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Lagrangian analysis of multiple scatter in acoustic and electromagnetic reflexion

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Abstract. — A new method is developed for the analysis of multiple scatter in the reflection of acoustic waves from a rough surface. It is based on Lagrangian equations derived from Hamilton's principle and applied to a layer of fluid adjacent to the rough surface. The reflection is then considered to occur on this fictitions fluid layer. It is characterized by a second order two-dimensional symmetric tensor which embodies the distribution of the scattening elements. The procedure yields the coherent field as a first approximation under the assumption of large wavelengths. The energy of the incoherent scatter may then be evaluated as a second approximation by classical procedures. Results are more general than those derived from integral equations which express the interaction of scattering centers. Quantitative agreement is obtained by comparing the two methods. Approximate validity of the theory is also discussed for wavelengths which are not « large » relative to the size of the roughness. The theory is developed in the context of acoustic waves but the same procedure is applicable to the reflection of electromagnetic waves on a rough perfectly conducting surface

1. Introduction

The problem of multiple scatter in wave reflection from a rough surface has been the subject of numerous studies. Because of intrinsic analytical difficulties there is no straightforward solution available. Essentially a wave incident upon a rough surface induces a large number of scattered wavelets. Each protuberance acts as a scattering center. However these scattered waves all interact with each other, since each scattered wavelet induces in turn additional scattered waves at the other scattering centers. Hence the problem is expressed by integral equations.

The interaction between scattering centers is of two types. One is non-radiative and corresponds to a close-range quasistatic interaction. The other which is a long range radiative one, is cumulative and is the source of one of the main difficulties in the analysis.

The author has shown that for the limiting case of large wavelenghts the effect of the multiple scatter is obtained by simply introducing a wave boundary condition at a fictitious plane reflecting surface. This was done in the context of electromagnetic waves [1, 3] and acoustic waves [2, 4, 5].

The purpose here is to show that this boundary condition for multiple scatter may be obtained and further extended by using Hamilton's principle and Lagrangian dynamics, thus by-passing completely the integral equation formulation. The analysis is carried out in the context of acoustic reflection but the same procedure is applicable to electromagnetic waves by using the Lagrangian of the electromagnetic field. The principle of the method is to consider the reflection to occur at the plane boundary of a fictitious fluid layer of small thickness which is adjacent to the rough surface. The roughness size is assumed sufficiently small relative to the layer thickness. The dynamic properties of the layer are then determined by Lagrangian analysis. These properties are different from those of the fluid because of the interactions of the fluid and the rough surface through what is generally called the apparent mass effect. This refers to the apparent increase of mass of an immersed solid accelerated relative to the fluid. Conversely a fluid moving relative to a solid also displays an apparent mass.

The result thus obtained agrees entirely with the earlier ones [5] based on the integral equation formulation, thus showing that the apparent mass effect represents an essential feature of multiple scatter. In addition the equations obtained are more general and establish rigorously the fact that the reflection properties are represented by a second order two-dimensional symmetric tensor. This is valid for any arbitrary shape of roughness.

The Lagragian equations for the layer dynamics are established in section 2 and the basic boundary condition for the reflection is derived. Fundamental properties already described earlier such as the grazing incidence phase reversal are briefly discussed in section 3.

The foregoing analysis deals only with the evaluation of the coherent portion of the reflected wave. As pointed out in section 4, the incoherent scatter in all directions may be evaluated directly by classical methods as a second approximation once the coherent reflection has been evaluated. The conclusion follows that the incoherent reflection tends to dissapear at grazing incidence. In fact it is shown how the present Lagrangian analysis should remain approximately valid in a range of wavelengths which are not physically large relative to the scattering centers, in which case the energy in the incoherent part of the reflected wave is not negligible.

2. Lagragian equations for coherent scatter

We consider the problem of acoustic reflection on a rough surface. The roughness is represented by small protuberances located on top of the x, y plane. They are arbitrary except for the fact that they are small relative to the wavelength. The half-space countains a compressible fluid in which acoustic waves may propagate. The equation of propagation of acoustic waves is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \tag{2.1}$$

where

$$c = \sqrt{\frac{\lambda}{\rho}}$$

is the sound velocity, λ is the fluid bulk modulus and ρ its specific mass. The gradient of the scalar ϕ represents the fluid displacement. For periodic waves of circular frequency ω, the scalar ϕ contains the time factor exp $(i\omega t)$. Omitting this factor, equation (2.1) is written

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + k^2 \phi = 0 \tag{2.2}$$

where $k = \omega/c$ is the wave number.

The essence of the procedure is to distinguish two parts of the fluid separated by the fictitious plane boundary

$$z = h \tag{2.3}$$

Hence one part of the fluid is a layer of thickness h, countaining the surface roughness, while the other is in the remaining half space. The problem of reflection of acoustic waves on the rough surface will be analysed by considering the reflection to occur on the boundary $z = h$ of the fluid layer instead of the solid and to consider the layer as having its own reflecting properties when the roughness is taken into account. The reflecting properties of the layer are then obtained by deriving the Lagrangian dynamics of the layer. The layer thickness h is assumed to be small relative to the wavelength of the acoustic waves but at the same time of sufficient magnitude so that the microvelocity disturbance due to the roughness is negligible at the boundary $z = h$. Hence at this boundary the fluid displacement is approximately parallel to the x, y plane. The x and y components of this displacement are denote by U_x and U_y and the normal displacement along z at $z = h$ is denote by w. The component w is small with respect to U_x and U_y. The velocity field in the layer is composed of two parts. The first part is represented by the field

$$\dot{U}_x = \frac{\partial U_x}{\partial t} \qquad \dot{U}_y = \frac{\partial U_y}{\partial t} \tag{2.4}$$

parallel to the x, y plane. The second part is a small scale velocity field v_x v_y v_z which represents a disturbance due to the roughness. There is of course an additional z component of the velocity proportional to $\dot{w} = \partial w/\partial t$. However it may be neglected in comparison with v_x v_y v_z. This is a consequence of the fact that the microvelocity disturbance is of the same order as $\dot{U}_x$ $\dot{U}_y$ while $\dot{w}$ is smaller of a higher order.

We shall consider a domain D of the fluid layer limited by the rough surface and the planes $z = h$, $x = x_1$, $x = x_2$, $y = y_1$, $y = y_2$. The size of the domain is assumed small relative to the wavelength so that the undisturbed velocity U_x U_y may be assumed constant within this domain. The actual velocity distribution is obtained by adding a small scale velocity field v_x v_y v_z due to the geometry of the roughness. In the subsequent derivation we shall make use of an important property of this field which may be described as follows.

First it should be pointed out that when an acoustic wave is incident upon a fixed scattering center of small size relative to the wavelength, the velocity field induced by the scattering solid in its immediate neighborhood is the same as if the solid were immersed in a perfect incompressible fluid at rest, and moving relative to the fluid with a velocity equal and opposite that of the unperturbed incident acoustic wave. The same property is of course applicable to the rough surface. Hence the small scale velocity field v_x v_y v_z is given by the potential flow in a perfect incompressible fluid at rest when the rough surface moves with a velocity of components $-\dot{U}_x$ and $-\dot{U}_y$. We conclude that the velocity disturbance due to the roughness is expressed by the following linear functions of $\dot{U}_x$ and $\dot{U}_y$.

$$\begin{aligned} v_x &= \alpha_1 \dot{U}_x + \beta_1 \dot{U}_y \\ v_y &= \alpha_2 \dot{U}_x + \beta_2 \dot{U}_y \\ v_z &= \alpha_3 \dot{U}_x + \beta_3 \dot{U}_y \end{aligned} \tag{2.5}$$

where α_i and β_i are functions of x, y, z determined by potential flow theory and by the geometry of the roughness.

The kinetic energy of the fluid in the domain D is

$$T = \frac{1}{2}\rho \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{h'}^{h} [(\dot{U}_x + v_x)^2 + (\dot{U}_y + v_y)^2 + v_z^2]\, dz \tag{2.6}$$

where

$$z = h'(x,y) \tag{2.7}$$

is the equation representing the rough surface.

The kinetic energy (2.6) may be written

$$T = \frac{1}{2}\rho(I_1 + I_2 + I_3) \tag{2.8}$$

where

$$\begin{aligned} I_1 &= (\dot{U}_x^2 + \dot{U}_y^2) \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{h'}^{h} dz \\ I_2 &= 2\dot{U}_x \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{h'}^{h} v_x dz \\ &\quad + 2\dot{U}_y \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{h'}^{h} v_y dz \end{aligned} \tag{2.9}$$

$$I_3 = \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{h'}^{h} (v_x^2 + v_y^2 + v_z^2) dz$$

The values of these integrals are found as follows

$$I_1 = S(h - \tau)(\dot{U}_x^2 + \dot{U}_y^2) \tag{2.10}$$

where S is the area of the domain D in the x, y plane

$$S = \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy$$

and τ is the average volume of the roughness per unit area as defined by

$$S\tau = \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{0}^{h} dz \tag{2.11}$$

In order to determine I_2, we take into account the incompressibility property of the small scale velocity field v_x v_y v_z. This leads to the following relation

$$\int_{y_1}^{y_2} dy \int_{h'}^{h} (\dot{U}_x + v_x) dz = \int_{y_1}^{y_2} dy \int_{0}^{h} \dot{U}_x dz \tag{2.12}$$

The left side integral is assumed to be evaluated at an abscissa x which constitutes the axis of a strip of width Δy on which the roughness is distributed while the remaining surface is flat and coincides with the x, y plane. The width Δy is assumed adequate to generate locally the correct velocity disturbance due to the protuberances of the surface and their interactions. The integral on the right side of equation (2.12) is evaluated on the plane surface $z = 0$ at a certain distance from the rough strip. Hence equation (2.12) is a consequence of the incompressibility of the flow. From equation (2.12) we derive

$$\int_{y_1}^{y_2} dy \int_{h'}^{h} v_x dz = \dot{U}_x \int_{y_1}^{y_2} dy \int_{0}^{h'} dz \tag{2.13}$$

Hence

$$\int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{h'}^{h} v_x dz = \dot{U}_x S\tau \tag{2.14}$$

A similar result is obtained replacing v_x and $\dot{U}_x$ by v_y and $\dot{U}_y$. Using these results we may write

$$I_2 = 2S\tau(\dot{U}_x^2 + \dot{U}_y^2) \tag{2.15}$$

The term I_3 is determined by substituting expressions (2.5) for v_x v_y v_z. The value obtained may be written in the form

$$I_3 = S\tau(\mu_{11}\dot{U}_x^2 + 2\mu_{12}\dot{U}_x\dot{U}_y + \mu_{22}\dot{U}_y^2) \tag{2.16}$$

The corresponding term in the kinetic energy (2.8) is

$$T_3 = \frac{1}{2}\rho I_3 = \frac{1}{2}\rho S\tau(\mu_{11}\dot{U}_x^2 + 2\mu_{12}\dot{U}_x\dot{U}_y + \mu_{22}\dot{U}_y^2) \tag{2.17}$$

This expression has a fundamental physical significance. We may choose principal directions for x and y such that $\mu_{12} = 0$. In this case the quantities $\rho\tau\mu_{11}$ and $\rho\tau\mu_{22}$ are the apparent masses of the rough surface due to the fluid, per unit area in the x, y plane and for motions respectively in the x and y directions. In general this apparent mass will be anisotropic (i.e. $\mu_{11} \neq \mu_{22}$).

With the foregoing values of I_1, I_2, I_3, the total kinetic energy (2.8) is

$$T = \frac{1}{2}\rho S h(a_{11}\dot{U}_x^2 + 2a_{12}\dot{U}_x\dot{U}_y + a_{22}\dot{U}_y^2) \tag{2.18}$$

with

$$\begin{aligned} a_{11} &= 1 + \frac{\tau}{h}(1+\mu_{11}) \\ a_{22} &= 1 + \frac{\tau}{h}(1+\mu_{22}) \\ a_{12} &= \frac{\tau}{h}\mu_{12} \end{aligned} \tag{2.19}$$

In order to obtain Lagrangian equations of motion of the fluid layer we must also evaluate the elastic potential energy V of the fluid domain D of the layer. The hydrostatic stress in the fluid is

$$\sigma = \lambda e \tag{2.20}$$

where e is the change of volume per unit volume.

Its evaluation is obtained by considering the rate of volume change of the fluid domain D. Its value is

$$\dot{\Delta} = \left[(x_2 - x_1)\frac{\partial \dot{F}_x}{\partial x} + (y_2 - y_1)\frac{\partial \dot{F}_y}{\partial y} + S\dot{w} \right] \tag{2.21}$$

where

$$\dot{F}_x = \int_{y_1}^{y_2} dy \int_{h'}^{h} (\dot{U}_x + v_x)dz = \dot{U}_x h(y_2 - y_1) \tag{2.22}$$

$$\dot{F}_y = \int_{x_1}^{x_2} dx \int_{h'}^{h} (\dot{U}_y + v_y)dz = \dot{U}_y h(x_2 - x_1) \tag{2.23}$$

The latter equations are based on relation (2.12) and the similar one for the y direction expressing *local* conservation of volume. Since $(h-\tau)$ S is the initial volume of the domain D, we derive

$$e = \frac{\Delta}{(h-\tau)S} = \frac{h}{h-\tau}\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{w}{h}\right) \tag{2.24}$$

Hence

$$\sigma = \frac{\lambda h}{h-\tau}\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{w}{h}\right) \tag{2.25}$$

In the evaluation of e we must take into account the slow variation of U_x and U_y with x and y. However, the values of e and σ may be considered as constant within the domain D. The potential energy of the fluid in this domain is

$$V = \frac{1}{2} S(h-\tau)\lambda e^2 \tag{2.26}$$

From equations (2.18) and (2.26) we derive the kinetic and potential energies of the layer per unit area of the reflecting surface. They are

$$T' = \frac{1}{2}\rho h(a_{11}\dot{U}^2 + 2a_{12}\dot{U}_x\dot{U}_y + a_{22}\dot{U}_y^2) \tag{2.27}$$

$$V' = \frac{1}{2}(h-\tau)\lambda e^2 = \frac{1}{2\lambda}(h-\tau)\sigma^2$$

Using these results we may now formulate Hamilton's principle for the fluid layer, in the form

$$\int dt \iint [\delta(\mathrm{T}'-\mathrm{V}') + \sigma'\delta w]\, dxdy = 0 \tag{2.28}$$

The integral is extended to an arbitrary area of the reflecting surface, and equation (2.28) expresses that it is stationary for arbitrary variations $\delta \mathrm{U}_x$ $\delta \mathrm{U}_y$ δw. The term $\sigma'\delta w$ is the virtual work of the traction exerted by the incident and reflected wave on the boundary $z = h$ of the layer. By the usual procedure of integration by parts used in the variational calculus the variational principle (2.28) yields the three equations

$$\begin{aligned} \rho(a_{11}\ddot{\mathrm{U}}_x + a_{12}\ddot{\mathrm{U}}_y) &= \frac{\partial \sigma}{\partial x} \\ \rho(a_{12}\ddot{\mathrm{U}}_x + a_{22}\ddot{\mathrm{U}}_y) &= \frac{\partial \sigma}{\partial y} \end{aligned} \tag{2.29}$$

$$\sigma = \sigma'$$

If we restrict the application to harmonic time dependence and omit the time factor $\exp(i\omega t)$ the first two equations (2.29) become

$$\begin{aligned} -\rho\omega^2(a_{11}\mathrm{U}_x + a_{12}\mathrm{U}_y) &= \frac{\partial \sigma}{\partial x} \\ -\rho\omega^2(a_{12}\mathrm{U}_x + a_{22}\mathrm{U}_y) &= \frac{\partial \sigma}{\partial y} \end{aligned} \tag{2.30}$$

In the inverse form these equations are written

$$\begin{aligned} -\rho\omega^2\mathrm{U}_x &= b_{11}\frac{\partial \sigma}{\partial x} + b_{12}\frac{\partial \sigma}{\partial y} \\ -\rho\omega^2\mathrm{U}_y &= b_{12}\frac{\partial \sigma}{\partial x} + b_{22}\frac{\partial \sigma}{\partial y} \end{aligned} \tag{2.31}$$

where

$$\begin{aligned} b_{11} &= a_{22}/(a_{11}a_{22} - a_{12}^2) \\ b_{22} &= a_{11}/(a_{11}a_{22} - a_{12}^2) \\ b_{12} &= -\,a_{12}/(a_{11}a_{22} - a_{12}^2) \end{aligned} \tag{2.32}$$

Introduction of the values U_x and U_y from equations (2.31) into expression (2.25) for σ yields

$$- k^2(h-\tau)\sigma = h\left(b_{11}\frac{\partial^2\sigma}{\partial x^2} + 2b_{12}\frac{\partial^2\sigma}{\partial x\partial y} + b_{22}\frac{\partial^2\sigma}{\partial y^2}\right) - \rho\omega^2 w \tag{2.33}$$

The hydrostatic stress in the fluid outside the layer is

$$\sigma' = \lambda\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) \tag{2.34}$$

Using the wave propagation equation (2.2) it becomes

$$\sigma' = - \lambda k^2\phi \tag{2.35}$$

According to the third of equations (2.29) $\sigma = \sigma'$, hence (*at* $z = h$)

$$\sigma = - \lambda k^2\phi \tag{2.36}$$

Since ϕ is a displacement potential, w is (*at* $z = h$)

$$w = \frac{\partial\phi}{\partial z} \tag{2.37}$$

By substituting the values (2.36) and (2.37) in equation (2.33) we derive

$$\frac{\partial\phi}{\partial z} + k^2(h-\tau)\phi = - h\left(b_{11}\frac{\partial^2\phi}{\partial x^2} + 2b_{12}\frac{\partial^2\phi}{\partial x\partial y} + b_{22}\frac{\partial^2\phi}{\partial y^2}\right) \tag{2.38}$$

This relation constitutes a boundary condition for the potential ϕ of the incident acoustic wave at the boundary $z = h$ of the fluid layer. It may be transformed into a boundary condition at $z = 0$ by taking into account the identity (2.2) which constitutes the propagation equation for ϕ. The value of $k^2\phi$ derived from this equation is introduced into the boundary condition (2.38). This yields, at $z = h$,

$$\frac{\partial\phi}{\partial z} - h\frac{\partial^2\phi}{\partial z^2} = \alpha_{11}\frac{\partial^2\phi}{\partial x^2} + 2\alpha_{12}\frac{\partial^2\phi}{\partial x\partial y} + \alpha_{22}\frac{\partial^2\phi}{\partial z^2} + \tau k^2\phi \tag{2.39}$$

where

$$\alpha_{11} = h(1-b_{11}) \qquad \alpha_{22} = h(1-b_{22}) \qquad \alpha_{12} = - hb_{12} \tag{2.40}$$

In first approximation considering τ/h to be a small quantity in expressions (2.32) we derive

$$\alpha_{11} = \tau(1+\mu_{11}) \qquad \alpha_{22} = \tau(1+\mu_{22}) \qquad \alpha_{12} = \tau\mu_{12} \tag{2.41}$$

On the other hand to the first order we may write

$$\left[\frac{\partial\phi}{\partial z} - h\frac{\partial^2\phi}{\partial z^2}\right]_{z=h} = \left[\frac{\partial\phi}{\partial z}\right]_{z=0} \tag{2.42}$$

Furthermore because $\partial\phi/\partial z$ is a small quantity, the value of ϕ at $z = 0$ may be considered the same as the value at $z = h$. Under these conditions the boundary condition (2.39) at $z = h$ may be replaced by the following one at $z = 0$,

$$\frac{\partial\phi}{\partial z} = \alpha_{11}\frac{\partial^2\phi}{\partial x^2} + 2\alpha_{12}\frac{\partial^2\phi}{\partial x\partial y} + \alpha_{22}\frac{\partial^2\phi}{\partial y^2} + \tau k^2\phi \tag{2.43}$$

Note that in this equation the parameter h has disappeared. Hence, as should be, the boundary condition at $z = 0$ turns out to be independent of the choice of the thickness h of the fictitious layer.

The boundary condition (2.43) may be further simplified by again taking into account equation (2.2) of acoustic propagation. The condition (2.43) may then be written

$$\frac{\partial\phi}{\partial z} + \tau\frac{\partial^2\phi}{\partial z^2} = \tau\left(\mu_{11}\frac{\partial^2\phi}{\partial x^2} + 2\mu_{12}\frac{\partial^2\phi}{\partial x\partial y} + \mu_{22}\frac{\partial^2\phi}{\partial y^2}\right) \tag{2.44}$$

The coherentreflective properties of the rough surface are thus characterized by the symmetric tensor μ_{ij} which depends solely on the geometry of the roughness and represents physically an *apparent mass effect* of the roughness in the surrounding fluid.

3. Discussion and comparison with other methods

Equation (2.44) constitutes a boundary condition which takes into account, the multiple scatter i.e. the mutual long range and short range interaction of the scattering centers upon each other. As can be seen this is equivalent to solving the integral equation for the mutual interaction. A theory based on such an integral equation was developed by the author in the context or both acoustic and electromagnetic waves [1-5]. It was shown that the solution of the integral equation leads to a boundary condition of the same type as (2.44). However by that method the symmetry of the tensor μ_{ij} is derived only for a particular case of roughness while the present derivation is quite general.

Attention should be called to the fact that the reflection condition (2.44) yields only the coherent part of the reflection which occurs here without loss of energy. Actually a small amount of energy is scattered in other directions by incoherent scatter. This incoherent scatter may be evaluated by classical procedures once the coherent field has been determined.

The coherent reflection was already discussed previously in the case of two-dimensional reflection. It is useful to recall some of the important properties of such reflection and verify at the same time that the present Lagrangian analysis yields the same result quantitatively as obtained from the earlier formulation based on integral equations.

An interesting feature is readily brought out by considering a plane wave at normal incidence. In that case ϕ depends only on z and the boundary condition (2.44) at $z = 0$ reduces to

$$\frac{\partial \phi}{\partial z} + \tau \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{3.1}$$

If τ is small the left side is the value of $\partial\phi/\partial z$ at $z = \tau$. Hence in this case the reflection is the same as on a plane surface without roughness located at $z = \tau$. In other words it is the same as if the volume of the rough material were spread smoothly on the reflecting surface.

Consider now the case where x is a principal direction of the roughness tensor μ_{ij}. The two-dimensional reflection in the x, z plane is now governed by the boundary condition

$$\frac{\partial \phi}{\partial z} + \tau \frac{\partial^2 \phi}{\partial z^2} = \tau \mu_{11} \frac{\partial^2 \phi}{\partial x^2} \tag{3.2}$$

Consider a plane incident wave

$$\phi_i = \mathrm{A} \exp (inz - ilx) \tag{3.3}$$

The wave number components are

$$n = k \cos \theta$$

$$l = k \sin \theta \tag{3.4}$$

where θ is the angle of incidence. The coherent reflected wave may be written

$$\phi_r = \mathrm{A} \exp (- inz - ilx + 2\psi i) \tag{3.5}$$

where 2ψ is a phase shift to be determined. The total acoustic field

$$\phi = \phi_i + \phi_r \tag{3.6}$$

satisfies the reflection condition (2.44). Substitution of ϕ into this condition yields the value of ψ. We find

$$\exp(2\psi i) = \frac{n - \tau(\mu_{11} l^2 - n^2)i}{n + \tau(\mu_{11} l^2 - n^2)i} \tag{3.7}$$

Hence

$$\tan\psi = -\tau\left(\mu_{11}\frac{l^2}{n} - n\right) \tag{3.8}$$

or

$$\tan\psi = -\tau\left(\mu_{11}\frac{k\sin^2\theta}{\cos\theta} - k\cos\theta\right) \tag{3.9}$$

For grazing incidence $\theta = \pi/2$ the phase angle tends to

$$2\psi = -\pi \tag{3.10}$$

This corresponds to a retardation with phase reversal. Thus a large effect due to multiple scatter occurs in this case *no matter how small the roughness*. In this case the amplitude of the total wave, incident plus reflected, tends to vanish at the reflecting surface. Physically this may be considered analogous to an antiresonance generated by the cumulative superposition of the scattered waves along the surface in the direction of propagation. This effect was already derived earlier by a different method [1-5].

It is important to show that the present result also agrees quantitatively with the previous analysis based on the integral equation formulation.

The agreement is readily brought out by considering a surface composed of hemispherical protuberances on a plane. All we need to do is to introduce the apparent mass of a hemispherical solid. For the spherical solid, it is well known that the apparent mass is half the mass of fluid displaced by the solid [6]. Because of the symmetry the same factor applies to the hemispherical bosses. Hence in this case $\mu_{11} = 1/2$. With this value equation (3.9) coincides quantitatively with the earlier result [5]. We have assumed here that the spacing between hemispherical bosses is such that the non radiative close range interaction is negligible. This interaction has been evaluated in the earlier analysis [2-5].

We also note that no phase change occurs ($\psi = 0$) for an angle of incidence given by

$$\tan\theta = \sqrt{\frac{1}{\mu_{11}}} \tag{3.10}$$

In this case the phase retardation due to the apparent mass effect compensates the phase advance illustrated by equation (3.1) due to the volume τ of the roughness.

The existence of a *surface wave* may also be derived from the boundary condition (2.44). The derivation will not be repeated here since this was analyzed and discussed in detail earlier [4-5].

When the plane of incidence is not in a principal direction of μ_{ij} the incident wave is respresented by

$$\phi_i = \mathrm{A} \exp(inz - imy - ilx) \tag{3.10}$$

where

$$k^2 = l^2 + m^2 + n^2$$

and $-l$, $-m$, n are proportional to the direction cosines of the incident ray in the reversed direction of propagation. The reflected wave is

$$\phi_r = \mathrm{A} \exp(-inz - imy - ilx + 2\psi i) \tag{3.11}$$

and contains a phase ψ to be determined. Substitution of $\phi = \phi_i + \phi_r$ in the boundary condition (2.44) yields the value

$$\tan\psi = -\tau\left[\frac{1}{n}(\mu_{11}l^2 + 2\mu_{12}lm + \mu_{22}m^2) - n\right] \tag{3.12}$$

which generalizes equation (3.8). Again at grazing incidence ($n \to 0$) the phase angle 2ψ tends to the value $-\pi$.

The evaluation of the parameters $\tau\mu_{ij}$ in the boundary condition (2.44) results from the determination of the apparent mass effect. In the principal directions, this apparent mass is represented by the mass of fluid of volumes $\tau\mu_{11}$ $\tau\mu_{22}$. They are not the same if the roughness is not statistically isotropic. For a certain number of geometric shapes such as hemispheres, half ellipsoids, for example, these apparent masses are available from the classical litterature. Note that the volume of the roughness may be zero $\tau = 0$, while

$\tau\mu_{ij}$ remains finite as for example in the case of half circular discs distributed normally to the reflecting surface. The apparent mass is zero in directions parallel to the discs, while in the normal direction it is equal to the mass of a fluid sphere of the same radius [6]. It was also pointed out earlier [5] that an isotropic shape of individual scattering protuberances such as hemispheres may lead to anisotropic reflecting properties if the distances between hemispheres in a given direction is small enough.

In the foregoing analysis we have assumed that the roughness is uniformely distributed statistically. If this is not the case the boundary condition (2.44) must be replaced by

$$\frac{\partial\phi}{\partial z} + \tau\frac{\partial^2\phi}{\partial z^2} = \sum^{ij} \frac{\partial}{\partial x_i}\left(\tau\mu_{ij}\frac{\partial\phi}{\partial x_j}\right) \tag{3.13}$$

where $x_1 = x$, $x_2 = y$, while τ and μ_{ij} are functions of x_1, x_2. If the distribution of roughness is periodic with a wavelength of the order of that of the acoustic wave, the boundary condition (3.13) should lead to selective reflection of the coherent wave in certain directions in analogy with an optical grating.

4. Incoherent scatter

Until now we have only dealt with the coherent wave generated by the reflection and multiple scatter. This coherent reflection involves no energy loss since only a change of phase is produced. Actually a small amount of first *order* energy is scattered in all directions and the amplitude of the coherent reflected wave is diminished by a corresponding amount.

The incoherent scatter may be readily evaluated from the amplitudes U_x and U_y of the coherent wave at $z = 0$. This displacement induces a source and dipole distribution proportional to U_x and U_y, which generates a radiation with an incoherent component. In is of interest to point out that for grazing incidence the components U_x and U_y at $z = 0$ tend to vanish because of the phase reversal. As a consequence the incoherent scatter in this case also tends to vanish. There are classical methods for the evaluation of the incoherent radiation from more or less randomly distributed sources and dipoles, which involve correlation coefficients for the geometry of the roughness. These methods being well known we shall not carry out this analysis.

Extension and range of validity of the analysis. Throughout sections 2 and 3, it has been assumed implicitly that the size of the roughness is small relative to the wavelength and that this smallness is understood in the mathematical sense. In addition the thickness of the fictitious fluid layer is also assumed large relative to the roughness and at the same time small relative to the wavelength. However the nature of the physical model used in the foregoing Lagrangian analysis leads to a possible extension of the applicability of the theory to a range of wavelengths which is not necessarily very large compared to the size of the roughness and the thickness h of the fluid layer. In the first place if we examine streamlines of flow patterns derived for potential flow around an obstacle such as a sphere we notice that the disturbance of the velocity field due to the obstacle is not important beyond a distance of about two diameters. Hence on the basis of physical intuition we may conclude that the Lagrangian analysis is approximately valid if we choose a layer thickness h of the order of two or three times the roughness size. Furthermore, instead of applying the boundary condition (2.44) at $z = 0$, we may use the boundary condition (2.3) at $z = h$ with the values (2.40) for the coefficients α_{ij}. By this procedure and on the basis of the foregoing physical reasoning the theory should remain approximately valid for wavelengths equal to only several times the roughness size along the reflecting surface. However in such a case the energy loss of the reflected coherent wave may be significant.

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