|/|\zeta - \tau| . \tag{2.29}$$

In these expressions the values of ζ and τ are defined by Eqs. (2.24) and correspond to points on the circle. Hence they may be expressed in terms of the angles θ and θ' by means of the following relations, which have an obvious geometrical interpretation:

$$\begin{aligned} |\zeta - \tau| &= |(\zeta - \tau)(\zeta^* - \tau^*)|^{1/2} = 2 \sin |\tfrac{1}{2}(\theta - \theta')|, \\ |\zeta - \tau^*| &= |(\zeta - \tau^*)(\zeta^* - \tau)|^{1/2} = 2 \sin |\tfrac{1}{2}(\theta + \theta')|. \end{aligned} \tag{2.30}$$

Therefore

$$R(x, t) = \sin |\tfrac{1}{2}(\theta + \theta')|/\sin |\tfrac{1}{2}(\theta - \theta')| . \tag{2.31}$$

Since θ and θ' may be interchanged we derive the reciprocity property

$$R(x, t) = R(t, x) \tag{2.32}$$

as required by the theory of elasticity. For a continuous distribution $p(x)$ of the load along the crack the width is obviously derived by superposition of solutions (2.28) for the point load. Hence

$$w = \frac{2}{\pi K} \int_{-c}^{+c} p(t) \log [R(x, t)]\, dt. \tag{2.33}$$

In many problems the pressure distribution is symmetric, i.e.

$$p(x) = p(-x).$$

In this case the width distribution (2.33) becomes

$$w = \frac{2}{\pi K} \int_0^c p(t) \log [R_1(x, t)]\, dt \tag{2.34}$$

where

$$R_1 = \frac{\tan |\frac{1}{2}(\theta + \theta')|}{\tan |\frac{1}{2}(\theta - \theta')|} = \frac{|\sin \theta + \sin \theta'|}{|\sin \theta - \sin \theta'|}. \tag{2.35}$$

It is interesting to compare this expression with results obtained by other investigators. We make use of the identity

$$2 \int_0^c \frac{t' 1(t' - x) 1(t' - t)\, dt'}{\sqrt{(t'^2 - x^2)(t'^2 - t^2)}} = \log [R_1(x, t)] \tag{2.36}$$

where $x = c \cos \theta$ and $t = c \cos \theta'$ while

$$\begin{aligned} 1(x) &= 0 \quad \text{for} \quad x < 0 \\ &= 1 \quad \text{for} \quad x > 0 \end{aligned} \tag{2.37}$$

is the Heaviside unit step function. The identity is easily verified by the substitution of $z = (t'^2 - x^2)^{1/2}/(t'^2 - t^2)^{1/2}$ as the variable of integration. By introducing the value (2.36) into the integral (2.34) we derive

$$w = \frac{4}{\pi K} \int_x^c \frac{t'\, dt'}{(t'^2 - x^2)^{1/2}} \int_0^{t'} \frac{p(t)}{(t'^2 - t^2)^{1/2}}\, dt \tag{2.38}$$

which coincides with the expression given by Sneddon [8].

In the foregoing analysis we have assumed that the load applied to the crack is in the nature of a fluid pressure, i.e. that it acts normally to the surface of the crack. The same procedure is readily applied to the case of purely tangential tractions σ_{xy} equal and opposite acting on the bottom and upper faces of the crack. Because of symmetry the faces remain in contact, i.e. the width remains zero, but the surfaces slip relative to each other by an equal amount in opposite directions. To show this, consider a sinusoidally distributed tangential load

$$\sigma_{xy} = \tau \cos lx, \qquad \sigma_{yy} = 0 \tag{2.39}$$

applied to the halfspace $y < 0$ at $y = 0$. The corresponding tangential displacement is

$$u = U \cos lx \tag{2.40}$$

with

$$\tau = KlU \tag{2.41}$$

and the same value (2.4) for K. Hence, except for some sign differences, all expressions derived above for the normal load are applicable. For example, if the tangential traction σ_{xy} is a constant, the relative slip distribution of the crack surfaces is

$$\bar{u} = 2u = (2\sigma_{xy}/K)(c^2 - x^2)^{1/2}. \tag{2.42}$$

For an arbitrary distribution of $\sigma_{xy}(x)$ the relative slip is given by the integral (2.33) where w and $p(t)$ are replaced by $\bar{u}$ and $\sigma_{xy}(t)$ respectively.

Uniqueness of solution. Physical conditions on the x axis and the definition (2.9) require ϕ to vanish at infinity. As a consequence $Z = \phi + i\phi_1$ must also vanish at infinity since

$$\partial\phi/\partial x = \partial\phi_1/\partial y, \qquad \partial\phi/\partial y = -\partial\phi_1/\partial x. \tag{2.43}$$

The condition $\partial\phi/\partial x = \partial\phi/\partial y = 0$ implies $\partial\phi_1/\partial x = \partial\phi_1/\partial y = 0$. Hence at infinity ϕ_1 is a constant which may be chosen equal to zero without affecting the physical problem. The function Z is defined in the upper halfplane. We may extend it to the lower halfplane by a Schwarz reflection. At the point z^* the value in the lower halfplane is defined as $-Z^*(z)$. The conditions $Z + Z^* = v = 0$ or $Z = -Z^*$ on the x axis for $|x| > c$ imply that the function Z is holomorphic throughout except at a cut $|x| < c$ on the x axis where it is discontinuous. According to Eqs. (2.12) the value of the discontinuity is

$$v(x) = Z(x) + Z^*(x). \tag{2.44}$$

Since Z vanishes at infinity its value is expressed by the Cauchy integral (see [23])

$$Z(z) = \frac{1}{2\pi i}\int_{-c}^{+c} \frac{v(x)}{x - z}\, dx. \tag{2.45}$$

For $|z| > c$ we may write the expansion

$$Z(z) = -\frac{z^{-1}}{2\pi i}\int_{-c}^{+c} v(x)\, dx - \frac{z^{-2}}{2\pi i}\int_{-c}^{+c} xv(x)\, dx - \cdots . \tag{2.46}$$

Since the volume of the crack is not zero, the coefficient of z^{-1} does not vanish. Hence Z is of the order z^{-1} at $|z| = \infty$. That these results imply uniqueness of the foregoing solutions can be seen as follows. Consider two solutions ϕ satisfying the boundary conditions

$$\begin{aligned} v = \phi(x, 0) &= 0 \quad \text{for} \quad |x| > c, \\ \sigma_{yy} &= -K(\partial\phi/\partial y) \quad \text{for} \quad |x| < c,\ y = 0. \end{aligned} \tag{2.47}$$

The difference ϕ_d of these two solutions satisfies the conditions

$$\begin{aligned} \phi_d(x, 0) &= 0 \quad \text{for} \quad |x| > c, \\ \partial\phi_d/\partial y &= 0 \quad \text{for} \quad |x| < 0,\ y = 0. \end{aligned} \tag{2.48}$$

Let us apply Green's theorem to ϕ_d. We may write

$$\iint_A (\text{grad}\,\phi_d)^2\, dx\, dy = \int_C \phi_d \frac{\partial\phi_d}{\partial n}\, ds \tag{2.49}$$

where C is a contour composed of the x axis and an infinite half-circle centered at the origin in the half-plane $y > 0$, $\partial\phi_d/\partial n$ is the normal outward derivative on the contour and A is the area inside the contour. On the infinite half-circle ϕ_d and $\partial\phi_d/\partial n$ are of the order $1/z$ and $1/z^2$ respectively; therefore the line integral vanishes on this half-circle. On the x axis it also vanishes because of the boundary conditions (2.48). Hence grad $\phi_d = 0$, and since ϕ_d is zero at infinity it vanishes everywhere in the half-plane. Therefore the solution ϕ is unique. The same conclusion holds for the solution corresponding to a given distribution $\sigma_{xy}(x)$ of shear stress at the crack.

3. The physics of crack propagation. The classical Griffith theory of crack propagation [1] [2] assumed a brittle isotropic perfectly elastic material and derived a propagation criterion under static conditions, based on energy balance. Along the same lines, the influence of plasticity was considered by Irwin [21], Orowan [22], Dugdale [26], and Goodier and Field [27]. The physical analysis may be extended to include the influence of anisotropy and initial stress on the plastic deformation. Similarity properties also provide some new insights.

We assume an isotropic material with a crack subject to a uniform fluid pressure p. The volume of the crack per unit thickness measured normally to the x, y plane is

$$\mathcal{V} = \pi p c^2/K \tag{3.1}$$

where K is the coefficient (2.4). This value of $\mathcal{V}$ is derived from Eq. (2.17). The elastic energy stored in the medium is

$$W = \tfrac{1}{2} p \mathcal{V}. \tag{3.2}$$

When the crack size progresses by an amount dc we may write the following energy balance equation:

$$p(\partial \mathcal{V}/\partial c) - (\partial W/\partial c) = 2\mathcal{E}_n. \tag{3.3}$$

The left-hand side is the work done by the pressure minus the change of energy stored elastically. On the right the quantity $\mathcal{E}_n$ is the work necessary to separate a unit area of the medium. We have introduced a subscript n to indicate that $\mathcal{E}_n$ represents the energy required for separation of the crack surfaces in a direction normal to the crack. This is to distinguish it from the case where the separation occurs by shear which will be considered below. By substituting expressions (3.1) and (3.2), Eq. (3.3) yields

$$p = (2\mathcal{E}_n K/\pi c)^{1/2}. \tag{3.4}$$

This is the critical pressure required for crack propagation.

An important aspect of the problem resides in the significance of $\mathcal{E}_n$, which we shall call the energy of normal separation. For physical reasons it is obvious that in general it will be a function of the size c of the crack.

Let us start with the case of a perfectly brittle material. The separation requires the creation of two free surfaces, each with a surface energy equal to the surface tension. Hence we may write

$$\mathcal{E}_n = 2T. \tag{3.5}$$

With this value the critical pressure (3.4) becomes

$$p = (4TK/\pi c)^{1/2}, \tag{3.6}$$

which coincides with the classical Griffith result [2]. A slight correction to this result may be introduced by taking into account the acoustic energy radiated during the cracking process. We denote this energy by $A(c)$ to indicate that it may depend on the size of the crack. Hence

$$\mathcal{E}_n = 2T + A(c). \tag{3.7}$$

For a large category of materials plasticity plays an important if not dominant role. Therefore we write

$$\mathcal{E}_n = 2T + \Omega_n(c), \tag{3.8}$$

where $\Omega_n(c)$ includes the energy dissipated in plastic deformation and acoustic radiation for normal separation. In a first approximation it is natural to assume that $\Omega_n(c)$ is proportional to the size of the crack, i.e.

$$\Omega_n(c) = a_n c. \tag{3.9}$$

The physical justification of this assumption is based on the fact that the size of the plastic region surrounding the crack tip must be proportional to the crack size, as required by the *principle of similitude.* If these assumptions are valid it is of interest to note that the critical pressure (3.4) required for crack propagation does not decrease indefinitely with increasing size of the crack but tends toward a constant value given by

$$p = (2a_n K/\pi)^{1/2} \tag{3.10}$$

Actually, the value of a_n also depends on p (see [27]). However, throughout the present analysis this dependence will be considered as a separate problem.

Before proceeding any further, a remark is in order regarding the value (3.2) of the elastic energy which is based on the assumption that the material behaves throughout according to the linear theory of elasticity. This is obviously not the case, because of the existence of the plastic region surrounding the crack tip. Actually the value of W should be corrected by substracting the elastic energy which would be present in the plastic region if it remained elastic. While such a correction may be introduced formally, we shall not do so explicitly in the present analysis.

The influence of the state of initial stress on crack propagation is twofold. In order to avoid confusion it is important to distinguish two entirely different aspects of the problem. One aspect resides in the influence of initial stress on the purely elastic portion of the stress field. This requires a more elaborate analysis based on the theory of incremental deformations [20] which will be developed in Sec. 5. The other is the dependence of the separation energy $\mathcal{E}_n$ on the magnitude of the initial stress. In the present section we shall limit ourselves to a preliminary discussion which considers only this second aspect of the problem. This will provide a clear physical basis for the more elaborate analysis of Sec. 5.

Since the classical test results obtained by von Kármán [25], it is wellknown that the ductility of materials increases considerably under large hydrostatic pressure. Materials which are brittle originally may become plastic under pressure. As the latter increases the work required to reach failure may be multiplied by a large factor. This means that the separation energy (3.8) should be written

$$\mathcal{E}_n = 2T + \Omega_n(p_i, c) \tag{3.11}$$

where the plastic energy Ω_n is a function of the initial hydrostatic stress p_i. In particular, expression (3.9) becomes

$$\Omega_n = ca_n(p_i) \tag{3.12}$$

where the coefficient a_n is now a function of p_i. The value of this coefficient may greatly increase and become dominant for increasing pressure p_i.

When a fluid pressure $p > p_i$ is applied inside the crack the crack tends to propagate. An important feature here is due to the fact that for a moderate pressure increment $p - p_i$ the elastic portion of the stress field behaves exactly as in the classical linear theory of elasticity for isotropic materials. This property, which is evident intuitively, was also derived rigourously by the author [20]. As shown in Sec. 5 below, the incremental elastic coefficient

$$K = K(p_i) \tag{3.13}$$

now depends on the initial pressure p_i. The shape of the crack remains elliptical and the volume $\mathcal{V}$ is given by an expression similar to (3.1), namely

$$\mathcal{V} = \pi(p - p_i)c^2/K. \tag{3.14}$$

The incremental elastic energy stored in the solid due to the incremental pressure $p - p_i$ is

$$W = \tfrac{1}{2}(p_i + p)\mathcal{V}. \tag{3.15}$$

With the values (3.15) and (3.14) the energy balance equation is the same as (3.3). We derive

$$p - p_i = (2\mathcal{E}_n K/\pi c)^{1/2} \tag{3.16}$$

where p is the critical propagation pressure while $\mathcal{E}_n$ and K are expressed by (3.11) and (3.13). Note again that the dependence of a_n on p is to be considered as a separate problem.

We now consider a more general case where S_{11}, S_{22}, S_{33} are principal stresses present initially along the x, y, z directions, the crack itself lying in the x, z plane and extending to infinity along z. In addition we also assume that an initial shear stress S_{12} is present along the x and y directions. In such a case the behavior of the incremental stress field is fundamentally different from that assumed in the foregoing analysis. As shown below in Sec. 5, a medium initially isotropic becomes generally anisotropic. In addition, certain new physical features, related to the existence of surface instability, enter into play.

Under certain assumptions, however, it is possible to neglect these more sophisticated features, thus providing an approximate preliminary analysis. Let

$$p_i = -(S_{11} + S_{22} + S_{33}) \tag{3.17}$$

be the average hydrostatic component of the initial stress. We assume that $p_i + S_{11}$, $p_i + S_{22}$, $p_i + S_{33}$ and S_{12} are sufficiently small that the material remains isotropic for incremental deformations. Moreover, the excess $p + S_{22}$ of the pressure p of the fluid in the crack should not exceed a magnitude beyond which a linear theory of incremental deformations breaks down. Keeping these limitations in mind, we may proceed as follows. As in Eqs. (3.1) and (3.14), the volume of the crack generated by the fluid pressure is

$$\mathcal{V} = \pi(p + S_{22})c^2/K. \tag{3.18}$$

We also evaluate

$$\mathcal{U} = \int_{-c}^{+c} \bar{u}\, dx = \frac{\pi S_{12} c^2}{K}, \tag{3.19}$$

where $\bar{u}$ is the relative slip (2.42) of the faces of the crack under application of a shear load.

The increase in elastic energy due to the application of the fluid pressure p and the disappearance of the tangential load S_{12} at the faces of the crack is

$$W = \tfrac{1}{2}(p - S_{22})\mathcal{V} - \tfrac{1}{2}S_{12}\mathcal{U}. \tag{3.20}$$

It is important to note that there is no coupling term between the tangential and normal displacements. This follows from the fact that the load $p - S_{22}$ is associated with a displacement u which is an odd function of x on which S_{12} produces no work. Similarly, the load S_{12} produces a normal displacement v which is also an odd function of x on which $p - S_{22}$ produces no work.

The energy balance equation is

$$p(\partial \mathcal{V}/\partial c) - (\partial W/\partial c) = 2\mathcal{E}_{ns} \tag{3.21}$$

where $\mathcal{E}_{ns}$ is the energy for combined normal and shear separation of the crack surfaces. Substitution of the values (3.18) and (3.20) yields

$$(p + S_{22})^2 + S_{12}^2 = 2\mathcal{E}_{ns}K/\pi c. \tag{3.22}$$

This is the condition for crack propagation. The energy $\mathcal{E}_{ns}$ of combined separation will in general be a function of $S_{11}S_{22}S_{33}S_{12}$ and c. In analogy with (3.11) and (3.12) we may write

$$\begin{aligned} \mathcal{E}_{ns} &= 2T + \Omega_{ns}(S_{11}, S_{22}, S_{33}, S_{12}, c), \\ \Omega_{ns} &= c a_{ns}(S_{11}S_{22}S_{33}S_{12}). \end{aligned} \tag{3.23}$$

Again we must keep in mind the foregoing remark concerning the possible dependence of a_{ns} on p. The incremental elastic coefficient $K = K(p_i)$ may also be assumed to depend on the average initial pressure p_i.

In particular, if $p = S_{12} = 0$ Eq. (3.22) becomes

$$S_{22} = (2\mathcal{E}_n K/\pi)^{1/2}. \tag{3.24}$$

This is the value of the critical tensile stress acting normally to the crack which produces spontaneous crack propagation. Similarly, for $p = S_{22} = 0$ Eq. (3.22) becomes

$$S_{12} = (2\mathcal{E}_s K/\pi)^{1/2} \tag{3.25}$$

which is the initial shear stress for spontaneous crack propagation. In this case $\mathcal{E}_s$ is the energy for pure shear separation.

A final remark is in order here regarding Eqs. (3.24) and (3.25). We have assumed implicitly that we are dealing with a static propagation. Spontaneous propagation may occur under dynamic conditions in which case plastic materials may become brittle, with a corresponding drop in the values of $\mathcal{E}_n$ and $\mathcal{E}_s$ as soon as the propagation starts. The failure thus acquires a more or less explosive character, an occurrence which is not infrequent in prestressed structures.

4. Crack propagation in orthotropic media. In the preceding section the problem of crack propagation under initial stress was analyzed in preliminary form for an isotropic material. As was pointed out, under certain conditions the linear theory of elasticity remains applicable. Under the same assumptions this preliminary analysis may be extended to orthotropic media. Such a simplified preliminary treatment is useful in order to bring out more clearly certain essential features which result from anisotropy and are distinct from those brought out by the more elaborate treatment in Sec. 5. A separate analysis for orthotropic media is also of particular importance because a medium originally isotropic acquires anisotropic properties under a non-hydrostatic state of initial stress.

Consider the problem in the context of the linear theory of elasticity for orthotropic symmetry. The method outlined in Sec. 2 is based entirely on the validity of Eq. (2.3) which itself is a consequence of a basic similarity law. Hence it should be valid for a large class of homogeneous materials. In particular, it is applicable to orthotropic media for which the stress-strain relations in plane strain are

$$\sigma_{xx} = C_{11}e_{xx} + C_{12}e_{yy}\,, \quad \sigma_{yy} = C_{12}e_{xx} + C_{22}e_{yy}\,, \quad \sigma_{xy} = 2Qe_{xy}\,. \tag{4.1}$$

With the displacements u, v the strain components are

$$e_{xx} = \frac{\partial u}{\partial x}\,, \qquad e_{yy} = \frac{\partial v}{\partial y}\,, \qquad e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \tag{4.2}$$

Again, we may solve the problem of the halfspace for the region $y < 0$ by applying at the surface ($y = 0$) the sinusoidally distributed stresses

$$\sigma_{xy} = \tau \sin lx, \qquad \sigma_{yy} = q \cos lx. \tag{4.3}$$

The corresponding surface displacements are of the form

$$u = U \sin lx, \qquad v = V \cos lx. \tag{4.4}$$

The displacement amplitudes U and V are related to the stress amplitudes τ and q by the equations

$$\tau = (a_{11}U + a_{12}V)Ql, \qquad q = (a_{12}U + a_{22}V)Ql, \tag{4.5}$$

where

$$a_{11} = \frac{C_{11}(\beta_1 + \beta_2)}{C_{11} + Q\beta_1\beta_2}\,, \qquad a_{22} = \frac{C_{22}(\beta_1 + \beta_2)\beta_1\beta_2}{C_{11} + Q\beta_1\beta_2}\,, \qquad a_{12} = \frac{C_{12}\beta_1\beta_2 - C_{11}}{C_{11} + Q\beta_1\beta_2}. \tag{4.6}$$

With positive values of the square roots the quantities β_1 and β_2 are given by

$$\begin{aligned} \beta_1 &= (m + [m^2 - k^2]^{1/2})^{1/2}, \\ \beta_2 &= (m - [m^2 - k^2]^{1/2})^{1/2} \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} 2m &= (1/QC_{22})[C_{11}C_{22} - 2QC_{12} - C_{12}^2], \\ k &= (C_{11}/C_{22})^{1/2}. \end{aligned} \tag{4.8}$$

The derivation of the values (4.6) of a_{ij} is obtained in routine manner from the two-dimensional theory of anisotropic elasticity. It may also be obtained as a particular case of the more general results established by the author for anisotropic elasticity with initial stress [18] [20]. This amounts to putting equal to zero the initial stress in Eqs. (5.14) below.

For our purpose we need the solution for the case where the shear stress at the surface is put equal to zero ($\tau = 0$). Introducing this condition in Eqs. (4.5) yields

$$q = K_1 lV \tag{4.9}$$

with

$$K_1 = ((a_{11}a_{22} - a_{12}^2)/a_{11})Q. \tag{4.10}$$

With the value K_1 instead of K the result is exactly the same as that of Eq. (2.3).

Similarly, we obtain the solution for the case where the normal stress is put equal to zero ($\sigma_{yy} = q = 0$) at the surface. For this case Eqs. (4.5) yield

$$\tau = K_2 lU \tag{4.11}$$

with

$$K_2 = ((a_{11}a_{22} - a_{12}^2)/a_{22})Q. \tag{4.12}$$

Note that we may shift the origin along x, thus replacing $\sin lx$ by $\cos lx$. Eqs. (4.11) and (4.12) are therefore valid for the shear distribution

$$\sigma_{xy} = \tau \cos lx \tag{4.13}$$

and the corresponding displacement

$$u = U \cos lx \tag{4.14}$$

More explicitly, values of K_1 and K_2 are derived by substituting expressions (4.6) for a_{ij} taking into account the relations

$$(2(m + k))^{1/2} = \beta_1 + \beta_2 , \qquad k = \beta_1\beta_2 . \tag{4.15}$$

We obtain

$$\begin{aligned} K_1 &= \frac{2(m + k)kC_{11}C_{22} - (C_{12}k - C_{11})^2}{\sqrt{2(m + k)}C_{11}(C_{11} + Qk)} Q, \\ K_2 &= K_1/k. \end{aligned} \tag{4.16}$$

It is interesting to verify that this result yields the value (2.4) of K for an isotropic medium. In this case the elastic moduli are written

$$C_{11} = C_{22} = 2\mu + \lambda, \qquad C_{12} = \lambda, \qquad Q = \mu, \tag{4.17}$$

where λ and μ are Lamé's constants for isotropic elasticity. Expressions (4.16) become

$$K_1 = K_2 = (2(\mu + \lambda)/(2\mu + \lambda))\mu. \tag{4.18}$$

In terms of Poisson's ratio ν and Young's modulus E we find

$$K_1 = K_2 = K = E/2(1 - \nu^2), \tag{4.19}$$

which coincides with expression (2.4). Under an arbitrary distribution of the loads σ_{yy} and σ_{xy} applied inside the crack, the width w and slip $\bar{u}$ are given as in Eq. (2.33) except that K must be replaced by K_1 and K_2 for normal and tangential loading respectively.

The crack propagation analysis may now be carried out by following the same procedure as in the preceding section. The state of initial stress is represented by the components S_{11}, S_{22}, S_{33} and S_{12}. The principal stress components S_{11}, S_{22}, S_{33} are oriented along x, y, z, which are also the directions of orthotropic symmetry. The hydrostatic component (3.17) of these initial stresses may be large. However, their differences, as well as the initial shear component S_{12}, are sufficiently small that the foregoing results derived from linear elasticity are not modified. The volume of the crack generated by the fluid pressure p is obtained as in Eq. (3.18). Its value is

$$\mathcal{V} = \pi(p + S_{22})c^2/K_1 . \tag{4.20}$$

The quantity $p + S_{22}$ represents the excess fluid pressure over the initial compression $-S_{22}$ normal to the crack. As before, we must also assume that the magnitude of $p + S_{22}$ does not exceed a limit beyond which linear elasticity is not applicable. We also need the value $\mathcal{U}$ defined by Eq. (3.19). In this case it becomes

$$\mathcal{U} = \pi S_{12}c^2/K_2. \tag{4.21}$$

Eqs. (3.20) and (3.21) for energy balance are formally the same in this case. We derive the crack propagation condition

$$(p + S_{22})^2/K_1 + S_{12}^2/K_2 = 2\mathcal{E}_{ns}/\pi c \tag{4.22}$$

or

$$(p + S_{22})^2 + kS_{12}^2 = (2\mathcal{E}_{ns}/\pi c)K_1. \tag{4.23}$$

The combined separation energy $\mathcal{E}_{ns}$ is an expression of the form (3.23), while k and K_1 may be functions of the initial hydrostatic component p_i. The case $p = 0$ while S_{22} is positive represents a state of initial tension normal to the crack.

Incompressible medium. By assuming incompressibility the crack propagation condition (4.22) is considerably simplified. For this case the stress-strain relations (4.1) are replaced by

$$\sigma_{xx} - \sigma = 2Ne_{xx} , \qquad \sigma_{yy} - \sigma = 2Ne_{yy} , \qquad \sigma_{xy} = 2Qe_{xy} . \tag{4.24}$$

The condition of incompressibility is

$$e_{xx} + e_{yy} = 0; \tag{4.25}$$

hence $2\sigma = \sigma_{xx} + \sigma_{yy}$. Note that the left-hand side of Eqs. (4.24) is a two-dimensional form of the stress-deviator which is not the same as its usual three-dimensional definition. For the incompressible medium Eqs. (4.5) become

$$\tau = 2(NQ)^{1/2}lU, \qquad q = 2(NQ)^{1/2}lV. \tag{4.26}$$

Hence

$$a_{11} = a_{22} = 2(N/Q)^{1/2}, \qquad a_{12} = 0, \tag{4.27}$$

$$K_1 = K_2 = 2(NQ)^{1/2}. \tag{4.28}$$

These results were derived earlier [16, 18, 20]. With the values (4.28) the crack propagation condition (4.23) becomes

$$(p + S_{22})^2 + S_{12}^2 = (4\mathcal{E}_{ns}/\pi c)(NQ)^{1/2}. \tag{4.29}$$

Except for the replacement of $K = E/2(1 - \nu^2)$ by $2(NQ)^{1/2}$ it is exactly the same as condition (3.22) for the isotropic medium.

It is of interest to point out that the case of incompressibility may also be derived directly from expressions (4.16) following a procedure indicated by the author [18, 20]. We write

$$C_{11} = C_{22} = \mathcal{K} + N, \qquad C_{12} = \mathcal{K} - N, \tag{4.30}$$

and substitute these values in Eqs. (4.1), imposing the condition that

$$\sigma = \mathcal{K}(e_{xx} + e_{yy}) \tag{4.31}$$

remains finite while $\mathcal{K}$ goes to infinity. This yields the stress-strain relations (4.24). Moreover, when we substitute expressions (4.30) with $\mathcal{K} = \infty$ into Eqs. (4.16) we obtain the value (4.28) for K_1 and K_2.

5. Application of the theory of incremental deformations. The foregoing results were obtained by applying the linear theory of elasticity. However, beyond a certain range of initial stress the validity of the linear theory breaks down and the analysis must be based on the general theory of incremental deformations. Such a theory was developed by the author [16, 17, 18, 19, 20] for an initially-stressed continuum.

We consider an orthotropic medium with directions of symmetry along the x, y, z directions. The medium is initially stressed by three principal stresses S_{11}, S_{22}, S_{33} along the same directions. This includes the particular case of a medium which is isotropic before application of the initial stresses. Small incremental deformations may be superimposed with displacements u, v in the x, y plane. This produces a state of incremental plane strain defined by expressions (4.2). In addition, a rotation field is produced of magnitude

$$\omega = \tfrac{1}{2}((\partial v/\partial x) - (\partial u/\partial y)). \tag{5.1}$$

Each small element of the medium is rotated by this amount. Initially the stresses on this element are S_{11}, S_{22}, S_{33}. If the element is rotated rigidly by the amount ω about the z direction the stresses remain unchanged. In other words, the stress components referred to coordinate axes rotated along with the element retain the initial values S_{11}, S_{22}, S_{33}. However, since the element is deformed the stresses increase by an amount s_{ij} which is also referred to the locally rotated axes. Because we restrict ourselves to incremental plane strain we need only consider the incremental stress components s_{11}, s_{22} and s_{12}. We have shown [16, 18, 20] that they are expressed in terms of the incremental strains by the relations

$$s_{11} = B_{11}e_{xx} + B_{12}e_{yy}, \qquad s_{22} = B_{21}e_{xx} + B_{22}e_{yy}, \qquad s_{12} = 2Qe_{xy}, \tag{5.2}$$

The existence of an elastic potential for incremental deformations requires that the incremental elastic coefficients B_{ij} satisfy the relation

$$B_{12} = B_{21} + P, \tag{5.3}$$

where

$$P = S_{22} - S_{11}. \tag{5.4}$$

Again we consider the analysis of the halfspace $y < 0$ with the normal stress S_{22} applied initially at the boundary $y = 0$. An incremental plane strain is thus produced by applying additional tractions on this free surface. These tractions per unit *initial* area are represented by the components Δf_x, Δf_y along fixed directions x and y. In terms of the deformations, it was shown [20] that Δf_x and Δf_y are expressed as follows:

$$\Delta f_x = \Delta \bar{f}_x - S_{22}(\partial v/\partial x), \qquad \Delta f_y = \Delta \bar{f}_y + S_{22} e_{xx}, \tag{5.5}$$

where

$$\Delta \bar{f}_x = s_{12} + P e_{xy}, \qquad \Delta \bar{f}_y = s_{22}. \tag{5.6}$$

A physical interpretation of these expressions is obtained by assuming that a distributed hydrostatic stress $p(x)$ and a distributed tangential traction $T(x)$ are applied at the surface

$$\Delta f_x = p(\partial v/\partial x) + T, \qquad \Delta f_y = -p(1 + e_{xx}) - S_{22}. \tag{5.7}$$

Substitution into Eqs. (5.5) yields

$$\Delta \bar{f}_x = (p + S_{22})(\partial v/\partial x) + T, \qquad \Delta \bar{f}_y = -(p + S_{22})(1 + e_{xx}). \tag{5.8}$$

The quantity $p + S_{22}$ is the incremental pressure. In the context of incremental deformations the product of this incremental pressure by $\partial v/\partial x$ and e_{xx} is considered to be of a higher order, and hence may be neglected. Under these conditions Eqs. (5.8) become

$$\Delta \bar{f}_x = T, \qquad \Delta \bar{f}_y = -p - S_{22}. \tag{5.9}$$

Physically we may look upon the system as one composed of the solid and an adjacent fluid at a uniform pressure equal to $-S_{22}$. The forces $\Delta \bar{f}_x$, $\Delta \bar{f}_y$ are the additional tractions applied to the solid at the interface. Note that substitution of the last of Eqs. (5.2) into the first of Eqs. (5.6) yields

$$\Delta \bar{f}_x = T = 2L e_{xy} \tag{5.10}$$

with

$$L = Q + \tfrac{1}{2}P. \tag{5.11}$$

This coefficient which we have called the "slide modulus" has therefore a simple physical meaning.

As before we assume a sinusoidal distribution of $\Delta \bar{f}_x$ and $\Delta \bar{f}_y$ by putting

$$\Delta \bar{f}_x = T = \tau \sin lx, \qquad \Delta \bar{f}_y = -p - S_{22} = q \cos lx. \tag{5.12}$$

The surface displacements u and v are sinusoidal of the form (4.4). The relations between τ, q and U, V were derived in the more general case of a plate of thickness h oscillating at a frequency α [19, 20]. By putting $h = \infty$ and $\alpha = 0$ in these more general results we derive the solution for the static problem of the halfspace. The required relations are

$$\tau = (a_{11}U + a_{12}V)lL, \qquad q = (a_{12}U + a_{22}V)lL \tag{5.13}$$

where

$$a_{11} = [B_{11}(\beta_1 + \beta_2)]/(B_{11} + L\beta_1\beta_2), \qquad a_{22} = [B_{22}(\beta_1 + \beta_2)\beta_1\beta_2]/(B_{11} + L\beta_1\beta_2),$$
$$a_{12} = (B_{21}\beta_1\beta_2 - B_{11})/(B_{11} + L\beta_1\beta_2). \tag{5.14}$$

The quantities β_1 and β_2 are given in terms of m and k by the same expressions as (4.7); however, the values of $2m$ and k are now

$$2m = \frac{1}{LB_{22}}[B_{11}B_{22} - L(2B_{21} + P) - B_{21}^2], \qquad k = \left[\frac{B_{11}}{B_{22}}\left(1 - \frac{P}{L}\right)\right]^{1/2}. \tag{5.15}$$

In the plate theory the thickness h appears only in two parameters, $z_1 = \beta_1 \tanh(\frac{1}{2}\beta_1 lh)$ and $z_2 = \beta_2 \tanh(\frac{1}{2}\beta_2 lh)$ (see page 325 of the author's book [20], also [18]). For the halfspace $h = \infty$ we obtain $z_1 = \beta_1$, $z_2 = \beta_2$. Note also that expressions (5.14) for the coefficients are obtained by cancelling the common factor $\beta_1 - \beta_2$ in the numerator and denominator which appears in the limiting values after substituting $h = \infty$ in the more general plate theory.

In order to analyze crack propagation we proceed exactly as in the preceding section. Consider first the case $\tau = 0$, we find

$$q = K_1 lV \tag{5.16}$$

with

$$K_1 = ((a_{11}a_{22} - a_{12}^2)/a_{11})L. \tag{5.17}$$

Next we consider the case $q = 0$. We find

$$\tau = K_2 lU \tag{5.18}$$

with

$$K_2 = ((a_{11}a_{22} - a_{12}^2)/a_{22})L. \tag{5.19}$$

Introducing expressions (5.14) for a_{ij}, we derive

$$K_1 = \frac{2(m + k)kB_{11}B_{22} - (B_{21}k - B_{11})^2}{(2(m + k))^{1/2}B_{11}(B_{11} + Lk)}, \qquad K_2 = K_1/k. \tag{5.20}$$

We apply these results to crack propagation in a medium with the initial principal stresses S_{11}, S_{22}, S_{33}. In addition we assume the presence of an initial shear stress S_{12}. However, the latter is considered as an incremental perturbation which does not modify the values of the coefficients in the stress-strain relations (5.2). The crack is produced by injection of a fluid under a uniform pressure p. This amounts to applying constant values

$$\Delta\bar{f}_x = T = -S_{12}, \qquad \Delta\bar{f}_y = -p - S_{22}. \tag{5.21}$$

The first equation expresses the fact that cancellation of the tangential initial stress S_{12} at the faces of the crack amounts to applying a constant tangential traction $T = S_{12}$.

In order to express energy balance we remember that we may consider the system composed of the solid and the adjacent fluid as a system initially in equilibrium. The fluid is initially at the pressure $-S_{22}$ and the solid is under the initial stress S_{11}, S_{22}, S_{33}, S_{12}. The additional tractions (5.21) are then applied at the interface. The additional potential energy generated in the *solid-fluid system* by the tractions (5.21) is

$$W' = \tfrac{1}{2}(p + S_{22})\mathcal{V} - \tfrac{1}{2}S_{12}\mathcal{U}, \tag{5.22}$$

where $\mathcal{V}$ is the volume of the crack and $\mathcal{U}$ the total integrated slip expressed by Eqs. (4.20) and (4.21) after introducing new values (5.20) for K_1 and K_2. The second term in Eq. (5.22) is the loss of potential energy due to the cancellation of S_{12}. The energy balance equation is now

$$(p + S_{22})(\partial\mathcal{V}/\partial c) - (\partial W'/\partial c) = 2\mathcal{E}_{ns}. \tag{5.23}$$

We derive the crack propagation condition

$$(p + S_{22})^2 + kS_{12}^2 = (2\mathcal{E}_{ns}K_1/\pi c) \tag{5.24}$$

where k and K_1 are given by Eqs. (5.20). The only difference in the condition (4.23) derived previously lies in the values of k and K_1.

The width and slip distribution of the crack with an arbitrary increment load distribution $p(x) + S_{22}$ and $T(x)$ are given by an expression similar to Eq. (2.33) with suitable coefficients (5.20) replacing K.

6. Discussion of some special cases. We shall now discuss some particular applications which are based on the more general theory of incremental deformations as developed in the preceding section. For example we shall consider the case of a material isotropic for finite deformations and that of an orthotropic incompressible material. The latter case brings out more simply certain qualitative properties due to the initial stress which are related to the phenomenon of surface instability.

Material with finite isotropy. The general theory is applicable to this case and the incremental elastic coefficients appearing in Eqs. (5.2) may be evaluated quite simply from the finite stress-strain relations. The following results were derived in earlier work (see the author's book [20, p. 332] and [17]). An isotropic material is strained along principal directions x, y, z with finite extension ratios $\lambda_1, \lambda_2, \lambda_3$. Because of the property of isotropy the corresponding principal stresses are expressed by a single function $F(\lambda_1, \lambda_2, \lambda_3)$. They are

$$S_{11} = F(\lambda_1, \lambda_2, \lambda_3), \qquad S_{22} = F(\lambda_2, \lambda_3, \lambda_1), \qquad S_{33} = F(\lambda_3, \lambda_1, \lambda_2). \tag{6.1}$$

Isotropy requires that the function F must satisfy the identity

$$F(\lambda_1, \lambda_2, \lambda_3) = F(\lambda_1, \lambda_3, \lambda_2). \tag{6.2}$$

The incremental elastic coefficients are then given by

$$\begin{aligned} B_{11} &= \lambda_1(\partial S_{11}/\partial\lambda_1), & B_{12} &= \lambda_2(\partial S_{11}/\partial\lambda_2) \\ B_{21} &= \lambda_1(\partial S_{22}/\partial\lambda_1), & B_{22} &= \lambda_2(\partial S_{22}/\partial\lambda_2). \end{aligned} \tag{6.3}$$

Relation (5.3) between B_{12} and B_{21}, which is a consequence of the existence of an elastic potential, imposes an additional condition on the function F. It must satisfy the relation

$$(\partial/\partial\lambda_2)(\lambda_2 S_{11}) = (\partial/\partial\lambda_1)(\lambda_1 S_{22}). \tag{6.4}$$

As for the slide modulus L, we have shown (see [17] or the author's book [20, p. 93]) that its value is

$$L = (S_{11} - S_{22})(\lambda_2^2/(\lambda_1^2 - \lambda_2^2)). \tag{6.5}$$

With these results it is possible to evaluate the coefficients (5.20), and hence to obtain the crack propagation condition (5.24) along one of the principal directions in terms of a given state of finite initial strain. As already pointed out, an initial shear stress S_{12} may be introduced without modifying the values of the coefficients.

Orthotropic incompressible material. We shall discuss the case of an incompressible medium, orthotropic along x, y, z with principal initial stresses S_{11}, S_{22}, S_{33} along the same directions. It was shown [16], [17] (see also the author's book [20, p. 101]) that in this case the incremental stress-strain relations become

$$s_{11} - s = 2Ne_{xx}, \qquad s_{22} - s = 2Ne_{yy}, \qquad s_{12} = 2Qe_{xy}, \tag{6.6}$$

with the condition $e_{xx} + e_{yy} = 0$. Relations (5.13) retain the same form, but the coefficients a_{ij} determined by Eq. (5.14) are considerably simplified. They become [16, 20]

$$a_{11} = (2(m + k))^{1/2}, \qquad a_{12} = k - 1, \qquad a_{22} = k(2(m + k))^{1/2}, \tag{6.7}$$

with

$$\begin{aligned} m &= (2M/L) - 1, \qquad & M &= N + \tfrac{1}{4}P, \\ k &= (1 - (P/L))^{1/2}, \qquad & L &= Q + \tfrac{1}{2}P. \end{aligned} \tag{6.8}$$

By introducing expressions (6.7) into the values (5.17) and (5.19) we obtain

$$K_1 = 2\psi(ML)^{1/2}, \qquad K_2 = K_1/k, \tag{6.9}$$

where

$$\psi = \frac{2k(m + 1) + k^2 - 1}{(2(m + k))^{1/2}}. \tag{6.10}$$

Surface instability and crack propagation. According to the basic equation (5.22), a decrease in the value of K_1 corresponds to a smaller value of the fluid pressure p required for crack propagation. On the other hand an initial state of stress, for which $K_1 = 0$, represents a surface instability of the half space. For the incompressible material Eq. (6.9) shows that surface instability occurs if

$$2k(m + 1) + k^2 - 1 = 0. \tag{6.11}$$

Eq. (6.11) and the corresponding phenomenon of surface instability were discussed extensively in earlier work ([16], see also the author's book [20, p. 204]). It was shown that it occurs under a critical value of the compressive stress $P = -S_{11}$ active in a direction parallel to the crack. Hence a compressive stress in this direction lowers the value of K_1 and therefore tends to weaken the crack. On the other hand, a tensile stress along the same direction increases the value of K_1 and tends to strengthen the crack. Note that in the case of a triaxial initial stress the same conclusion holds provided the compressive stress is replaced by $P = S_{22} - S_{11}$. The stress S_{22} acting initially on the surface is obtained physically by applying a fluid pressure $-S_{22}$. It is interesting to compare two different cases. In the first the crack propagation is entirely due to the injection of a fluid with excess pressure $p + S_{22}$ while $S_{12} = 0$. The crack propagation condition (5.22) becomes

$$(p + S_{22})^2 = 2\mathcal{E}_{ns}K_1/\pi c. \tag{6.12}$$

As we approach surface instability K_1 tends to vanish and the excess pressure $p + S_{22}$ required for crack propagation tends to zero.

On the other hand, in the presence of an initial shear stress S_{12} and an excess pressure maintained at zero value ($p + S_{22} = 0$) the stress propagation condition (5.22) becomes

$$kS_{12}^2 = 2\mathcal{E}_{ns}K_1/\pi c. \tag{6.13}$$

A study of surface instability [16], [20] shows that when it is approached (hence when K_1 tends to zero) the value of k becomes small. However, the value $K_1 = 0$ is obtained before k vanishes. Since k and K_1 both diminish at the same time, Eq. (6.13) shows that the effect of initial stress on crack propagation is much less pronounced for this case. The same conclusions regarding the qualitative influence of initial stress on crack strength remain valid for the more general case of a compressible material. According to Eq. (5.20) the condition $K_1 = 0$ for surface instability is

$$2(m + k)kB_{11}B_{22} - (B_{21}k - B_{11})^2 = 0. \tag{6.14}$$

When this condition is approached the crack is weakened, while it is strengthened in the opposite direction. Eq. (6.14) for surface instability is equivalent to the result derived in the context of dynamics for surface waves in a halfspace with initial stress [19] (see also the author's book [20, p. 334]). Surface instability is derived for zero value of the frequency.

7. Crack propagation between dissimilar media. We shall analyze the case of two adjacent orthotropic media with directions of elastic symmetry along x, y, z. The lower medium occupies the halfspace $y < 0$. The upper medium occupies the halfspace $y > 0$. There is complete adherence at the interface $y = 0$. Principal initial stresses S_{11}, S_{22}, S_{33} along x, y, z are present in the lower medium. In the upper medium similar initial stresses S'_{11}, S_{22}, S'_{33} are also present. For reasons of equilibrium the component S_{22} normal to the interface is the same in both media, but S_{11} and S_{33} may be different from S'_{11} and S'_{33}. The problem of crack propagation will be analyzed by applying the more accurate theory of incremental deformations of Sec. 5.

Consider first the lower halfspace. The incremental stress-strain relations (5.2) are applicable. As already explained in Sec. 5 we may reason on a physical model composed of this halfspace and fluid at the pressure $-S_{22}$ occupying the other halfspace. Additional tractions $\Delta\bar{f}_x$ and $\Delta\bar{f}_y$ are then applied to the lower halfspace at the interface. We assume a sinusoidal distribution and write as before

$$\Delta\bar{f}_x = \tau \sin lx, \qquad \Delta\bar{f}_y = q \cos lx. \tag{7.1}$$

The corresponding displacements at $y = 0$ are

$$u = U \sin lx, \qquad v = V \cos lx. \tag{7.2}$$

Eqs. (5.13) are valid, i.e.

$$\tau = (a_{11}U + a_{12}V)lL, \qquad q = (a_{12}U + a_{22}V)lL, \tag{7.3}$$

where a_{ij} are given by expressions (5.14).

Similarly, equal and opposite surface tractions $-\Delta\bar{f}_x$ and $-\Delta\bar{f}_y$ are applied at a solid-fluid interface of the upper medium. The displacements at the interface are now

$$u' = U' \sin lx, \qquad v' = V' \cos lx \tag{7.4}$$

with

$$\tau = (-a'_{11}U' + a'_{12}V')lL', \qquad q = (a'_{12}U' - a'_{22}V')lL'. \tag{7.5}$$

These equations are the same as derived earlier ([16] [18] [19] [20]). The coefficients L' and a'_{ij} are given by the same expressions (5.11) and (5.14) where the elastic coefficients are replaced by those of the upper medium and P is replaced by

$$P' = S_{22} - S'_{11}. \tag{7.6}$$

We now invert Eqs. (7.3) and (7.5). They become

$$UlL = A_{11}\tau + A_{12}q, \qquad VlL = A_{12}\tau + A_{22}q \tag{7.7}$$

and

$$U'lL' = -A'_{11}\tau + A'_{12}q, \qquad V'lL' = A'_{12}\tau - A'_{22}q. \tag{7.8}$$

We derive

$$\begin{aligned} \bar{U}l &= \left(\frac{A_{11}}{L} + \frac{A'_{11}}{L'}\right)\tau + \left(\frac{A_{12}}{L} - \frac{A'_{12}}{L'}\right)q, \\ \bar{V}l &= \left(\frac{A_{12}}{L} - \frac{A'_{12}}{L'}\right)\tau + \left(\frac{A_{22}}{L} + \frac{A'_{22}}{L'}\right)q, \end{aligned} \tag{7.9}$$

where

$$\bar{U} = U - U', \qquad \bar{V} = V - V'. \tag{7.10}$$

By inversion, Eqs. (7.9) are written

$$\tau = (D_{11}\bar{U} + D_{12}\bar{V})l, \qquad q = (D_{12}\bar{U} + D_{22}\bar{V})l \tag{7.11}$$

By taking into account expressions (7.1), (7.2), (7.4) and (7.10), Eqs. (7.11) may be written

$$\begin{aligned} \Delta\bar{f}_x &= lD_{11}\bar{U}\sin lx + lD_{12}\bar{V}\sin lx, \\ \Delta\bar{f}_y &= lD_{12}\bar{U}\cos lx + lD_{22}\bar{V}\cos lx. \end{aligned} \tag{7.12}$$

A similar set of equations is obtained by shifting the origin of x replacing lx by $lx - (\pi/2)$. This amounts to the substitution for $\sin lx$ and $\cos lx$ respectively of $-\cos lx$ and $\sin lx$. Hence

$$\begin{aligned} \Delta\bar{f}_x &= -lD_{11}\bar{U}_1\cos lx - lD_{12}\bar{V}_1\cos lx, \\ \Delta\bar{f}_y &= lD_{12}\bar{U}_1\sin lx + lD_{22}\bar{V}_1\sin lx. \end{aligned} \tag{7.13}$$

Adding expressions (7.12) and (7.13), we obtain

$$\begin{aligned} \Delta\bar{f}_x &= lD_{11}\bar{u}(l, x) - D_{12}(\partial/\partial x)\bar{v}(l, x), \\ \Delta\bar{f}_y &= D_{12}(\partial/\partial x)\bar{u}(l, x) + lD_{22}\bar{v}(l, x) \end{aligned} \tag{7.14}$$

where

$$\begin{aligned} \bar{u}(l, x) &= \bar{U}(l)\sin lx - \bar{U}_1(l)\cos lx, \\ \bar{v}(l, x) &= \bar{V}(l)\cos lx + \bar{V}_1(l)\sin lx, \end{aligned} \tag{7.15}$$

with $\bar{U}$, $\bar{U}_1$, $\bar{V}$, $\bar{V}_1$ denoting arbitrary functions of l. We now introduce the functions $\psi(x, y)$ and $\phi(x, y)$ defined by the Fourier integrals in the region $y > 0$:

$$\psi(x, y) = \int_0^\infty \bar{u}(l, x)e^{-ly}\, dl, \qquad \phi(x, y) = \int_0^\infty \bar{v}(l, x)e^{-ly}\, dl. \tag{7.16}$$

They vanish at $y = \infty$ and satisfy Laplace's equation

$$(\partial^2\psi/\partial x^2) + (\partial^2\psi/\partial y^2) = 0, \qquad (\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = 0. \tag{7.17}$$

According to Eqs. (7.16) the corresponding values of the displacement differences (at $y = 0$)

$$\bar{u} = u - u', \qquad \bar{v} = v - v', \tag{7.18}$$

are

$$\bar{u} = \psi(x, 0), \qquad \bar{v} = \phi(x, 0). \tag{7.19}$$

From Eqs. (7.14) we derive the corresponding values of $\Delta\bar{f}_x$ and $\Delta\bar{f}_y$:

$$\begin{aligned} \Delta\bar{f}_x &= -D_{11}(\partial\psi/\partial y) - D_{12}(\partial\phi/\partial x), \\ \Delta\bar{f}_y &= D_{12}(\partial\psi/\partial x) - D_{22}(\partial\phi/\partial y), \end{aligned} \tag{7.20}$$

where the values are taken at $y = 0$.

The problem of determining $\bar{u}$ and $\bar{v}$ for a given distribution of tractions $\Delta\bar{f}_x$ and $\Delta\bar{f}_y$ is solved if we can find the functions ψ and ϕ. This may be accomplished by introducing two analytic functions $Z_1(z)$ and $Z_2(z)$ of $z = x + iy$ and putting

$$\bar{u} = \psi = Z_1 + Z_1^*, \qquad \bar{v} = \phi = Z_2 + Z_2^*, \tag{7.21}$$

where the asterisk denotes the complex conjugate quantities. By introducing the complex derivatives

$$Z_1' = (dZ_1/dz), \qquad Z_2' = (dZ_2/dz), \tag{7.22}$$

Eqs. (7.20) may be written in the form

$$i\Delta\bar{f}_k = M_{kj}Z_j' - M_{kj}^*Z_j'^* \tag{7.23}$$

where $\Delta\bar{f}_x = \Delta\bar{f}_1$, $\Delta\bar{f}_y = \Delta\bar{f}_2$ and M_{kj} are the elements of the matrix

$$[M_{kj}] = \begin{bmatrix} D_{11} & -iD_{12} \\ iD_{12} & D_{22} \end{bmatrix}. \tag{7.24}$$

It is Hermitian, since

$$M_{kj} = M_{jk}^* . \tag{7.25}$$

The matrix is also positive definite. This can be shown by considering the sinusoidal displacements corresponding to Eqs. (7.12). The energy input into the system over one wavelength is

$$\frac{1}{2}\int_0^{2\pi/l} (\Delta\bar{f}_x\bar{U}\sin lx + \Delta\bar{f}_y\bar{V}\cos lx)\, dx = \frac{\pi}{2}(D_{11}\bar{U}^2 + 2D_{12}\bar{U}\bar{V} + D_{22}\bar{V}^2). \tag{7.26}$$

Note that this represents the energy input by the interfacial forces $\Delta\bar{f}_x$ and $\Delta\bar{f}_y$ into a solid-fluid system represented by the two solids and a fluid at the pressure $-S_{22}$ in between. We may assume that the fluid is connected to a large reservoir so that its pressure remains constant. If the magnitude of the initial stresses is below the critical value of surface instability of either the lower or upper medium, expression (7.26) is positive definite. Hence

$$D_{11} > 0 \; D_{22} > 0, \qquad D_{11}D_{22} - D_{12}^2 > 0. \tag{7.27}$$

Until now the functions Z_1 and Z_2 were defined in the upper halfspace $y > 0$. They may be extended to the total plane by analytical prolongation. The values in the upper halfspace at a point z are denoted by Z_i^+. In the lower halfspace at the symmetric point z^* we define Z_i as

$$Z_i^- = -Z_i^{+*}. \tag{7.28}$$

In addition, it is assumed that on the x axis in the range $|x| > c$ we have

$$Z_i^+ = Z_i^- . \tag{7.29}$$

This insures that the functions Z_i defined by Z_i^+ and Z_i^- are analytical throughout except along a cut $|x| < c$, $y = 0$, where they may be discontinuous. From (7.28) and (7.29) we derive

$$Z_i^+ + Z_i^{+*} = 0; \tag{7.30}$$

hence

$$\bar{u} = \bar{v} = 0 \quad \text{for} \quad |x| > c, \qquad y = 0. \tag{7.31}$$

This corresponds to the problem of a crack along $|x| < c$. From Eq. (7.28) we derive for $y = 0$

$$(\partial Z_i^-/\partial x) = -(\partial Z_i^{+*}/\partial x) = -(\partial Z_i^{+}*/\partial x) \tag{7.32}$$

or

$$Z_i'^* = -Z_i'^- . \tag{7.33}$$

Hence Eqs. (7.28) may be written

$$i\Delta\bar{f}_k = M_{ki}Z_i'^+ + M_{ki}^*Z_i'^- . \tag{7.34}$$

This equation may be simplified by diagonalizing the matrix $[M_{ki}]$. This requires the solution of the equations

$$M_{ki}^*\xi_i = \kappa M_{ki}\xi_i . \tag{7.35}$$

The characteristic equation is quadratic with two characteristic roots $\kappa = \kappa_1$ and $\kappa = \kappa_2$ and corresponding values $\xi_i = \xi_i^1$, $\xi_i = \xi_i^2$ of the characteristic vector. Since M_{ki} is positive definite the roots κ_1 and κ_2 are real and positive. A fundamental property of the characteristic solution is obtained by writing Eq. (7.35) in the form

$$(1/\kappa)M_{ki}\xi_i^* = M_{ki}^*\xi_i^* . \tag{7.36}$$

This shows that the two characteristic roots and the corresponding vectors satisfy the relations

$$\kappa_2 = 1/\kappa_1 , \qquad \xi_i^2 = \xi_i^{1*}. \tag{7.37}$$

The characteristic vectors also satisfy orthogonality relations

$$M_{kj}\xi_j^s\xi_k^{r*} = M_{kj}^*\xi_j^s\xi_k^{r*} = 0 \tag{7.38}$$

and may be normalized so as to satisfy the equations

$$M_{kj}\xi_j^\rho\xi_k^{\rho*} = 1 \quad \text{(no summation for } \rho). \tag{7.39}$$

From Eqs. (7.35) and (7.39) we derive

$$M_{kj}^*\xi_j^\rho\xi_k^{\rho*} = \kappa_\rho \quad \text{(no summation for } \rho). \tag{7.40}$$

We now introduce the transformation

$$Z_j' = \xi_j^s\mathfrak{z}_s \tag{7.41}$$

with two new holomorphic functions $\mathfrak{z}_1$ and $\mathfrak{z}_2$. With this substitution Eqs. (7.34) lead to

$$M_{kj}\xi_j^s\xi_k^{r*}\mathfrak{z}_s^+ + M_{kj}\xi_j^s\xi_k^{r*}\mathfrak{z}_s^- = i\xi_k^{r*}\Delta\bar{f}_k. \tag{7.42}$$

Taking into account the orthogonality property (7.38), the normalization (7.39) and Eq. (7.40) we obtain

$$\mathfrak{z}_1^+ + \kappa_1\mathfrak{z}_1^- = i\xi_k^{1*}\Delta\bar{f}_k\,, \qquad \mathfrak{z}_2^+ + \kappa_2\mathfrak{z}_2^- = i\xi_k^{2*}\Delta\bar{f}_k\,. \tag{7.43}$$

The values of κ_1 and κ_2 may be written explicitly by solving the characteristic equation. We find

$$\kappa_1 = 1/\kappa_2 = (1 + \alpha)/(1 - \alpha), \qquad \alpha = |D_{12}|/(D_{11}D_{22})^{1/2}. \tag{7.44}$$

According to the inequalities (7.27) we have $0 < \alpha < 1$. Hence $\kappa_1 > 1$. Solving Eqs. (7.43) is a classical Hilbert problem. The solution is

$$\begin{aligned}\mathfrak{z}_1 &= \frac{X_1(z)}{2\pi}\int_{-c}^{+c}\frac{\xi_k^{1*}\Delta\bar{f}_k(t)}{X_1^+(t)(t - z)}\,dt,\\ \mathfrak{z}_2 &= \frac{X_2(z)}{2\pi}\int_{-c}^{+c}\frac{\xi_k^{2*}\Delta\bar{f}_k(t)}{X_2^+(t)(t - z)}\,dt,\end{aligned} \tag{7.45}$$

with $X_1(z)$ and $X_2(z)$, holomorphic functions except on the cut ($y = 0$, $|x| < c$), where they satisfy the condition

$$-\kappa_1 = X_1^+/X_1^-\,, \qquad -\kappa_2 = -(1/\kappa_1) = X_2^+/X_2^-\,. \tag{7.46}$$

Since Z_1 and Z_2 vanish as $1/z$ at infinity, $\mathfrak{z}_1$ and $\mathfrak{z}_2$ vanish as $1/z^2$. Therefore the functions X_1 and X_2 must be chosen to vanish as $1/z$. As follows from the remark in the subsequent paragraph, this insures uniqueness for the solutions ψ and ϕ. The required values of X_1 and X_2 are

$$\begin{aligned}X_1 &= (z - c)^{-1/2 - i\gamma}(z + c)^{-1/2 + i\gamma}\\ X_2 &= (z - c)^{-1/2 + i\gamma}(z + c)^{-1/2 - i\gamma},\end{aligned} \tag{7.47}$$

with

$$\gamma = (1/2\pi)\log\kappa_1\,. \tag{7.48}$$

A particular branch of the functions must be chosen defined by values

$$z - c = r_1 \exp(i\theta_1), \qquad z + c = r_2 \exp(i\theta_2) \tag{7.49}$$

where θ_1 and θ_2 are between zero and 2π. This insures that X_1 and X_2 behave like $1/z$ at infinity.

The values (7.45) and Eqs. (7.41) determine Z_1 and Z_2 by integration. In principle we may therefore determine the shape of the crack for constant values of $\Delta\bar{f}_x$ $\Delta\bar{f}_y$. By the same reasoning as previously we may derive a crack propagation condition of the type (5.24). However, in this case attention should be called to the following remarks. First, we note that the values (7.47) contain a factor of the type

$$(z - c)^{-i\gamma}(z + c)^{i\gamma} = \exp[i\gamma \log(r_2/r_1) + \gamma(\theta_1 - \theta_2)]. \tag{7.50}$$

Near the crack tip, i.e. near the points $z = c$, $z = -c$ this factor oscillates violently. As already shown by Erdogan [11], this behavior also occurs for the case of two dissimilar isotropic media and involves interpenetration of the two faces of the crack. However, in practice it may be disregarded, since it occurs in an extremely small region where the linear theory breaks down.

A second remark concerns another type of interpenetration. If a constant distribution of tangential forces $\Delta\bar{f}_x$ is applied to the faces, the normal displacement of the faces produces a s-shaped curve with an inflection at $x = 0$. However, the amplitudes are not the same if the materials are dissimilar so that there is interpenetration of the two faces. This will not occur, however, if simultaneously a normal force $\Delta\bar{f}_y$ of sufficient magnitude is also applied.

Uniqueness of solution. The ψ and ϕ as defined by expressions (7.16) vanish at infinity in the halfplane $y > 0$. The argument developed in the last paragraph of Sec. 2 is applicable here and shows that ψ and ϕ vanish as $1/z$. This also implies uniqueness under the conditions $\psi(x, 0) = \phi(x, 0) = 0$ for $|x| > c$ while $\Delta\bar{f}_x$ and $\Delta\bar{f}_y$ of Eqs. (7.20) are given functions of x for $|x| < c$. To show this, consider two functions ψ and two functions ϕ satisfying the same boundary conditions, and denote the differences of these functions by ψ_d and ϕ_d. They satisfy Eqs. (7.20) with $\Delta\bar{f}_x = \Delta\bar{f}_y = 0$. Hence for $|x| < c$ on the x axis we derive

$$D_{11}\psi_d \frac{\partial \psi_d}{\partial y} + D_{12}\left(\psi_d \frac{\partial \phi_d}{\partial x} - \phi_d \frac{\partial \psi_d}{\partial x}\right) + D_{22}\phi_d \frac{\partial \phi_d}{\partial y} = 0. \tag{7.51}$$

On the other hand, applying Green's and Stokes' theorems we may write

$$\iint_A (F_1 + F_2)\, dx\, dy = \int_C F_3\, ds, \tag{7.52}$$

where

$$\begin{aligned} F_1 &= D_{11}\left(\frac{\partial \psi_d}{\partial y}\right)^2 + 2D_{12}\frac{\partial \psi_d}{\partial y}\frac{\partial \phi_d}{\partial x} + D_{22}\left(\frac{\partial \phi_d}{\partial x}\right)^2, \\ F_2 &= D_{11}\left(\frac{\partial \psi_d}{\partial x}\right)^2 - 2D_{12}\frac{\partial \psi_d}{\partial x}\frac{\partial \phi_d}{\partial y} + D_{22}\left(\frac{\partial \phi_d}{\partial y}\right)^2, \\ F_3 &= D_{11}\psi_d \frac{\partial \psi_d}{\partial n} + D_{22}\phi_d \frac{\partial \phi_d}{\partial n} - D_{12}\left(\psi_d \frac{\partial \phi_d}{\partial s} - \phi_d \frac{\partial \psi_d}{\partial s}\right). \end{aligned} \tag{7.53}$$

The counterclockwise contour C is composed of the x axis and an infinite half circle in the halfplane $y > 0$ centered at the origin. The domain A lies inside this contour. The derivatives along the outward normal at C and along the arc s are denoted by $\partial/\partial n$ and $\partial/\partial s$. The value of the contour integral is zero. This follows from the fact that ψ_d and ϕ_d vanish as $1/z$ on the half circle and in addition on the x axis $\phi_d = \psi_d = 0$ for $|x| > c$, while for $|x| < c$ the integrand vanishes according to Eq. (7.51). Hence the surface integral of Eq. (7.52) also vanishes. Because of the inequalities (7.52) the quadratic forms F_1 and F_2 are positive-definite. Therefore the functions ψ_d and ϕ_d must be constants and, in addition, equal to zero, since they vanish at infinity. We have thus shown that under the assumed boundary conditions any two solutions must be identical.

Simplified cases. It can be seen that considerable formal complications arise if $\gamma \neq 0$ in expressions (7.47). This is due to the coupling term D_{12} in Eqs. (7.20). If the materials are such that

$$A_{12}/L = A'_{12}/L', \tag{7.54}$$

then $D_{12} = 0$ and $\gamma = 0$. In this case the oscillatory behavior disappears. From Eqs. (7.9) and (7.11) we also write

$$\begin{aligned} 1/D_{11} &= (A_{11}/L) + (A'_{11}/L'), \\ 1/D_{22} &= (A_{22}/L) + (A'_{22}/L'). \end{aligned} \tag{7.55}$$

Under these conditions the integrands in Eqs. (7.45) are identical to expression (2.19) except for the coefficient K and the difference in sign, which is due to the negative value (2.18) of σ_{yy}. In practice it may also happen that Eq. (7.54) is approximately valid. The solution is thus drastically simplified and becomes the same as for an homogeneous medium, with the substitution of the values (7.55) of D_{11} and D_{22} in place of $\frac{1}{2}K_2$ and $\frac{1}{2}K_1$. Condition (5.24) for crack propagation becomes in the present case

$$(p + S_{22})^2/2D_{22} + S_{12}^2/2D_{11} = 2\mathcal{E}_{ns}/\pi c. \tag{7.56}$$

There is a case where the condition $D_{12} = 0$ is always rigorously verified. This is for two orthotropic *incompressible* materials if the initial stress in each material satisfies the conditions

$$P = S_{22} - S_{11} = 0, \qquad P' = S_{22} - S'_{11} = 0. \tag{7.57}$$

According to Eqs. (6.7) and (6.8) it follows that $a_{12} = a'_{12} = 0$; hence, also, $D_{12} = 0$. We derive

$$\frac{1}{D_{11}} = \frac{1}{D_{22}} = \frac{1}{2(NQ)^{1/2}} + \frac{1}{2(N'Q')^{1/2}} \tag{7.58}$$

where N, Q and N', Q' are the coefficients in the incremental stress-strain relations (6.6) for each of the two materials. The crack propagation condition is obtained by substituting the values (7.58) in Eq. (7.56). Conditions (7.57) are also verified for zero values of the initial stresses or if we neglect the effect of initial stress on incremental deformations. The problem is then similar to the one treated in Sec. 4. The drastic simplification of the theory in this case is worth noting.

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