# *Non-linear Thermoelasticity, Irreversible Thermodynamics and Elastic Instability*

M. A. BIOT

*Communicated by the Editors*

**1. Introduction.** In the linear range a general theory of thermoelasticity based on irreversible thermodynamics was developed and discussed extensively in some earlier work ([3], [4], [9]). It was assumed that the elastic solid is initially in a stress-free equilibrium state at uniform temperature. Completely linearized perturbations in the vicinity of this equilibrium state are then analyzed assuming the strains, the rotations, the displacements and temperature variations to be small of the first order. The case of similar first order perturbations for a solid in equilibrium at uniform temperature but in a state of initial stress was also analyzed in the context of the analogous case of a fluid saturated porous medium ([14], [15]).

The general irreversible thermodynamics of linearized perturbations in the vicinity of equilibrium was developed by the author [7] for the case of either stable or unstable equilibrium. Instability is the consequence of the property that a new thermodynamic function of the perturbation is not a positive quadratic form. Moreover this thermodynamic function has been shown to represent the total entropy change of the system and its environment. The general mechanics of an initially stressed medium including unstable behavior was developed for elastic and viscoelastic media in the general context of irreversible thermodynamics. For an elastic solid under initial stress isothermal and adiabatic behavior were discussed [7] but heat conduction is not introduced explicitly although it is implicit in the thermodynamic context.

Our purpose here is to develop a non-linear theory of thermoelasticity which takes into account the non-linear geometry of the deformation while maintaining at the same time the validity of the linear irreversible thermodynamics. In a vast category of practical problems such a non-linear theory embodies the major physical features without complicating unduly the analytical treatment. The theory includes the non-linear thermoelasticity for an elastic solid either stress-free or already stressed in the initial state. The latter case leads to the non-linear thermoelastic analysis of *post-buckling* behavior. As a consequence a more thorough analysis is also derived of elastic instability for completely linearized perturbations in the vicinity of an unstable equilibrium state. The existence of isothermal and adiabatic buckling as well as the associated thermoelastic creep buckling already discovered earlier by the author ([7], [14], [15]) are further confirmed and clarified. The phenomenon is illustrated by the analogy with the linearized perturbation theory of an initially stressed porous medium containing a viscous fluid, which itself is a particular case of a recently developed thermodynamic theory of porous solids for finite deformations [16]. The non-oscillatory character of elastic instability is also established as a particular case of a general property of thermodynamic instability. The instability is related to the statistical probabilistic concept of entropy for a system with minimum entropy at equilibrium.

A variational formulation analogous to the principle of virtual work and d'Alembert's principle with corresponding Lagrangian equations provide a complete description of the behavior of the thermoelastic system by means of generalized coordinates. As already pointed out [9] such generalized coordinates are sufficient to describe a physical system macroscopically if we take into account the fact that there is a "resolution threshold". The Lagrangian equations also lead to an equivalent formulation analogous to Hamilton's principle.

An important simplification is obtained by using a cartesian definition of finite stress and strain ([5], [6], [7]) referred to locally rotated axes. This approach, in conjunction with the principle of virtual work, avoids the more ponderous techniques of the classical tensor calculus. In the context of irreversible thermodynamics the cartesian description of finite strain puts the problems of elastic stability in a new perspective. It should also provide a new impetus for a unified treatment of non-linear problems of thermal stress analysis. The Lagrangian equations are particularly well suited to obtain approximate solutions for complex technological systems. They also lead to a variety of formulations by means of "finite elements".

As already pointed out [7, p. 483] the choice of the local rotation is not unique. A number of alternative choices are discussed in the Appendix.

From the thermodynamic viewpoint the present treatment goes beyond classical concepts by taking into account Onsager's principle, and by using a thermoelastic potential [3] which differs from the classical thermodynamic functions. This is in contrast to some other attempts to formulate non-linear theories of thermoelasticity in the context of classical thermodynamics ([17], [18], [19], [20], [21]). One exception is represented by the work of Schapery [22], who has considered applications of irreversible thermodynamics to non-linear rheology.

## 2. Stress and strain referred to locally rotated axes.

For our purposes we shall first present an outline and further extensions of the cartesian definition of finite stress and strain as introduced by the author ([5], [6], [7]).

Let an elastic medium be deformed in a continuous fashion so that a material point initially at the point $x_i$ is displaced to a point of coordinates.

$$\xi_i = x_i + u_i(x_k) \tag{2.1}$$

The displacement is $u_i(x_k)$. The local differential transformation is

$$d\xi_i = (\delta_{ij} + a_{ij})\, dx_j \ , \tag{2.2}$$

where

$$a_{ij} = \frac{\partial u_i}{\partial x_j} \ . \tag{2.3}$$

From the standpoint of local behavior it is equivalent to consider the finite linear transformation

$$\xi_i = (\delta_{ij} + a_{ij})x_j \tag{2.4}$$

with nine coefficients $a_{ij}$. Hence it is sufficient to analyze a finite element originally a cube of unit volume oriented along the coordinate axes with one vertex at the origin subject to the linear transformation (2.4). We shall refer to this cube of unit volume as the *unit element.*

The transformation (2.4) is now considered to be the result of two successive transformations. The unit element is first deformed according to the linear transformation

$$\xi_i = (\delta_{ij} + \epsilon_{ij})x_j \ , \tag{2.5}$$

where

$$\epsilon_{ij} = \epsilon_{ji} \tag{2.6}$$

constitute six distinct coefficients. The transformation has the property that there are three orthogonal (principal) directions which do not change during the deformation. The deformed cube is now a parallelepiped on whose faces forces are applied represented by the nine components $\tau_{ij}'$. When the deformation is varied the virtual work of these forces is

$$\delta W_e = \tau_{ij}'\delta\epsilon_{ij} = \tau_{ij}\delta\epsilon_{ij} \ , \tag{2.7}$$

where

$$\tau_{ij} = \tau_{ji} = \tfrac{1}{2}(\tau_{ij}' + \tau_{ji}') \ . \tag{2.8}$$

We define the finite strain by the six quantities $\epsilon_{ij}$ and the corresponding stress by the six components $\tau_{ij}$. The second transformation is a rotation of the deformed element and of the coordinate axes, as if they both constituted a rigid system. The new axes after rotation become $x_i'$. With respect to these rotated axes, the deformation is still defined by a linear transformation with the same coefficients $\epsilon_{ij}$ as in (2.5) while the stress components $\tau_{ij}$ are unchanged.

As already pointed out [7] the present choice of a cartesian definition of finite strain is not unique. Other alternatives are discussed in the Appendix.

With the present definitions the first question which arises is how to express the stress components with respect to the original unrotated axes. The forces acting on opposite faces of the deformed unit element are equal and opposite. Referred to the unrotated axes $x_i$ these forces are represented by nine components $f_{ij}$ . A particular component $f_{ij}$ denotes the projection on the $x_i$ axis of the force acting on the face originally in the plane $x_j = 1$ perpendicular to the axis $x_j$ . The six strain components $\epsilon_{ij}$ are functions of the nine coefficients $a_{ij}$ . These nine coefficients include three degrees of freedom corresponding to a rigid rotation, hence for which the six strain components remain invariant. Variations $\delta a_{ij}$ define corresponding variations $\delta\epsilon_{ij}$ and the virtual work of the stresses on the unit element is

$$\delta W_e = f_{ij}\,\delta a_{ij} = \tau_{ij}\,\delta\epsilon_{ij} = \tau_{lk}\frac{\partial\epsilon_{lk}}{\partial a_{ij}}\,\delta a_{ij}\,. \tag{2.9}$$

We conclude

$$f_{ij} = \tau_{lk}\frac{\partial\epsilon_{lk}}{\partial a_{ij}}. \tag{2.10}$$

Note that these nine components are not independent, since their virtual work dissappears for a rigid rotation about each of the coordinate axis. Hence they satisfy identically the three conditions of equilibrium of moments about the axes.

Relations (2.10) assume that we know $\epsilon_{ij}$ as functions of $a_{ij}$ . In principle this relationship may be determined as follows. Because the transformations (2.4) and (2.5) differ only by a rigid rotation, the length of the vector $\xi_i$ is the same for both. As a consequence we may write

$$\begin{aligned}\xi_i\xi_i &= (\delta_{ki} + \epsilon_{ki})(\delta_{kj} + \epsilon_{kj})x_ix_j\\ &= (\delta_{ki} + a_{ki})(\delta_{kj} + a_{kj})x_ix_j\,.\end{aligned} \tag{2.11}$$

Identity of the two quadratic forms imply the six equations

$$2\epsilon_{ij} + \epsilon_{ki}\epsilon_{kj} = a_{ij} + a_{ji} + a_{ki}a_{kj} \tag{2.12}$$

which may be solved for the six unknowns $\epsilon_{ij}$ . The inconvenience of course is the non-linear character of the equations. This difficulty may be circumvented by a procedure introduced earlier ([5], [6], [7]). We put

$$\begin{aligned}e_{ij} &= e_{ji} = \tfrac{1}{2}(a_{ij} + a_{ji}),\\ \omega_{ij} &= -\omega_{ji} = \tfrac{1}{2}(a_{ij} - a_{ji}).\end{aligned} \tag{2.13}$$

With these quantities equations (2.12) become

$$2\epsilon_{ij} = 2e_{ij} + e_{kj}\omega_{ki} + e_{ki}\omega_{kj} + \omega_{ki}\omega_{kj} + e_{ki}e_{kj} - \epsilon_{ki}\epsilon_{kj}\,. \tag{2.14}$$

Considering $a_{ij}$ to be of the first order, a first order solution is

$$\epsilon_{ij} = e_{ij} . \tag{2.15}$$

Substitution of this value into $\epsilon_{ki}\epsilon_{kj}$ on the right side of equations (2.14) yields the second order approximation

$$\epsilon_{ij} = e_{ij} + \tfrac{1}{2}e_{kj}\omega_{ki} + \tfrac{1}{2}e_{ki}\omega_{kj} + \tfrac{1}{2}\omega_{ki}\omega_{kj} . \tag{2.16}$$

This process may be continued, by substituting again the value (2.16) of $\epsilon_{ij}$ into the term $\epsilon_{ki}\epsilon_{kj}$ of equations (2.14) and so on. Thus we may obtain $\epsilon_{ij}$ to any order.

The second order approximation (2.16) has been used extensively by the author in non-linear elasticity ([5], [6]) and in the mechanics of incremental deformations [7]. In the vast majority of problems, expression (2.16) is quite satisfactory. It has the crucial advantage over the tensor (2.12) of the classical theory, that the spurious terms $\epsilon_{ki}\epsilon_{kj}$ have been eliminated. By contrast, in expressions (2.16) the higher order terms are due only to the rotation, thus providing a clear distinction between the non-linear effects resulting from the geometry and those resulting from purely physical properties.

The dynamical equilibrium equations are obviously,

$$\frac{\partial f_{ij}}{\partial x_j} + \rho\mathcal{B}_i(x_k + u_k) = \rho\,\frac{\partial^2 u_i}{\partial t^2} , \tag{2.17}$$

where $\mathcal{B}_i(\xi_k)$ is the body force field per unit mass at the point $\xi_i = x_i + u_i$ and $\rho$ is the mass per unit initial volume at the initial point $x_i$ . Introducing the values (2.10) of $f_{ij}$ , equations (2.17) become

$$\frac{\partial}{\partial x_j}\left(\tau_{lk}\,\frac{\partial \epsilon_{lk}}{\partial a_{ij}}\right) + \rho\mathcal{B}_i(x_k + u_k) = \rho\,\frac{\partial^2 u_i}{\partial t^2}, \tag{2.18}$$

where $a_{ij}$ are the nine partial derivatives (2.3). The equilibrium equations (2.18) are thus expressed by means of the stress components $\tau_{ij}$ referred to locally rotated axes. The traction $f_i$ per unit initial area on a surface element whose outward unit normal is initially $n_i$ is given by

$$f_i = f_{ij}n_j = \tau_{lk}\,\frac{\partial \epsilon_{lk}}{\partial a_{ij}}\,n_j . \tag{2.19}$$

Equations (2.18) may also be derived from the following principle of virtual work

$$\int_\Omega \left(\tau_{lk}\,\delta\epsilon_{lk} + \rho\mathcal{B}_i\,\delta u_i - \rho\,\frac{\partial^2 u_i}{\partial t^2}\,\delta u_i\right) d\Omega = \int_A f_i\,\delta u_i\,dA , \tag{2.20}$$

where the integrals are extended to the domain $\Omega$ and its boundary $A$ in the space of initial coordinates $x_i$ . The variational principle (2.20) is the same as obtained earlier ([5], [6], [7]). It leads to the equilibrium equations in curvilinear coordinates by substituting values of $e_{ij}$ and $\omega_{ij}$ for curvilinear coordinates ([7], [24]).

**3. Thermoelastic potential.** During the deformation and heating of the unit element, its internal energy increases by the amount $u$ and its entropy by the amount $s$. In the initial state the medium is assumed stress-free and at a uniform temperature $T_r$. The thermodynamic function

$$v = u - T_r s \tag{3.1}$$

was proposed by the author ([1], [2], [3], [4]) in order to provide a general analysis of systems which are not at uniform temperature. In the present context in accordance with the previously introduced terminology [3] we shall refer to $v$ as the specific *thermoelastic potential* per unit initial volume. The heat absorbed by the unit element in a reversible process is denoted by $h$. Conservation of energy implies

$$du = \tau_{ij}\, d\epsilon_{ij} + dh \tag{3.2}$$

and the differential entropy is

$$ds = \frac{dh}{T}, \tag{3.3}$$

where $T = T_r + \theta$ is the temperature of the unit element and $\theta$ the temperature increase. Combining (3.1) (3.2) and (3.3) we derive

$$dv = \tau_{ij}\, d\epsilon_{ij} + \theta\, ds. \tag{3.4}$$

The thermoelastic potential $v$ is a function of the seven state variables $\epsilon_{ij}$ and $s$

$$v = v(\epsilon_{ij}, s). \tag{3.5}$$

Hence according to (3.4) the thermodynamic stress-strain relations are

$$\begin{aligned} \tau_{ij} &= \frac{\partial v}{\partial \epsilon_{ij}}, \\ \theta &= \frac{\partial v}{\partial s}. \end{aligned} \tag{3.6}$$

We may compare $v$ with the classical Helmholtz free energy

$$A = u - (T_r + \theta)s = v - \theta s \tag{3.7}$$

more frequently used by other investigators. One advantage of the function $v$ is the fact that it is positive definite while $A$ is not. It also leads quite naturally to Lagrangian equations and variational principles as part of a broader treatment of irreversible thermodynamics. In addition it will be shown in the next section that a fundamental property is obtained for the total thermoelastic potential defined as

Considering $a_{ij}$ to be of the first order, a first order solution is

$$\epsilon_{ij} = e_{ij} . \tag{2.15}$$

Substitution of this value into $\epsilon_{ki}\epsilon_{kj}$ on the right side of equations (2.14) yields the second order approximation

$$\epsilon_{ij} = e_{ij} + \tfrac{1}{2}e_{kj}\omega_{ki} + \tfrac{1}{2}e_{ki}\omega_{kj} + \tfrac{1}{2}\omega_{ki}\omega_{kj} . \tag{2.16}$$

This process may be continued, by substituting again the value (2.16) of $\epsilon_{ij}$ into the term $\epsilon_{ki}\epsilon_{kj}$ of equations (2.14) and so on. Thus we may obtain $\epsilon_{ij}$ to any order.

The second order approximation (2.16) has been used extensively by the author in non-linear elasticity ([5], [6]) and in the mechanics of incremental deformations [7]. In the vast majority of problems, expression (2.16) is quite satisfactory. It has the crucial advantage over the tensor (2.12) of the classical theory, that the spurious terms $\epsilon_{ki}\epsilon_{kj}$ have been eliminated. By contrast, in expressions (2.16) the higher order terms are due only to the rotation, thus providing a clear distinction between the non-linear effects resulting from the geometry and those resulting from purely physical properties.

The dynamical equilibrium equations are obviously,

$$\frac{\partial f_{ij}}{\partial x_j} + \rho\mathcal{B}_i(x_k + u_k) = \rho\frac{\partial^2 u_i}{\partial t^2} , \tag{2.17}$$

where $\mathcal{B}_i(\xi_k)$ is the body force field per unit mass at the point $\xi_i = x_i + u_i$ and $\rho$ is the mass per unit initial volume at the initial point $x_i$ . Introducing the values (2.10) of $f_{ij}$ , equations (2.17) become

$$\frac{\partial}{\partial x_j}\left(\tau_{lk}\frac{\partial \epsilon_{lk}}{\partial a_{ij}}\right) + \rho\mathcal{B}_i(x_k + u_k) = \rho\frac{\partial^2 u_i}{\partial t^2}, \tag{2.18}$$

where $a_{ij}$ are the nine partial derivatives (2.3). The equilibrium equations (2.18) are thus expressed by means of the stress components $\tau_{ij}$ referred to locally rotated axes. The traction $f_i$ per unit initial area on a surface element whose outward unit normal is initially $n_i$ is given by

$$f_i = f_{ij}n_j = \tau_{lk}\frac{\partial \epsilon_{lk}}{\partial a_{ij}} n_j . \tag{2.19}$$

Equations (2.18) may also be derived from the following principle of virtual work

$$\int_\Omega \left(\tau_{lk}\, \delta\epsilon_{lk} + \rho\mathcal{B}_i\, \delta u_i - \rho\frac{\partial^2 u_i}{\partial t^2}\, \delta u_i\right) d\Omega = \int_A f_i\, \delta u_i\, dA , \tag{2.20}$$

where the integrals are extended to the domain $\Omega$ and its boundary $A$ in the space of initial coordinates $x_i$ . The variational principle (2.20) is the same as obtained earlier ([5], [6], [7]). It leads to the equilibrium equations in curvilinear coordinates by substituting values of $e_{ij}$ and $\omega_{ij}$ for curvilinear coordinates ([7], [24]).

**3. Thermoelastic potential.** During the deformation and heating of the unit element, its internal energy increases by the amount $u$ and its entropy by the amount $s$. In the initial state the medium is assumed stress-free and at a uniform temperature $T_r$. The thermodynamic function

$$v = u - T_r s \tag{3.1}$$

was proposed by the author ([1], [2], [3], [4]) in order to provide a general analysis of systems which are not at uniform temperature. In the present context in accordance with the previously introduced terminology [3] we shall refer to $v$ as the specific *thermoelastic potential* per unit initial volume. The heat absorbed by the unit element in a reversible process is denoted by $h$. Conservation of energy implies

$$du = \tau_{ij}\, d\epsilon_{ij} + dh \tag{3.2}$$

and the differential entropy is

$$ds = \frac{dh}{T}, \tag{3.3}$$

where $T = T_r + \theta$ is the temperature of the unit element and $\theta$ the temperature increase. Combining (3.1) (3.2) and (3.3) we derive

$$dv = \tau_{ij}\, d\epsilon_{ij} + \theta\, ds. \tag{3.4}$$

The thermoelastic potential $v$ is a function of the seven state variables $\epsilon_{ij}$ and $s$

$$v = v(\epsilon_{ij}, s). \tag{3.5}$$

Hence according to (3.4) the thermodynamic stress-strain relations are

$$\tau_{ij} = \frac{\partial v}{\partial \epsilon_{ij}}, \qquad \theta = \frac{\partial v}{\partial s}. \tag{3.6}$$

We may compare $v$ with the classical Helmholtz free energy

$$A = u - (T_r + \theta)s = v - \theta s \tag{3.7}$$

more frequently used by other investigators. One advantage of the function $v$ is the fact that it is positive definite while $A$ is not. It also leads quite naturally to Lagrangian equations and variational principles as part of a broader treatment of irreversible thermodynamics. In addition it will be shown in the next section that a fundamental property is obtained for the total thermoelastic potential defined as

$$V = \int_\Omega v \, d\Omega = U - T_r S \tag{3.8}$$

with the integral over the domain $\Omega$ before deformation. In expression (3.8), $U$ and $S$ are the total internal energy and total entropy of the elastic medium. They are expressed by

$$U = \int_\Omega u \, d\Omega, \qquad S = \int_\Omega s \, d\Omega. \tag{3.9}$$

We should note that while (3.8) is formally similar to a classical expression for a system at uniform temperature it is physically quite different since here the temperature varies from point to point. Equation (3.8) also exhibits the important additive property of $v$ in contrast to A.

**Linear thermodynamics.** In the large majority of technological problems the elastic range is confined to strains $\epsilon_{ij}$ which are only a fraction of a percent. We may also assume that the relative temperature increase $\theta/T_r$ remains small. In such a case the stress-strain relations (3.6) may be linearized, which implies that $v$ is a quadratic form in the seven variables $\epsilon_{ij}$ and $s$. In order to evaluate $v$, we note that the integration of (3.4) is independent of the path of integration. We first integrate it at constant temperature $\theta = 0$ by varying only $\epsilon_{ij}$ , then at constant deformation we increase the temperature to $\theta$. By this process we obtain

$$v = \tfrac{1}{2} C_{ij}^{\mu\nu} \epsilon_{ij} \epsilon_{\mu\nu} + \int_0^\theta \theta \, ds. \tag{3.10}$$

The first group of terms,

$$v_r = \tfrac{1}{2} C_{ij}^{\mu\nu} \epsilon_{ij} \epsilon_{\mu\nu} , \tag{3.11}$$

represent the isothermal free energy at constant temperature where

$$C_{ij}^{\mu\nu} = C_{ji}^{\mu\nu} = C_{ij}^{\nu\mu} = C_{\mu\nu}^{ij} \tag{3.12}$$

are the isothermal elastic coefficients. In equation (3.10) the value of $s$ must be restricted to constant deformation. Because of linearization its value is

$$s = \frac{c\theta}{T_r} \tag{3.13}$$

where $c$ is the specific heat per unit volume, for constant deformation. Again because of linearization $c$ may be identified with its constant value for zero deformation. By substituting the value (3.13) of $s$ into (3.10) we obtain ([3], [4])

$$v = \tfrac{1}{2} C_{ij}^{\mu\nu} \epsilon_{ij} \epsilon_{\mu\nu} + \frac{1}{2} \frac{c\theta^2}{T_r} = v_r + \frac{1}{2} \frac{c\theta^2}{T_r}. \tag{3.14}$$

Hence we verify that $v$ is positive-definite with a zero minimum value in the initial state ($\epsilon_{ij} = \theta = 0$).

In order to apply relations (3.6) we must express $v$ in terms of $\epsilon_{ij}$ and $s$. This requires the knowledge of the following relation

$$T_r s = \gamma_{ij}\epsilon_{ij} + c\theta , \tag{3.15}$$

where $\gamma_{ij}$ are six physical coefficients, with the dimensions of elastic moduli. The physical significance of relation (3.15) is quite evident since in the linearized theory

$$h = T_r s \tag{3.16}$$

represents the heat absorbed by the unit element. We solve (3.15) for $\theta$ and substitute this value in expression (3.14) which becomes

$$v = \tfrac{1}{2}\bar{C}_{ij}^{\mu\nu}\epsilon_{ij}\epsilon_{\mu\nu} - \frac{1}{c}\gamma_{ij}\epsilon_{ij}s + \frac{1}{2}\frac{T_r}{c}s^2 , \tag{3.17}$$

where

$$\bar{C}_{ij}^{\mu\nu} = C_{ij}^{\mu\nu} + \frac{1}{cT_r}\gamma_{ij}\gamma_{\mu\nu} \tag{3.18}$$

are the adiabatic elastic moduli ([4], [7]). With the value (3.17) the thermodynamic stress-strain relations (3.6) are written

$$\begin{aligned} \tau_{ij} &= \bar{C}_{ij}^{\mu\nu}\epsilon_{\mu\nu} - \frac{1}{c}\gamma_{ij}s, \\ \theta &= -\frac{1}{c}\gamma_{ij}\epsilon_{ij} + \frac{T_r}{c}s. \end{aligned} \tag{3.19}$$

Although these relations are linear in $\epsilon_{ij}$, they remain non-linear with respect to the geometry since $\epsilon_{ij}$ are non-linear functions of $a_{ij}$. In addition the equilibrium equations (2.18) for $\tau_{ij}$ are also non-linear with respect to the geometry.

In the context of linearization the results derived in this section coincide with those obtained earlier ([3], [4]).

**4. Entropy production and dissipation function.** The Lagrangian analysis of dissipative systems as formulated by the author ([1], [2], [3], [4], [8], [9]) involves the generalization of the concept of dissipation function as an invariant proportional to the rate of entropy production. Its importance is derived from a dual role. As a physical concept it leads immediately to general phenomenological properties of dissipative systems. As a mathematical concept its invariance provides a powerful tool for the analytical formulation in arbitrary coordinate systems.

The dissipation function in the present analyses involves the entropy production associated with heat conduction. In this context we shall follow a procedure similar but somewhat more general than in the earlier analysis ([4], [8], [9]). We consider the rate of heat flow in the elastic medium. Let $\dot{h}$ be the rate of heat absorption in the unit element. The rate of heat flow outward through the face of the unit element originally normal to axis $x_i$ is denoted by $\dot{H}_i$. If there are no heat sources we may write the conservation relation

$$\dot{h} = -\frac{\partial \dot{H}_i}{\partial x_i}. \tag{4.1}$$

The rate of entropy increase of the unit element is

$$\dot{s} = \frac{\dot{h}}{T} = -\frac{1}{T}\frac{\partial \dot{H}_i}{\partial x_i}. \tag{4.2}$$

As shown by Meixner [23] this conservation condition may be written

$$\dot{s} = -\frac{\partial}{\partial x_i}\left(\frac{\dot{H}_i}{T}\right) - \frac{1}{T^2}\dot{H}_i\frac{\partial T}{\partial x_i}. \tag{4.3}$$

The first term is interpreted as due to an entropy flow $\dot{H}_i/T$ and the second term as representing an entropy production.

We integrate equation (4.3) over the domain $\Omega$ in the initial coordinates $x_i$. After integration by parts we obtain

$$\dot{S} + \int_A \frac{\dot{H}_i}{T} n_i \, dA = -\int_\Omega \frac{1}{T^2}\dot{H}_i \frac{\partial T}{\partial x_i}\, d\Omega \tag{4.4}$$

where $n_i$ in the unit outward normal at the boundary $A$ of $\Omega$, and

$$\dot{S} = \int_\Omega \dot{s}\, d\Omega \tag{4.5}$$

is the rate of change of the entropy $S$ of the elastic medium. The surface integral may be interpreted as follows. We assume that the environment of the elastic body is constituted by a large number of thermal reservoirs which are in thermal contact with it. One of these reservoirs is a large isothermal container at the constant temperature $T_r$. We shall call it the *thermal well* $R_w$. The others are a collection of *driving thermal reservoirs* $\sum R_T$ at various temperatures $T$. We denote by $H_w$ the heat energy acquired by the thermal well and by $H_T$ the heat energy acquired by the particular thermal reservoir $R_T$. We may write

$$\int_A \frac{\dot{H}_i}{T} n_i \, dA = \frac{\dot{H}_w}{T_r} + \sum \frac{\dot{H}_T}{T} = \sum \dot{S}_R . \tag{4.6}$$

As can be seen this represents the rate of change $\sum \dot{S}_R$ of the total entropy $\sum S_R$ of the reservoirs. With relation (4.6) equation (4.4) becomes

$$\dot{S}' = \dot{S} + \sum \dot{S}_R = -\int_\Omega \frac{1}{T^2}\dot{H}_i\frac{\partial T}{\partial x_i}\, d\Omega. \tag{4.7}$$

This is the rate of change of the total entropy $S' = S + \sum S_R$ of the elastic medium and its surrounding reservoirs. The rate of entropy production in this combined isolated system is

$$\dot{S}' = -\int_\Omega \frac{1}{T^2} \dot{H}_i \frac{\partial \theta}{\partial x_i} \, d\Omega. \tag{4.8}$$

We have replaced $\partial T/\partial x_i$ by $\partial\theta/\partial x_i$ since $T = T_r + \theta$. The integral (4.8) is applicable to an arbitrary domain. In particular we may apply it to a unit element surrounded by its own driving thermal reservoirs. Consider it to be in a state of frozen deformation with a uniform heat flow. The rate of entropy production of this element is

$$\dot{s}' = -\frac{\dot{H}_i}{T^2} \frac{\partial \theta}{\partial x_i}. \tag{4.9}$$

In the context of irreversible thermodynamics $\dot{H}_i/T^2$ are the fluxes conjugate to the forces $\partial\theta/\partial x_i$. According to Onsager's principle ([10], [11], [12], [13]) we may write

$$-\dot{H}_i = k_{ij} \frac{\partial \theta}{\partial x_j}, \tag{4.10}$$

where $k_{ij} = k_{ji}$ is the thermal conductivity tensor. By inversion, equations (4.10) become

$$-\frac{\partial \theta}{\partial x_i} = \lambda_{ij} \dot{H}_i, \tag{4.11}$$

where $\lambda_{ij} = \lambda_{ji}$. Hence the value (4.9) becomes

$$\dot{s}' = \frac{1}{T^2} k_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} = \frac{1}{T^2} \lambda_{ij} \dot{H}_i \dot{H}_j. \tag{4.12}$$

Since $\dot{s}'$ is always positive, expressions (4.12) are positive-definite quadratic forms. It follows that the total entropy production (4.8)

$$\dot{S}' = \int_\Omega \dot{s}' \, d\Omega = \int_\Omega \frac{k_{ij}}{T^2} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} \, d\Omega = \int_\Omega \frac{\lambda_{ij}}{T^2} \dot{H}_i \dot{H}_j \, d\Omega \tag{4.13}$$

is also positive definite.

An important relation may be obtained which relates $\dot{S}'$ to the thermoelastic potential $V$. According to equations (4.6) and (4.7) we may write

$$\dot{S}' = \dot{S} + \frac{H_w}{T_r} + \sum \frac{\dot{H}_T}{T}. \tag{4.14}$$

On the other hand conservation of energy for the elastic medium is expressed by

$$\dot{\mathfrak{I}} + \dot{U} = \dot{W} - \dot{H}_w - \Sigma \dot{H}_T, \tag{4.15}$$

where $\mathfrak{I}$ is the kinetic energy

$$\mathfrak{J} = \frac{1}{2}\int_\Omega \rho \dot{u}_i \dot{u}_i \, d\Omega \tag{4.16}$$

and $\dot{W}$ the power input of the body forces and surface tractions

$$\dot{W} = \int_\Omega \mathcal{B}_i \dot{u}_i \, d\Omega + \int_A f_i \dot{u}_i \, dA. \tag{4.17}$$

In addition equation (3.8) may be written

$$\dot{V} = \dot{U} - T_r \dot{S}. \tag{4.18}$$

Elimination of $\dot{U}$ and $\dot{S}$ between equations (4.14) (4.15) and (4.18) yields

$$-T_r \dot{S}' = \dot{V} + \dot{\mathfrak{J}} + \sum \frac{\theta}{T} \dot{H}_T - \dot{W}. \tag{4.19}$$

We define a dissipation function by

$$D = \tfrac{1}{2} T_r \dot{S}' = \frac{T_r}{2} \int_\Omega \frac{k_{ij}}{T^2} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} \, d\Omega = \frac{T_r}{2} \int_\Omega \frac{\lambda_{ij}}{T^2} \dot{H}_i \dot{H}_j \, d\Omega. \tag{4.20}$$

With this value equation (4.19) becomes

$$\dot{W} = \dot{\mathfrak{J}} + 2D + \dot{V} + \sum \frac{\theta}{T} \dot{H}_T \,. \tag{4.21}$$

The case where all thermal reservoirs are at the same temperature $\theta = 0$ is of particular interest. If the elastic medium returns to the same initial state, the value of $V$ and $\mathfrak{J}$ return to the same value. The total energy input of the body forces and surface traction obtained by the time integration of equation (4.21) is

$$W = \int_0^t \dot{W} \, dt = 2 \int_0^t D \, dt. \tag{4.22}$$

This is the energy dissipated. Again a time integration of equation (4.15) yields

$$W = H_W + \sum H_T \,. \tag{4.23}$$

Hence the energy dissipated appears in the form of heat injected into the surrounding.

**Dissipation function for linear thermodynamics.** For small deviations from equilibrium the temperature increases $\theta$ are small and we may replace $T$ by $T_r$ in expressions (4.20). They become

$$D = \frac{1}{2T_r} \int_\Omega k_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} \, d\Omega = \frac{1}{2T_r} \int_\Omega \lambda_{ij} \dot{H}_i \dot{H}_j \, d\Omega \tag{4.24}$$

which coincides with previous results ([3], [4], [9]). If the heat transfer at the boundary of the elastic medium is represented by a local heat transfer coefficient

$K$ we may write

$$\dot{H}_n = K(\theta - \theta_a) , \tag{4.25}$$

where $\dot{H}_n$ is the rate of heat flow outward per unit area, $\theta = T - T_r$ is the excess temperature of the surface $A$ of the solid and $\theta_a = T_a - T_r$ the excess temperature in the local thermal reservoir in the vicinity of the surface. In this case it was shown ([4], [8], [9]) that the dissipation function becomes

$$D = \frac{1}{2T_r} \int_\Omega \lambda_{ij} \dot{H}_i \dot{H}_j \, d\Omega + \frac{1}{2T_r} \int_A \frac{\dot{H}_n^2}{K} \, dA. \tag{4.26}$$

This includes the boundary dissipation which is represented by the second term.

**5. Variational principles and Lagrangian equations.** The Lagrangian equations and its associated variational principles will be derived here in the context of linear thermodynamics and non linear geometry. Although the derivation hereafter is based on the field equations it should be pointed out that the Lagrangian equations may also be derived *directly from Onsager's principle* as already shown previously in the context of the linear theory ([1], [2], [4]).

Consider the equilibrium equations (2.18) and the law of heat conduction (4.11). They are

$$\frac{\partial}{\partial x_j}\left(\tau_{lk} \frac{\partial \epsilon_{lk}}{\partial a_{ij}}\right) + \rho \mathcal{B}_i(x_k + u_k) = \rho \frac{\partial^2 u_i}{\partial t^2} , \qquad -\frac{\partial \theta}{\partial x_i} = \lambda_{ij} \dot{H}_j , \tag{5.1}$$

where $a_{ij}$ is defined as

$$a_{ij} = \frac{\partial u_i}{\partial x_j}. \tag{5.2}$$

This equation may be considered as a holonomic constraint implying continuity of the deformations. The strains $\epsilon_{ij}$ are thermodynamic state variables which are function of $a_{ij}$ , through expressions (2.16) and by the same token functions of $u_i$ through equation (5.2). Hence the strains $\epsilon_{ij}$ are functions of the displacement field $u_i$ .

The same considerations apply to the heat *displacement field* $H_i$ provided we limit ourselves to linear thermodynamics. The heat conservation equation (4.1) may be integrated with respect to the time t with zero initial conditions

$$h = -\frac{\partial H_i}{\partial x_i}. \tag{5.3}$$

Although this constitutes a holonomic constraint, the heat $h$ absorbed per unit initial volume is not a state variable. However it becomes one if the approximate relation (3.16)

$$s = \frac{h}{T_r} \tag{5.3}$$

is valid, *i.e.*, if we limit ourselves to small first order values for $\theta$, $h$ and $s$. Note that expression (5.4) for $s$ may be used in the value (3.4) of $v$. However it is not valid in the value (3.1) of $v$ because the latter requires a second order evaluation of $s$. Under these conditions the fields $u_i$ and $H_i$ may be considered as six unknowns governed by the six equations (5.1).

Variational principles may then be derived as follows: We multiply the first three equations (5.1) by variations $\delta u_i$ and the last three equations by $\delta H_i/T_r$. Addition of the results and integration over the initial domain $\Omega$ yield

$$\begin{aligned} &\int_\Omega \left[\frac{\partial}{\partial x_j}\left(\tau_{lk}\frac{\partial \epsilon_{lk}}{\partial a_{ij}}\right)\delta u_i - \frac{1}{T_r}\frac{\partial \theta}{\partial x_i}\,\delta H_i\right] d\Omega \\ &\qquad = \int_\Omega \left(\rho\frac{\partial^2 u_i}{\partial t^2}\,\delta u_i - \rho\mathcal{B}_i\,\delta u_i + \frac{1}{T_r}\lambda_{ij}\dot{H}_j\,\delta H_i\right) d\Omega. \end{aligned} \tag{5.5}$$

We integrate by parts the left side of this equation taking into account the continuity constraints (5.2) and (5.3). In addition account is taken of equations (3.4) which we may write

$$\delta v = \tau_{ij}\,\delta\epsilon_{ij} + \theta\,\delta s = \tau_{ij}\,\delta\epsilon_{ij} + \frac{\theta}{T_r}\,\delta h. \tag{5.6}$$

By this procedure equation (5.5) yields the variational principle

$$\begin{aligned} &\delta V + \int_\Omega \left(\rho\frac{\partial^2 u_i}{\partial t^2}\,\delta u_i + \frac{1}{T_r}\lambda_{ij}\dot{H}_j\,\delta H_i - \rho\mathcal{B}_i\,\delta u_i\right) d\Omega \\ &\qquad = \int_A \left(f_i\,\delta u_i - \frac{\theta}{T_r}n_i\,\delta H_i\right) dA \end{aligned} \tag{5.7}$$

where $V$ is the total thermoelastic potential (3.8) and $f_i$ is the boundary traction (2.19).

This variational principle leads to Lagrangian equations if we express the fields $u_i$ and $H_i$ by means of $n$ generalized coordinates $q_i$,

$$\begin{aligned} u_i &= u_i(q_1, q_2, \cdots q_n, x_k, t), \\ H_i &= H_i(q_1, q_2, \cdots q_n, x_k, t). \end{aligned} \tag{5.8}$$

The variations are

$$\begin{aligned} \delta u_k &= \frac{\partial u_k}{\partial q_i}\,\delta q_i, \\ \delta H_k &= \frac{\partial H_k}{\partial q_i}\,\delta q_i. \end{aligned} \tag{5.9}$$

Also

(5.10) $$\dot{H}_k = \frac{\partial H_k}{\partial q_i}\dot{q}_i + \frac{\partial H_k}{\partial t};$$

hence

(5.11) $$\frac{\partial \dot{H}_k}{\partial \dot{q}_i} = \frac{\partial H_k}{\partial q_i}.$$

In terms of the variations $\delta q_i$ we obtain the following results. First we write

(5.12) $$\int_\Omega \rho\mathfrak{B}_i\,\delta u_i\,d\Omega + \int_A \left(f_i\,\delta u_i - \frac{\theta}{T_r}n_i\,\delta H_i\right)dA = Q_i\,\delta q_i$$

with

(5.13) $$Q_i = \int_\Omega \rho\mathfrak{B}_k\frac{\partial u_k}{\partial q_i}\,d\Omega + \int_A \left(f_k\frac{\partial u_k}{\partial q_i} - \frac{\theta}{T_r}n_k\frac{\partial H_k}{\partial q_i}\right)dA.$$

By a well known classical procedure we obtain

(5.14) $$\int_\Omega \rho\frac{\partial^2 u_i}{\partial t^2}\,\delta u_i\,d\Omega = I_i\,\delta q_i,$$

where

(5.15) $$I_i = \frac{d}{dt}\left(\frac{\partial\mathfrak{T}}{\partial\dot{q}_i}\right) - \frac{\partial\mathfrak{T}}{\partial q_i}$$

and $\mathfrak{T}$ is the kinetic energy (4.16). Using relation (5.11) we derive

(5.16) $$\frac{1}{T_r}\int_\Omega \lambda_{ij}\dot{H}_j\,\delta H_i\,d\Omega = \frac{1}{T_r}\int_\Omega \lambda_{kj}\dot{H}_j\frac{\partial\dot{H}_k}{\partial\dot{q}_i}\,d\Omega\,\delta q_i = \frac{\partial D}{\partial\dot{q}_i}\,\delta q_i,$$

where $D$ is the dissipation function (4.24). Finally we also have the relation

(5.17) $$\delta V = \frac{\partial V}{\partial q_i}\,\delta q_i.$$

By substituting expressions (5.12), (5.14), (5.16) and (5.17) into the variational principle (5.7) and taking into account that $\delta q_i$ are arbitrary variations we obtain

(5.18) $$\frac{d}{dt}\left(\frac{\partial\mathfrak{T}}{\partial\dot{q}_i}\right) - \frac{\partial\mathfrak{T}}{\partial q_i} + \frac{\partial D}{\partial\dot{q}_i} + \frac{\partial V}{\partial q_i} = Q_i$$

which are $n$ Lagrangian equations for $q_i$. They govern the combined non-linear mechanical and thermodynamic system in a unified way including the thermoelastic dissipation.

**Generalized d'Alembert's principle and its Hamiltonian form.** If we put

(5.19) $$X = \frac{\partial D}{\partial\dot{q}_i},$$

the Lagrangian equations are obtained by writing that the variational equation

$$\delta V = (-I_i - X_i + Q_i)\delta q_i \tag{5.20}$$

is verified for arbitrary variations. This may be considered as a generalization of d'Alemberts principle where a dissipative force $-X_i$ must be added to the inertia force $-I_i$. As can be seen $X_i$ plays the role of a generalized "disequilibrium force" as already pointed out earlier [2]. By integration of equation (5.20) with respect to time we obtain

$$\int_0^t (\delta V + I_i\,\delta q_i + X_i\,\delta q_i - Q_i\,\delta q_i)\,dt = 0 \tag{5.21}$$

which is a Hamiltonian form of the variational principle.

**System with body force potential and surface heat transfer coefficient.** We consider the case where there is a body force potential $\phi(x_k + u_k)$ such that

$$\mathcal{B}_i = -\frac{\partial \phi}{\partial u_i}. \tag{5.22}$$

In addition we may assume that the surface heat transfer obeys the local condition (4.25). The Lagrangian equations (5.18) retain the same form and are written

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{T}}{\partial \dot{q}_i}\right) - \frac{\partial \mathfrak{T}}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} + \frac{\partial \mathcal{G}}{\partial q_i} = Q_i\,, \tag{5.23}$$

where $D$ is given by (4.26) and includes the boundary dissipation, while $\mathcal{G}$ is a total potential

$$\mathcal{G} = V + \int_\Omega \rho\phi\,d\Omega. \tag{5.24}$$

The value of $Q_i$ is also different and expressed by

$$Q_i = \int_A \left(f_k \frac{\partial u_k}{\partial q_i} - \frac{\theta_a}{T_r} n_k \frac{\partial H_k}{\partial q_i}\right) dA\,, \tag{5.25}$$

where $\theta_a$ is the excess temperature in the adjacent environment.

**Completeness of generalized coordinates and finite element analysis.** The description of a continuum by a finite number of generalized coordinates is of course not mathematically complete. However, as already pointed out [9], in physical systems it is possible to use the concept of *resolution threshold.* We may imagine that the system is composed of a large number of cells, each of which is described by a small number of generalized coordinates. In order for the description to be *physically complete* the cells need to be sufficiently small but not smaller than a resolution threshold beyond which the macroscopic laws involved in the formulation break down due to the atomistic structure of matter. Hence in such a case the physical description with a large but finite number of generalized coordinates may be said to be complete. The Lagrangian equations therefore embody the total physical properties of the system in a general unified form. They also provide the foundation for a large number of methods which have become known as "finite element" methods. The system is divided into a certain number of cells each of which is described by a small number of generalized coordinates, in such a way that interconnections constraints are satisfied. The Lagrangian equations for the total system of cells are then formulated with a limited number of unknowns. For the present case of thermoelasticity the results may be used to obtain a large variety of finite element methods in non-linear problems of thermal stress analysis. Finite element methods based on a Lagrangian approach were also briefly discussed earlier [9].

**6. Non-linear thermoelasticity with initial stress.** Consider now that in the initial state the solid is in equilibrium while already stressed at a uniform temperature $T_r$. The initial stress field is denoted by

$$S_{ij} = S_{ji} . \tag{6.1}$$

The initial state is one of thermodynamic equilibrium and we may expect that superimposed deformations obey the same thermodynamic principles as for the solid initially stress-free. The difference here is due to the fact that the initial equilibrium state may be unstable.

The initial stress field satisfies the equilibrium conditions

$$\frac{\partial S_{ij}}{\partial x_j} + \mathcal{B}_i(x_k) = 0, \tag{6.2}$$

and

$$f_i = S_{ij} n_j \tag{6.3}$$

is the corresponding boundary traction. Additional stresses and temperatures are superimposed associated with a displacement field $u_i$ and a heat displacement field $H_i$ defined as previously. The strain $\epsilon_{ij} = \epsilon_{ji}$ and the stress

$$\tau_{ij} = t_{ij} + S_{ij} \tag{6.4}$$

are referred, as above, to locally rotated axes. Therefore

$$t_{ij} = t_{ji} \tag{6.5}$$

is the incremental stress per unit initial area, referred to the same rotated axes. It is evident that without deformation or heating a unit element undergoes only a rigid rotation and the stress relative to the rotated axes remains the same. Hence the incremental stress $t_{ij}$ depends only on the strain $\epsilon_{ij}$ and on the thermodynamic variable.

A unit element as defined in Section 2 is now a unit cube initially with stresses

$S_{ij}$ acting on its faces. Under deformation and heating the internal energy and entropy of the element are increased by amounts $u$ and $s$. The corresponding specific thermoelastic potential is

$$v = u - T_r s. \tag{6.6}$$

This time we have the property

$$\tau_{ij} = t_{ij} + S_{ij} = \frac{\partial v}{\partial \epsilon_{ij}}. \tag{6.7}$$

This may be written

$$t_{ij} = \frac{\partial v'}{\partial \epsilon_{ij}}, \tag{6.8}$$

where

$$v' = v - S_{ij}\epsilon_{ij}. \tag{6.9}$$

This quantity $v'$ plays the role of a local incremental thermoelastic potential of a unit element provided we consider the initial stress as part of the elastic system with its own local potential $S_{ij}\epsilon_{ij}$. Using $v'$ we may analyze the thermodynamic stress strain relations as in the foregoing analysis of Section 3. We assume $\epsilon_{ij}$ and $s$ to be suitably small. In this case $v'$ is of the same form as (3.14), *i.e.*,

$$\begin{aligned} v' &= v_r' + \frac{1}{2}\frac{c\theta^2}{T_r}, \\ v_r' &= \tfrac{1}{2}C_{ij}^{\mu\nu}\epsilon_{ij}\epsilon_{\mu\nu}. \end{aligned} \tag{6.10}$$

The coefficients $C_{ij}^{\mu\nu}$ are the isothermal incremental elastic coefficients with the same symmetry properties as (3.12). We shall assume material stability of the initially stressed unit element, hence $v_r'$ and $v_r$ are positive definite. In terms of $\epsilon_{ij}$ and $s$ the incremental stress-strain relations are similar to (3.19), *i.e.*,

$$\begin{aligned} t_{ij} &= \bar{C}_{ij}^{\mu\nu}\epsilon_{\mu\nu} - \frac{1}{c}\gamma_{ij}s, \\ \theta &= -\frac{1}{c}\gamma_{ij}\epsilon_{ij} + \frac{T_r}{c}s, \end{aligned} \tag{6.11}$$

where $\bar{C}_{ij}^{\mu\nu}$ are the adiabatic incremental coefficients. They are given in terms of the isothermal incremental coefficients by relations of the same form as (3.18), [7]. These coefficients refer to incremental stresses. They depend on the state of initial stress and are not the same as those of Section 3.

Note that the initial stress $S_{ij}$ may not be the result of an initial elastic deformation. Only the incremental deformation are assumed to be elastic.

Also note that equations (6.11) are non-linear since $\epsilon_{ij}$ are non-linear functions of $a_{ij}$.

The equilibrium equations are obtained by substituting $\tau_{ij} = t_{ij} + S_{ij}$ in equations (5.1). In addition they may be put in incremental form by subtracting the equilibrium conditions (6.2) of the initial stress from the first three equations (5.1). The result is

$$\frac{\partial}{\partial x_j}\left[t_{lk}\frac{\partial \epsilon_{lk}}{\partial a_{ij}} + S_{lk}\frac{\partial \eta_{lk}}{\partial a_{ij}}\right] + \rho\Delta\mathcal{B}_i = \rho\frac{\partial^2 u_i}{\partial t^2}, \qquad -\frac{\partial \theta}{\partial x_i} = \lambda_{ij}\dot{H}_j, \tag{6.12}$$

where

$$\begin{aligned} \eta_{ij} &= \epsilon_{ij} - e_{ij} = \tfrac{1}{2}(e_{kj}\omega_{ki} + e_{ki}\omega_{kj} + \omega_{ki}\omega_{kj}), \\ \Delta\mathcal{B}_i &= \mathcal{B}_i(x_k + u_k) - \mathcal{B}(x_k). \end{aligned} \tag{6.13}$$

In deriving these equations use has been made of the identity $S_{ij} = S_{lk}\,\partial e_{lk}/\partial a_{ij}$. Equations (6.12) are six equations for the unknown fields $u_i$ and $H_i$. Note that the approximation (3.16) is also valid here. Corresponding variational principles may be derived as previously from equations (6.12). However they may be derived more directly as follows.

The variational principle (5.7) is obviously valid, since it follows from equations (5.1) in which $\tau_{ij} = t_{ij} + S_{ij}$. On the other hand the initial stress field satisfies the variational principle

$$\int_\Omega (S_{ij}\,\delta e_{ij} + \mathcal{B}_i\,\delta u_i)\,d\Omega = \int_A S_{ij}n_j\,\delta u_i\,dA. \tag{6.14}$$

If we subtract this equation from the variational principle (5.7) we obtain

$$\begin{aligned} \delta\mathcal{V} + \int_\Omega \left(\rho\frac{\partial^2 u_i}{\partial t^2}\,\delta u_i + \frac{1}{T_r}\lambda_{ij}\dot{H}_j\,\delta H_i - \rho\Delta\mathcal{B}_i\,\delta u_i\right)d\Omega& \\ = \int_A \left(f_i'\,\delta u_i - \frac{\theta}{T_r}n_i\,\delta H_i\right)dA& \end{aligned} \tag{6.15}$$

where

$$\begin{aligned} \mathcal{V} &= \int_\Omega (v' + S_{ij}\eta_{ij})\,d\Omega, \\ f_i' &= f_i - S_{ij}n_j. \end{aligned} \tag{6.16}$$

The vector $f_i'$ is the incremental boundary traction per unit initial area.

Since the variational principle (6.15) is of the same form as (5.7) it leads to corresponding Lagrangian equations. Expressing $u_i$ and $H_i$ by means of generalized coordinates $q_i$ as in equations (5.8) we derive

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{T}}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} + \frac{\partial \mathcal{V}}{\partial q_i} = Q_i. \tag{6.17}$$

In the most general case with a surface heat transfer coefficient as defined by relation (4.25) the dissipation function is given by expression (4.26) which includes the boundary dissipation. The generalized thermo-mechanical driving force is then given by

$$(6.18) \qquad Q_i = \int_\Omega \rho \Delta \mathcal{B}_k \frac{\partial u_k}{\partial q_i}\, d\Omega + \int_A \left( f_k' \frac{\partial u_k}{\partial q_i} - \frac{\theta_a}{T_r} n_k \frac{\partial H_k}{\partial q_i} \right) dA\ ,$$

where $\theta_a$ is in the excess temperature in the local environment in the vicinity of the boundary.

The Lagrangian equations (6.17) are now in incremental form. A fundamental difference with equations (5.18) is due to the fact that $\mathcal{V}$ may not be positive-definite in the vicinity of initial equilibrium. This corresponds to instability of the initial state.

**Thermodynamics of incremental deformations.** The initially stressed system is a system in thermodynamic equilibrium which is then deformed by additional stresses. This deformation shall include the effect of the non-linear geometry. Obviously from the thermodynamic viewpoint, the presence of initial stress does not introduce any essential difference. This can be seen in the variational principle (6.15) and in the Lagrangian equations (6.17) where the incremental tractions $f_i'$ and body forces $\Delta\mathcal{B}_i$ play the role of external forces applied to the system. The initial traction $f_i$ and body forces $\mathcal{B}_i$ are considered as belonging to the stressed system initially in equilibrium. In addition the function $\mathcal{V}$ plays the role of a thermoelastic potential due to deviations from the initial stressed state of equilibrium. That this is the case can be seen as follows. Using the value (6.13) of $\eta_{ij}$ we write equation (6.9) in the form

$$(6.19) \qquad v' + S_{ij}\eta_{ij} = v - S_{ij}e_{ij}\ .$$

By introducing the value (6.6) of $v$ and integrating over the volume we obtain

$$(6.20) \qquad \mathcal{V} = U - T_r S - \int_\Omega S_{ij} e_{ij}\, d\Omega\ ,$$

where

$$(6.21) \qquad U = \int_\Omega u\, d\Omega, \qquad S = \int_\Omega s\, d\Omega.$$

Integration by parts, taking into account the initial equilibrium conditions (6.2) and (6.3), yields

$$(6.22) \qquad \int_\Omega S_{ij} e_{ij}\, d\Omega = \int_\Omega \mathcal{B}_i(x_k) u_i\, d\Omega + \int_A f_i u_i\, dA.$$

With this value equation (6.20) becomes

$$(6.23) \qquad \mathcal{V} = U' - T_r S\ ,$$

where

$$U' = U - \int_{\Omega} \mathcal{B}_i(x_k)u_i \, d\Omega - \int_A f_i u_i \, dA. \tag{6.24}$$

This represents the total internal energy of the system including both the solid and the initially applied forces as part of the system. Hence from the thermodynamic viewpoint $\mathcal{V}$ is the total thermoelastic potential for incremental deformations of this combined system. In particular we may apply equation (4.19) by writing

$$-T_r \dot{S}' = \dot{\mathcal{V}} + \dot{\mathfrak{T}} + \sum \frac{\theta}{T} \dot{H}_T - \dot{\mathcal{W}} , \tag{6.25}$$

where $\dot{\mathcal{W}}$ is now the power input of the incremental forces

$$\dot{\mathcal{W}} = \int_{\Omega} \Delta\mathcal{B}_i \dot{u}_i \, d\Omega + \int_A f_i' \dot{u}_i \, dA \tag{6.26}$$

while $S'$ is the total entropy including the thermal environment.

**7. Thermodynamics of elastic instability.** The foregoing analysis of incremental deformations starting from an initial state of stress provides the foundation for a general discussion of elastic instability. Equations (6.12) are essentially non-linear with respect to the geometry of the deformation. However, for our purpose, the first three equations (6.12) may be further linearized with respect to incremental strains and displacements by substituting the first order approximations

$$\begin{aligned} \epsilon_{ij} &= e_{ij} , \\ \Delta\mathcal{B}_i &= \frac{\partial \mathcal{B}_i}{\partial x_j} u_j . \end{aligned} \tag{7.1}$$

To the first order the equilibrium equations (6.12) then become

$$\frac{\partial}{\partial x_j}\left(t_{ij} + S_{lk} \frac{\partial \eta_{lk}}{\partial a_{ij}}\right) + \rho \frac{\partial \mathcal{B}_i}{\partial x_j} u_j = \rho \frac{\partial^2 u_i}{\partial t^2}. \tag{7.2}$$

The stress-strain relations (6.11) are further linearized using the value (7.1) of $\epsilon_{ij}$ , *i.e.*,

$$\begin{aligned} t_{ij} &= \bar{C}_{ij}^{\mu\nu} e_{\mu\nu} - \frac{1}{c} \gamma_{ij} s, \\ \theta &= -\frac{1}{c} \gamma_{ij} e_{ij} + \frac{T_r}{c} s. \end{aligned} \tag{7.3}$$

We note that in equations (7.2) the influence of the initial stress is clearly brought out by the term $S_{lk}\, \partial\eta_{lk}/\partial a_{ij}$ which is due essentially to a non-linear property of the geometry.

Equations (7.2) developed by the author ([5], [6], [7]) have been extensively applied in the more general context of viscoelasticity [7]. For an elastic body the particular case of isothermal buckling ($\theta = 0$) or adiabatic buckling ($s = 0$) were discussed [7]. The characteristic features introduced by heat conduction in stability problems were also derived on the basis of a mathematical analogy with problems of stability of a fluid-saturated porous solid ([14], [15]). It was pointed out that in addition to the limiting cases of isothermal and adiabatic buckling there is an intermediate mixed behavior represented by thermoelastic creep. Further discussion of these features from a fundamental thermodynamic viewpoint is presented hereafter.

We consider the initially stressed solid to be in thermodynamic equilibrium at a constant temperature $T_r$. Let us investigate the stability of perturbations of this initial state on the basis of the linearized equations (7.2) and (7.3). For simplicity we shall assume that the body force is represented by a uniform gravity field; hence

$$\Delta \mathcal{B}_i = 0. \tag{7.4}$$

We also assume that the boundary of the elastic body is either adiabatic or that it may have a local heat transfer coefficient $K$ to an immediate environ maintained at the constant temperature $T_r$ (hence $\theta_a = 0$). Finally we assume that if the displacement at the boundary is not zero, then the incremental traction $f_i'$ at that point remains zero. Under these assumptions the generalized force (6.18) in the corresponding Lagrangian equations vanishes

$$Q_i = 0. \tag{7.5}$$

In order to investigate the stability we represent the perturbations as linear functions of generalized coordinates as follows

$$\begin{aligned} u_i &= u_{ij} q_j , \\ H_i &= H_{ij} q_j , \end{aligned} \tag{7.6}$$

where $u_{ij}$ and $H_{ij}$ are given functions of the coordinates $x_i$. With $\epsilon_{ij} = e_{ij}$ we may write

$$\mathcal{V} = \int_\Omega (v' + S_{ij}\eta_{ij})\, d\Omega, \tag{7.7}$$

where

$$\begin{aligned} v' &= v_r' + \frac{1}{2}\frac{c\theta^2}{T_r} , \\ v_r' &= \tfrac{1}{2} C_{ij}^{\mu\nu} e_{ij} e_{\mu\nu} . \end{aligned} \tag{7.8}$$

Hence $\mathcal{V}$ is the quadratic form

$$\mathcal{V} = \tfrac{1}{2} a_{ij} q_i q_j . \tag{7.9}$$

The following quadratic forms are also derived from the values (4.16) and (4.26):

$$D = \tfrac{1}{2} b_{ij} \dot{q}_i \dot{q}_j \, , \qquad \mathfrak{T} = \tfrac{1}{2} m_{ij} \dot{q}_i \dot{q}_j \, . \tag{7.10}$$

With these values, taking into account equation (7.5), the Lagrangian equations (6.17) become

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{T}}{\partial \dot{q}_i}\right) + \frac{\partial D}{\partial \dot{q}_i} + \frac{\partial \mathfrak{V}}{\partial q_i} = 0 \tag{7.11}$$

or

$$m_{ij}\ddot{q}_j + b_{ij}\dot{q}_j + a_{ij}q_j = 0. \tag{7.12}$$

This is a set of linear homogeneous equations for the unknowns $q_i$ .

**Non-oscillatory character of unstable solutions.** The characteristic solutions of the linear system (7.12) are of the type

$$q_i = C_i e^{pt}. \tag{7.13}$$

When $\mathfrak{V}$ is positive-definite these solutions are necessarily stable and the real part of $p$ is negative representing damped oscillations. Unstable solutions are possible if $\mathfrak{V}$ may acquire negative values. The important property here is that unstable solutions are non-oscillatory, corresponding to real positive values of $p$. This property is a consequence of a general theorem derived in the author's book, where it is discussed in a more general context and shown to apply to unstable thermodynamic systems in the vicinity of equilibrium [7].

**Isothermal and adiabatic buckling. Thermoelastic creep.** In order to clarify the nature of thermoelastic instability it is useful to assume that inertia forces are negligible. For example we may assume $p = 0$; hence $\mathfrak{T} = 0$. The effect of body forces also dissappears in this case and the equations become identical to those of an initially stressed porous solid containing a viscous fluid. In the context of this analogy the buckling of both the thermoelastic and the porous solid have been discussed previously ([14], [15]). There are two critical buckling loads. In the thermoelastic case the lower value corresponds to isothermal buckling. The rate of deformation in this case is sufficiently slow so that temperatures remain approximately uniform throughout. Obviously this represents a *creep buckling* of purely thermoelastic origin. On the other hand the upper critical load corresponds to purely adiabatic deformations where the rate of deformation is so large that no heat flow can occur in the solid. For a massless medium this adiabatic buckling corresponds to an instantaneous collapse. The same behavior occurs for a porous solid where the lower critical load leads to creep buckling with a slow diffusion of the fluid through the pores, while for the higher critical load fast deformation does not allow any exchange of fluid between pores.

A more explicit analytic analysis of these features is obtained as follows. As stated we put $p = 0$ for the purpose of discussion. We write $\mathcal{V}$ in the form

$$\mathcal{V} = \mathcal{V}_r + \mathcal{V}_\theta ,$$

$$\mathcal{V}_r = \int_\Omega (\tfrac{1}{2} C_{ij}{}^{\mu\nu} e_{ij} e_{\mu\nu} + S_{ij}\eta_{ij})\, d\Omega, \tag{7.14}$$

$$\mathcal{V}_\theta = \frac{1}{2} \int_\Omega \frac{c\theta^2}{T_r}\, d\Omega.$$

We also separate the thermal and mechanical variables by writing (7.6) in the form

$$\begin{aligned} u_k &= u_{ki} q_i , \\ H_k &= H_{kl} q_l , \end{aligned} \tag{7.15}$$

where $i$ assumes values from 1 to $\nu$ while $l$ assumes values from $\nu + 1$ to $n$. Since $\mathcal{V}_r$ depends only on $u_i$ , it is a quadratic form in $q_i$ while $\mathcal{V}_\theta$ is a quadratic form in $q_i$ and $q_l$ . The dissipation function $D$ is a quadratic form in $\dot{q}_l$ . We write the Lagrangian equations (7.12) for this case. After substituting an exponential solution of the type (7.13) the Lagrangian equations become

$$\begin{aligned} \frac{\partial \mathcal{V}_r}{\partial q_i} + \frac{\partial \mathcal{V}_\theta}{\partial q_i} &= 0, \\ \frac{\partial \mathcal{V}_\theta}{\partial q_l} + p \frac{\partial \mathfrak{D}}{\partial q_l} &= 0, \end{aligned} \tag{7.16}$$

where $\mathfrak{D}$ is the quadratic form obtained by substituting $q_l$ for $\dot{q}_l$ in the dissipation function $D$. The characteristic solutions for incipient instability are obtained for $p = 0$. In this case we add equations (7.16) after multiplying the first group by $q_i$ and the second by $q_l$ . This yields

$$\mathcal{V}_r + \mathcal{V}_\theta = 0. \tag{7.17}$$

For a physically stable material, $C_{ij}{}^{\mu\nu} e_{ij} e_{\mu\nu}$ is positive definite. Also $\mathcal{V}_\theta$ is positive definite. Hence equation (7.17) requires that $S_{ij}\eta_{ij}$ be negative of sufficient magnitude. Hence, for increasing initial stress, equation (7.17) is first verified when

$$\mathcal{V}_r = \mathcal{V}_\theta = 0. \tag{7.18}$$

This implies $\theta = 0$, *i.e.*, isothermal deformations. Furthermore in this case

$$\frac{\partial \mathcal{V}_\theta}{\partial q_i} = \int_\Omega \frac{c\theta}{T_r} \frac{\partial \theta}{\partial q_i}\, d\Omega = 0. \tag{7.19}$$

Hence the first of equations (7.16) becomes

(7.20) $$\frac{\partial \mathcal{V}_r}{\partial q_i} = 0.$$

Note that for $\theta = 0$, $\mathcal{V}_r$ represents the classical *free energy at constant temperature $T_r$ for perturbations of the initially stressed system* as is shown by expression (6.23). As a consequence of equation (7.20), incipient instability occurs for stationary values of this isothermal free energy $\mathcal{V}_r$. According to (7.14) $\mathcal{V}_r$ is expressed by means of the isothermal incremental coefficients $C_{ij}^{\mu\nu}$.

The other extreme case is obtained by putting $p = \infty$ in equations (7.16). Since $\mathfrak{D}$ is positive definite this requires $q_l = H_l = h = 0$. The buckling is then instantaneous and adiabatic. In this case we must substitute for $\theta$ the value (7.3) with $s = 0$. We obtain

(7.21) $$\mathcal{V} = \mathcal{V}_r + \mathcal{V}_\theta = \int_\Omega \tfrac{1}{2}\bar{C}_{ij}^{\mu\nu} e_{ij} e_{\mu\nu} + S_{ij}\eta_{ij})\, d\Omega$$

where $\bar{C}_{ij}^{\mu\nu}$ are the adiabatic incremental coefficients. From (7.16) we derive

(7.22) $$\frac{\partial}{\partial q_i}(\mathcal{V}_r + \mathcal{V}_\theta) = 0$$

which are the equations for adiabatic buckling.

For intermediate values of the initial stress between isothermal and adiabatic buckling the value of $p$ is finite corresponding to *creep instability.*

In the foregoing discussion we have assumed $\rho = 0$. Actually of course this is not the case and for large values of $p$ inertia enters into play. However, we may conclude that for critical loads between isothermal and adiabatic values, the instability is dominated by thermal diffusion, and will be in the nature of a creep buckling. Beyond the adiabatic value the phenomenon is dominated by inertia, and will be in the nature of a dynamic buckling, with a correction due to thermal diffusion.

**Thermoelastic buckling as a particular case of unstable thermodynamic equilibrium.** Elastic instability may be related directly to the properties of entropy. The general theory of instability of a thermodynamic system in the vicinity of an equilibrium state was developed by the author [7] mainly in the context of elasticity and viscoelasticity. The properties are represented by the thermodynamic function $\mathcal{V}$. Instability occurs when the latter is not positive definite. The relation of $\mathcal{V}$ to entropy was already derived in some earlier work ([1], [2], [4]). The result is fundamentally the same as in equation (6.25). In the present discussion we have assumed that there are no incremental applied forces and that the kinetic energy is negligible ($\dot{\mathfrak{J}} = \dot{\mathcal{W}} = 0$). In addition the excess temperature $\theta_a$ of the environment is also assumed to vanish. In equation (6.25) $\theta$ may be replaced by $\theta_a$, hence we may put $\theta = 0$. Under these conditions equation (6.25) may be written

(7.23) $$-\mathcal{V} = T_r S'.$$

Hence $-\mathcal{V}$ is proportional to the total entropy of the isolated system composed of the solid and its isothermal environment. If $\mathcal{V} = 0$ is not a minimum at equilibrium, and may become negative for small perturbations, we have shown that the system is unstable. According to (7.23) negative values of $\mathcal{V}$ correspond to an increase in total entropy $S'$. Thus we have a typical case of thermodynamic instability since according to the statistical probability concept of entropy it will tend to increase spontaneously.

**Appendix—Alternative cartesian representations of finite deformations.** The cartesian definition of finite strain is by no means restricted to the transformation (2.5) as already pointed out [7, (page 483)]. For example we may first apply a transformation

$$\xi_i = (\delta_{ij} + \epsilon_{ij}')x_j , \tag{A.1}$$

where $\epsilon_{ij}' \neq \epsilon_{ji}'$ , which is then followed by a solid rotation. The non-symmetric tensor $\epsilon_{ij}'$ is just as adequate to describe the deformation, provided the nine components $\epsilon_{ij}'$ are constrained to satisfy three relations so that they contain only six degrees of freedom. These six degrees of freedom must of course be such that they still represent the most general deformation. In the particular case of the transformation (2.5) these three relations are

$$\epsilon_{ij}' = \epsilon_{ji}' . \tag{A.2}$$

However, we may just as well choose the relations to be

$$\epsilon_{31}' = \epsilon_{32}' = \epsilon_{21}' = 0. \tag{A.3}$$

The six remaining components $\epsilon_{11}'$ , $\epsilon_{22}'$ , $\epsilon_{33}'$ , $\epsilon_{13}'$ , $\epsilon_{23}'$ , $\epsilon_{12}'$ may then be considered as a measure of the strain. These values represent the deformation of a cube of unit size originally oriented along the axes, such that the edge along the $x_1$ axis remains on this axes, while the face in the $x_1x_2$ plane remains in this plane. After performing a solid rotation we obtain the transformation (2.1). The condition that the two transformations (2.1) and (A.1) represent the same deformation is expressed by a condition similar to equation (2.12) which in this case becomes

$$\epsilon_{ij}' + \epsilon_{ji}' = a_{ij} + a_{ji} + a_{ki}a_{kj} - \epsilon_{ki}'\epsilon_{kj}' . \tag{A.4}$$

These six equations determine the six components $\epsilon_{ij}'$ . A first approximation is

$$\begin{aligned} &\epsilon_{11}' = a_{11} , && \epsilon_{22}' = a_{22} , && \epsilon_{23}' = a_{33} , \\ &\epsilon_{12}' = a_{12} + a_{21} , && \epsilon_{13}' = a_{13} + a_{31} , && \epsilon_{23}' = a_{23} + a_{32} . \end{aligned} \tag{A.5}$$

Substitution of these values into $\epsilon_{ki}'\epsilon_{kj}'$ on the right side of equations (3.48) yields the second approximation. Continuation of this process yields $\epsilon_{ij}'$ to any order.

The great advantage of this procedure is the possibility of choosing material directions of reference for the deformation which are most suitable to describe the physical properties. For example if the material is transverse isotropic about the $x_3$ axis it is convenient to choose

$$\epsilon_{12}' = \epsilon_{21}', \quad \epsilon_{31}' = \epsilon_{32}' = 0. \tag{A.6}$$

Again in this case during the deformation the face in the $x_1x_2$ plane remains in this plane. The six independent components of $\epsilon_{ij}'$ are again determined by the six equations (A.4). Their evaluation up to any order is obtained by the same process of successive approximations.

The virtual work for a virtual deformation $\delta\epsilon_{ij}'$ is

$$\delta W = \tau_{ij}'\delta\epsilon_{ij}'. \tag{A.7}$$

This defines the six corresponding stress components $\tau_{ij}'$. As in the case of equation (2.9) we derive

$$f_{ij} = \tau_{lk}' \frac{\partial \epsilon_{lk}'}{\partial a_{ij}}. \tag{A.8}$$

The nine components $f_{ij}$ thus obtained in terms of $\tau_{ij}'$ satisfy the three conditions of equilibrium of moments.

For convenience we may introduce the following symmetric notation. For example in case (A.3) we write

$$\begin{array}{llll} \epsilon_{11} = \epsilon_{11}', & \epsilon_{12} = \epsilon_{21} = \tfrac{1}{2}\epsilon_{12}', & \tau_{11} = \tau_{11}', & \tau_{12} = \tau_{21} = \tau_{12}', \\ \epsilon_{22} = \epsilon_{22}', & \epsilon_{23} = \epsilon_{32} = \tfrac{1}{2}\epsilon_{23}', & \tau_{22} = \tau_{22}', & \tau_{23} = \tau_{32} = \tau_{23}', \\ \epsilon_{33} = \epsilon_{33}', & \epsilon_{31} = \epsilon_{13} = \tfrac{1}{2}\epsilon_{13}', & \tau_{33} = \tau_{33}', & \tau_{31} = \tau_{12} = \tau_{13}'. \end{array} \tag{A.9}$$

and in case (A.6) we write the same values for $\epsilon_{ij}$ and $\tau_{ij}$ except for $\epsilon_{12}$ and $\tau_{12}$ which are defined as

$$\tau_{12} = \tau_{21} = \tfrac{1}{2}(\tau_{12}' + \tau_{21}'), \quad \epsilon_{12} = \epsilon_{21} = \epsilon_{12}' = \epsilon_{21}'. \tag{A.10}$$

It should be understood that this is purely conventional and that the symmetric notation of $\tau_{ij}$ and $\epsilon_{ij}$ does not imply that they are symmetric tensors. With this notation, equations (2.10) and (2.19) for $f_{ij}$ and $f_i$ remain formally the same and the foregoing analysis of thermoelasticity remains the same.

## REFERENCES

1. M. A. BIOT, *Theory of stress-strain relations in anisotropic viscoelasticity and relaxation phenomena,* Jour. Appl. Phys. **25** (1954), 1385–1391.
2. M. A. BIOT, *Variational principles in irreversible thermodynamics with application to viscoelasticity,* The Physical Review **97** (1955), 1463–1469.
3. M. A. BIOT, *Thermoelasticity and irreversible thermodynamics,* Jour. Appl. Phys. **27** (1956), 240–253.
4. M. A. BIOT, *Linear thermodynamics and the mechanics of solids,* Proc. Third U. S. National Congress Appl. Mech. (Amer. Soc. Mech, Eng.) pp. 1–18, New York, 1958.

5. M. A. Biot, *Non-linear theory of Elasticity and the linearized case for a body under initial stress*, Phil. Mag. **27** (1939), 468–489.
6. M. A. Biot, *Elastizitätstheorie zweiter Ordnung mit Anwendungen*, Zeit. Angew Math. u. Mech. (2) **20** (1940), 89–99.
7. M. A. Biot, *Mechanics of incremental deformations*, John Wiley & Sons, Inc. New York, 1965.
8. M. A. Biot, *New methods in heat flow analysis with application to flight structures*, Jour. Aeronautical Sciences **24** (1957), 857–873.
9. M. A. Biot, *Variational principles in heat transfer—A unified Lagrangian analysis of dissipative phenomena*, Oxford Press, 1970.
10. L. Onsager, *Reciprocal relations in irreversible processes*, I, The Physical Rev. **37** (1930), 405–426.
11. L. Onsager, *Reciprocal relations in irreversible processes*, II, The Physical Rev. **38** (1931), 2265–2279.
12. L. Onsager & S. Machlup, *Fluctuations and irreversible processes I*, The Physical Rev. **91** (1953), 1505–1512.
13. S. Machlup & L. Onsager, *Fluctuations and irreversible processes* II, The Physical Rev. **91** (1953), 1512–1515.
14. M. A. Biot, *Theory of stability and consolidation of a porous medium under initial stress*, J. Math. Mech. **12** (1963), 521–542.
15. M. A. Biot, *Theory of buckling of a porous slab and its thermoelastic analogy*, Jour. of Applied Mech. (E.) **31** (1964), 194–198.
16. M. A. Biot, *Theory of finite deformations of porous solids*, Indiana University Mathematics Journal **21** (1972), 597–620.
17. O. W. Dillon, *A non-linear thermoelastic theory*, J. Mech. Phys. Solids **10** (1962), 128.
18. H. Parkus, *Methods of solution of thermoelastic boundary value problems*, in *High temperature structures and materials*, pp. 317–347, Pergamon Press, London, 1964.
19. A. E. Green, *Thermoelastic stresses in initially stressed bodies*, Proc. Roy. Soc. (A) **266** (1962), 1.
20. J. E. Adkins, *Large elastic deformations*, Progress in Solid Mechanics **II**, Amsterdam, 1961.
21. B. D. Coleman, *Thermodynamics of materials with memory*, Archives for Rational Mechanics and Analysis **17** (1964), 1–46.
22. R. A. Schapery, *A theory of non-linear thermoviscoelasticity, based on irreversible thermodynamics*, Proc. Fifth U. S. Congr. Appl. Mech. (1966), 511–530.
23. J. Meixner, *Zur Thermodynamik der Thermodiffusion*, Annalen der Physik **39** (1941), 333–356.
24. G. Temple, *Cartesian tensors*, Methuen & Co. Ltd. London, 1960.

This research was sponsored by the Air Force Office of Scientific Research (N. A. M.), through the European Office of Aerospace Research AFSC, United States Air Force, under contract F 44620-71-C-0092.

Royal Academy of Belgium

*Date communicated:* January 5, 1973